Winter School and Workshop on Polytopes 2025

A Brief Introduction to Random Polytopes

Florian Besau*

*Technische Universität Wien

florian.besau@tuwien.ac.at

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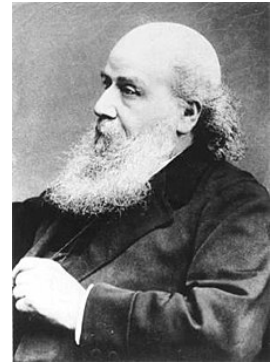
Overview

- Part I: Sylvester's Four Point Problem
- Part II: Asymptotic Results for the Expectation
- Part III: Concentration, Variance Bounds and Central Limit Theorems

Part I: **Sylvester's Four Point Problem**

"What is the probability that 4 random points in the plane are in convex position?"

paraphrasing J.J. Sylvester [Educational Times 1864, Question 1491]



James Joseph Sylvester
(1814–1897)

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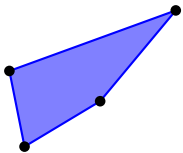
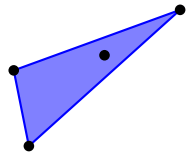
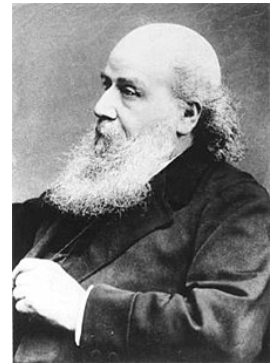


4 points in convex position.



Not in convex position!



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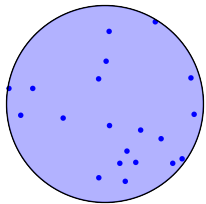
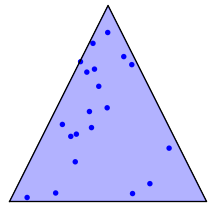
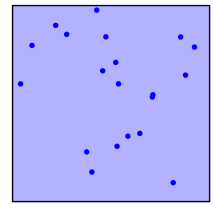
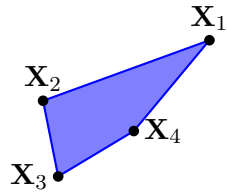
- The classical approach is to fix a convex body (compact and convex set with non-empty interior) $K \subset \mathbb{R}^2$ and consider points $\mathbf{X}_i$ drawn from K uniformly and independently, i.e., for measurable $A \subset K$ the probability that $\mathbf{X}_i$ is contained in A is

$$\mathrm{Prob}(\mathbf{X}_i \in A) = \frac{\mathrm{Vol}_2(A)}{\mathrm{Vol}_2(K)}.$$

Sylvester's question is to determine

$$p_4(K) = \mathrm{Prob}(\{\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \mathbf{X}_4\} \text{ are in convex position})$$



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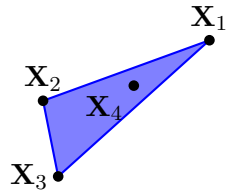

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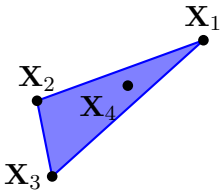
Note that $\mathrm{conv}\{\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3\}$ spans a proper triangle with probability 1!



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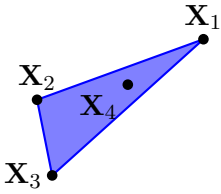

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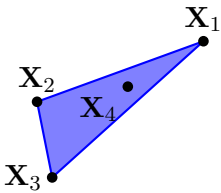

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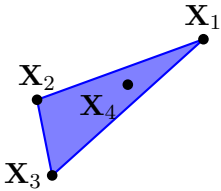
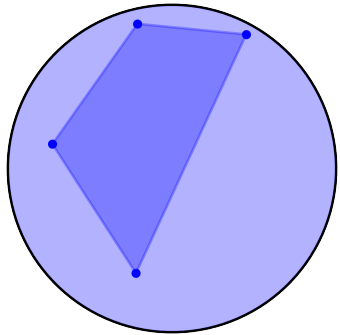


Solving $p_4(D)$ for the unit disk $D := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$.



- Since $\mathbf{X}_i$ are drawn uniformly from K, we find that

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  is an affine invariant, that is, for any linear map $A \in \mathrm{GL}(\mathbb{R}^2)$ and translation $x \in \mathbb{R}^2$ we have

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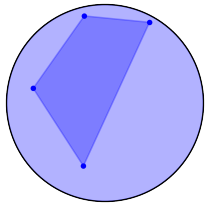
  - For a circular disk (or any ellipse) D we have $p_4(D) = 1 - \frac{35}{12\pi^2} = 70.447988..\%$.



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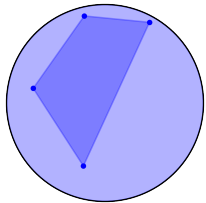

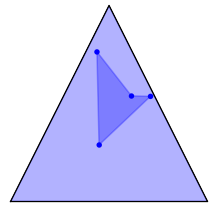

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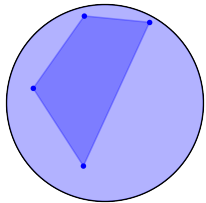
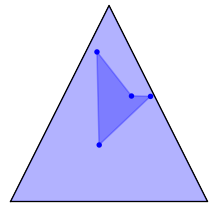
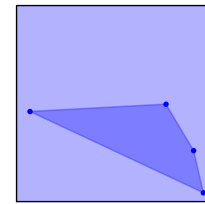
  - For a square (or any parallelogram) $S = [0,1]^2$ we have $p_4(S) = 25/36 = 69.\dot{4}\%$.







Theorem (Blaschke, 1923).
For any convex body $K \subset \mathbb{R}^2$ we have

$$p_4(K) \leq p_4(D).$$

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(1885–1962)

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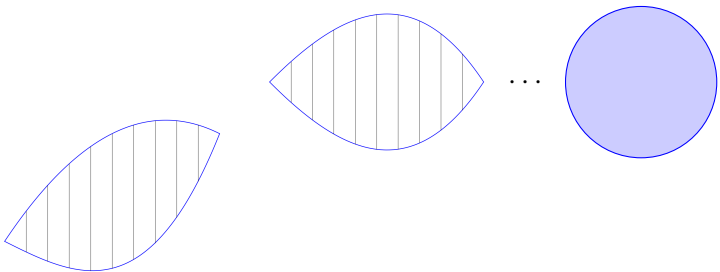
Idea of Proof: Steiner Symmetrization (disk D),



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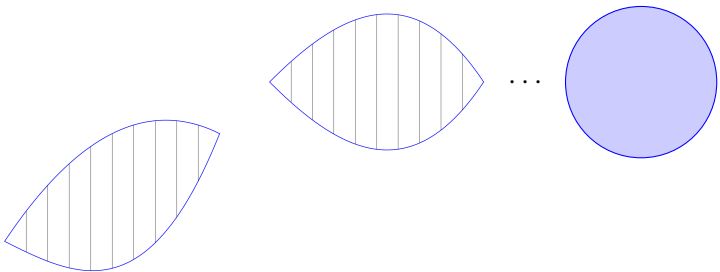

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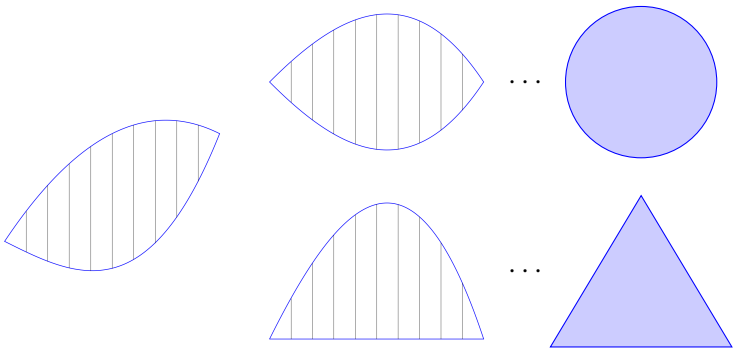
Idea of Proof: Steiner Symmetrization (disk D), Shakedown ("Schüttelung") (triangle T).



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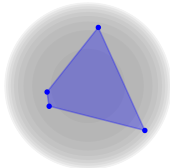
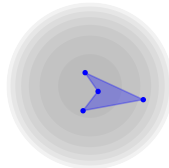
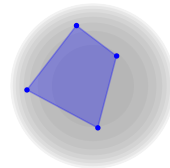
The lower bound $p_4(T) = 2/3 = 66.\dot{6}\%$ can be broken by non-uniform distribution:

- **H. Maehara** (1978), **C. Blatter** (2008) considered the standard normal distribution γ_2 on $\mathbb{R}^2$ and obtained

$$p_4(\gamma_2) = \frac{6}{\pi} \arcsin(1/3) = 64.904068..\%$$

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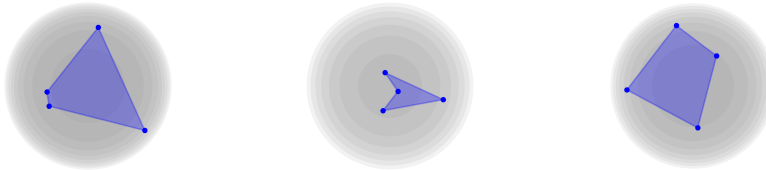

- **Z. Kabluchko** (2020) derives for 4 points on the half-sphere $\mathbb{S}^2_+$ (Cauchy distribution in $\mathbb{R}^2$) that

$$p_4(\mathbb{S}^2_+) = 3 - \frac{24}{\pi^2} = 56.829159..\%$$

See also **H. Maehara** & **H. Martini** (2018).

- **Blaschke** (1923): for uniform random points in a convex body $K \subset \mathbb{R}^2$

$$p_4(T) \leq p_4(K) = 1 - \frac{4}{\mathrm{Vol}_2(K)} \, \mathbb{E} \, \mathrm{Vol}_2(\mathrm{conv}\{\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3\}) \leq p_4(D).$$

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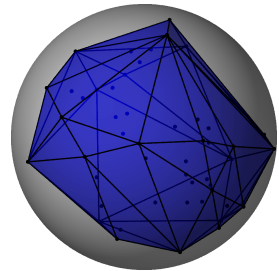

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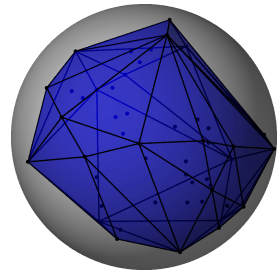

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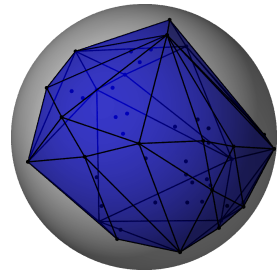
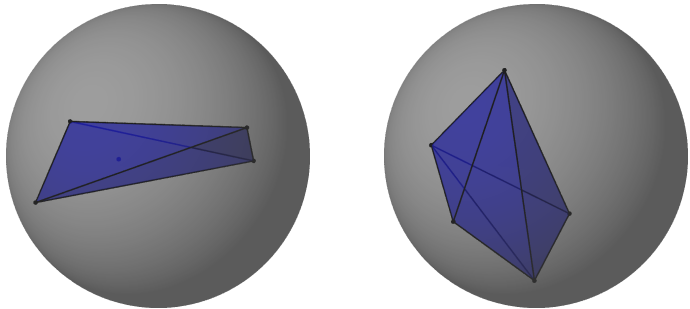


- Groemer's inequality is just one example of a randomized isoperimetric inequality that can be obtained using Steiner Symmetrization, or the more general theory of Shadow Systems.
- For a good overview on the **numerous** results on randomized isoperimetric inequalities see, for example, **G. Paouris & P. Pivovarov** (2017).

Groemer's inequality yields

$$p_{d+2}(K) = 1 - \frac{d+2}{\mathrm{Vol}_d(K)} \mathbb{E}\,\mathrm{Vol}_d(\mathrm{conv}\{\mathbf{X}_1, \dots, \mathbf{X}_{d+1}\}) \leq p_{d+2}(B_2^d).$$



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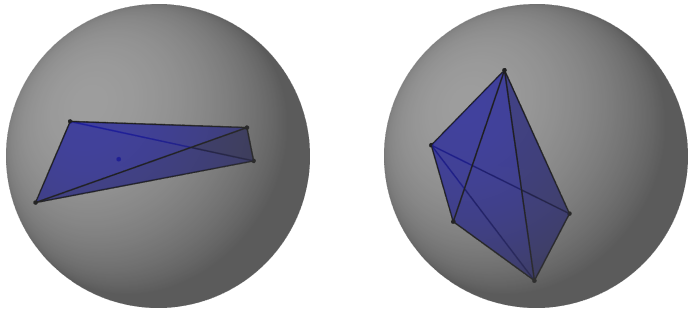
J.F.C. Kingman (1969) showed:

$$p_{d+2}(B_2^d) = 1 - \frac{d+2}{2^d} \binom{d+1}{(d+1)/2}^{d+1} \binom{(d+1)^2}{(d+1)^2/2}^{-1} = 1 - e^{-O(d \log d)}.$$



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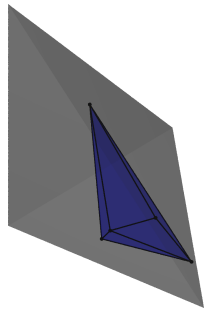
- Thus, for all $d \geq 2$ and any convex body $K \subset \mathbb{R}^d$: $p_{d+2}(K) \sim 1 - e^{-O(d \log d)}$.

Explicit values of $p_{d+2}(K)$ for $d \geq 3$ are mostly unavailable. A few noteworthy exceptions:

- **D. Mannion** (1994) and **C. Buchta & M. Reitzner** (2001) derive explicit values for $p_5(T^3)$ for the 3-dimensional simplex $T^3 \subset \mathbb{R}^3$.



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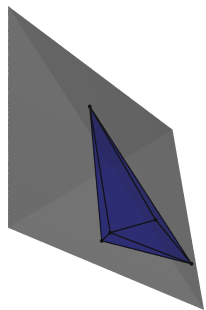

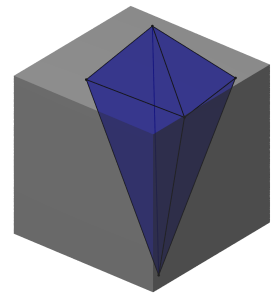

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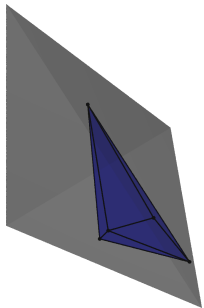
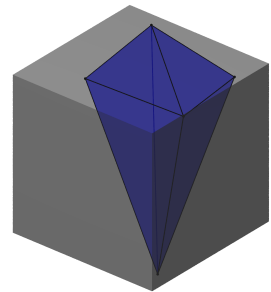
- **A. Gusakova & Z. Kabluchko** (2025+) give solutions for $p_{d+2}(\mu)$ for random points $\mathbf{X}_i$ distributed with respect to μ being either the Gaussian, a beta, or a beta-prime distribution in $\mathbb{R}^d$.



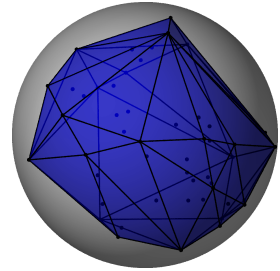


Part II: **Asymptotic Results**

- Let $K \subset \mathbb{R}^d$ be a convex body. Since for a convex body $K \subset \mathbb{R}^d$ the expected volume difference

$$\mathbb{E}\,\mathrm{Vol}_d(K \setminus \mathcal{P}(K,n))$$

between K and a random polytope $\mathcal{P}(K,n) := \mathrm{conv}\{\mathbf{X}_1, \dots, \mathbf{X}_n\}$ for uniform i.i.d. random points $\mathbf{X}_i$ in K is mostly inaccessible for $n \geq d+2$ one may wonder about asymptotic results as $n \to \infty$.



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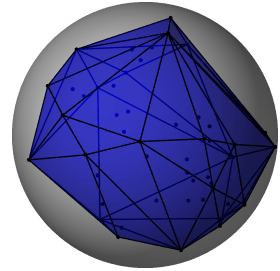
Question. For $n \to \infty$, can we determine the asymptotic behavior of $\dfrac{\mathbb{E}\,\mathrm{Vol}_d(K \setminus \mathcal{P}(K,n))}{\mathrm{Vol}_d(K)}$?



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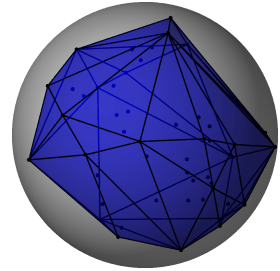
What about the expect number k-faces, for fixed $k \in \{0, \dots, d-1\}$, as $n \to \infty$, i.e., $\mathbb{E} f_k(\mathcal{P}(K,n))$?



Note again that $\frac{\mathbb{E}\,\mathrm{Vol}_d(K \setminus \mathcal{P}(K,n))}{\mathrm{Vol}_d(K)}$ and $\mathbb{E} f_k(\mathcal{P}(K,n))$ are affine invariant.

Efron's identity:

[**B. Efron** (1965)]

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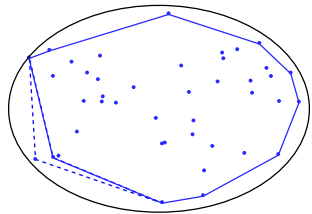

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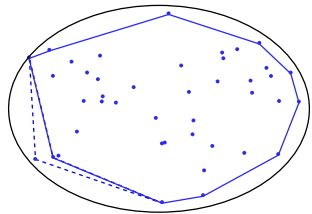

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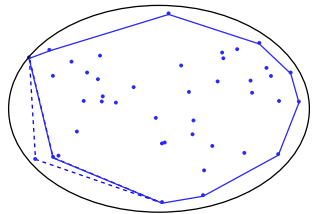

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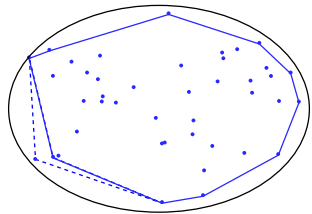

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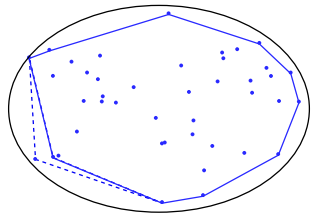
Thus

$$\frac{\mathbb{E}\,\mathrm{Vol}_d(K \setminus \mathcal{P}(K,n))}{\mathrm{Vol}_d(K)} \sim \frac{1}{n}\,\mathbb{E}\, f_0(\mathcal{P}(K,n)) \qquad \text{for } n \to \infty.$$



Asymptotic behavior for $\mathbb{E}f_0(\mathcal{P}(K,n)) = \mathbb{E}f_1(\mathcal{P}(K,n))$ for a smooth convex body $K \subset \mathbb{R}^2$.

Theorem (Rényi & Sulanke, 1963/64). Let $K \subset \mathbb{R}^2$ be a smooth convex body such that the boundary curve ∂K has curvature $\kappa(\mathbf{x}) > 0$ for all $\mathbf{x} \in \partial K$. Then

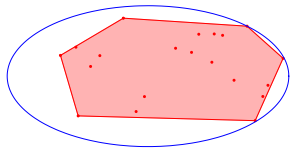
$$\mathbb{E} f_0(\mathcal{P}(K,n)) = \mathbb{E} f_1(\mathcal{P}(K,n)) \sim c_1\, \Omega(K) \left(\frac{n}{\mathrm{Vol}_2(K)} \right)^{1/3},$$

and

$$\mathbb{E}\, \mathrm{Vol}_2(K \setminus \mathcal{P}(K,n)) \sim c_2\, \Omega(K) \left(\frac{\mathrm{Vol}_2(K)}{n} \right)^{2/3},$$

where $c_1, c_2 > 0$ and $\Omega(K)$ is the affine perimeter of the boundary curve ∂K defined by

$$\Omega(K) := \int_{\partial K} \kappa(\mathbf{x})^{1/3}\, \mathrm{d}\mathbf{x}.$$



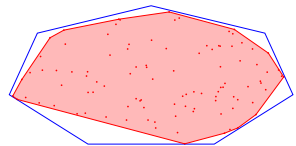
Theorem (Rényi & Sulanke, 1963/64). Let $P \subset \mathbb{R}^2$ be a convex 2-dimensional polygon. Then

$$\mathbb{E} f_0(\mathcal{P}(P,n)) = \mathbb{E} f_1(\mathcal{P}(P,n)) \sim \tilde{c}_1 \, f_0(P) \log\left(\frac{n}{\mathrm{Vol}_2(P)}\right),$$

and

$$\mathbb{E}\,\mathrm{Vol}_2(P \setminus \mathcal{P}(P,n)) \sim \tilde{c}_2 \, f_0(P) \, \frac{\mathrm{Vol}_2(P)}{n} \log\left(\frac{n}{\mathrm{Vol}_2(P)}\right),$$

where $\tilde{c}_1, \tilde{c}_2 > 0$.



I. Bárány (1992) and **M. Reitzner** (2005) gave extensions to all dimensions as follows:

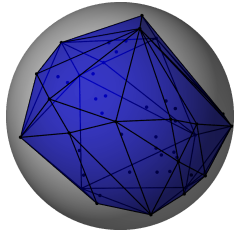
Theorem. Let $K \subset \mathbb{R}^d$ be a convex body of class C^2_+. Then, for $0 \leq k \leq d-1$,

$$\mathbb{E} f_k(\mathcal{P}(K,n)) \sim c_k \Omega(K) \left(\frac{n}{\mathrm{Vol}_d(K)} \right)^{\frac{d-1}{d+1}},$$

and

$$\mathbb{E}\, \mathrm{Vol}_d(K \setminus \mathcal{P}(K,n)) \sim c_n \Omega(K) \left(\frac{n}{\mathrm{Vol}_d(K)} \right)^{-\frac{2}{d+1}},$$

for $n \to \infty$. Here $c_i = c_i(d) > 0$ are constants that only depend on the dimension d and $\Omega(K)$ is the affine surface area of K.



Blaschke's affine surface area

- $K \subset \mathbb{R}^d$ a convex body of class C^2_+ that contains the origin in the interior.

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$$\Omega(K) = \int_{\partial K} H_{d-1}(K, \mathbf{x})^{\frac{1}{d+1}} \, \mathcal{H}^{d-1}(\mathrm{d}\mathbf{x})$$

- Extension of Ω to convex bodies K without curvature conditions:
  Petty (1985); **Leichtweiss** (1988/89); **Schütt & Werner** (1990);
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- Extensions to L_p- and Orlicz affine surface areas that are semi-continuous and centro-affine invariant valuations.

There are fewer results for a d-dimensional convex polytope P as a container body. Nevertheless, **I. Bárány & C. Buchta** (1993) and **M. Reitzner** (2005) obtain:

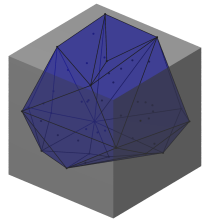
Theorem. Let $P \subset \mathbb{R}^d$ be a convex d-dimensional polytope. Then, for $0 \leq k \leq d-1$,

$$\mathbb{E} f_k(\mathcal{P}(P,n)) \sim \tilde{c}_k \, \mathrm{flag}(P) \left(\log \frac{n}{\mathrm{Vol}_d(P)} \right)^{d-1},$$

and

$$\mathbb{E} \, \mathrm{Vol}_d(P \setminus \mathcal{P}(P,n)) \sim \tilde{c}_n \, \mathrm{flag}(P) \frac{\mathrm{Vol}_d(P)}{n} \left(\log \frac{n}{\mathrm{Vol}_d(P)} \right)^{d-1}$$

for $n \to \infty$. Here $\tilde{c}_i = \tilde{c}_i(d) > 0$ are constants that only depend on the dimension d and $\mathrm{flag}(P)$ is the total number of complete flags (maximal chains in the face lattice).



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- $\mathcal{F}_i(P)$ ... i-dimensional faces of P, $i \in \{-1, 0, 1, \ldots, d\}$.
  - $\mathcal{F}_{-1}(P) = \{\emptyset\}$
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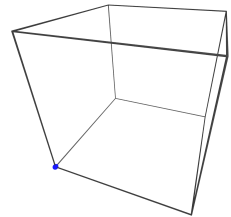

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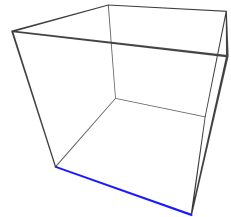

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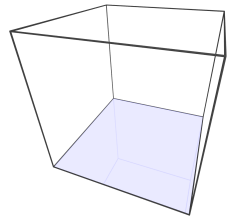

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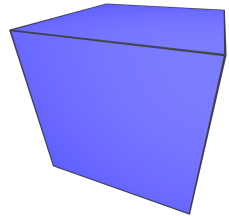

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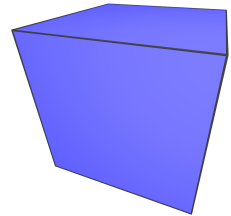

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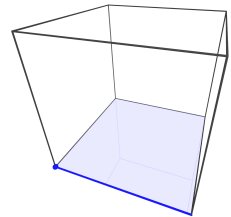

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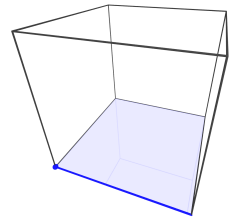

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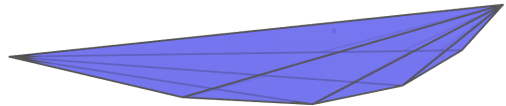
  ii) $F_i \subseteq F_{i+1}$.
- $\mathrm{flag}(P)$ ... total number of all (complete) flags $\mathbb{F}$ of P.

Bounds on $\mathrm{flag}(P)$

- The upper bound theorem for convex polytopes yields: $\mathrm{flag}(P) \leq c_d \, f_0(P)^{d/2}$, where the upper bound is realized by cyclic polytopes.



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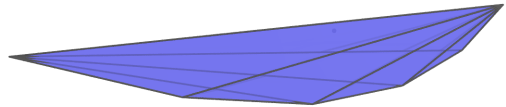

- An absolute lower bound is given by the simplex T^d: $\mathrm{flag}(P) \geq \mathrm{flag}(T^d) = (d+1)!$

- If $P \subset \mathbb{R}^d$ is centrally symmetric, then it is conjectured that:

  $$\mathrm{flag}(P) \overset{?}{\geq} \mathrm{flag}(C^d) = d!2^d,$$

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- Note that for any Hanner d-polytope H we have $\mathrm{flag}(H) = d!2^d$. Starting with $d = 4$ there are conjectured minimizers that are neither simple nor simplicial, e.g. octahedral 4-prism.

An (incomplete) list of asymptotic results for the expectation in the non-uniform case:

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- uniform (and block-beta) polytopes in cylinder bodies $K = B_2^{k_1} \times \cdots \times B_2^{k_m} \subset \mathbb{R}^d$: **FB, A. Gusakova, & C. Thäle** (2024+)

Part III: **Concentration, Variance and the Central Limit Theorem**

- Let $K \subset \mathbb{R}^d$ be a convex body and $\mathbf{x} \in K$ a fixed point. Among all half-spaces H^+ that contain $\mathbf{x}$ there is at least one $H^+(\mathbf{x})$ that gives minimal volume relative to K, i.e.,

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- First definition by **Donoho** (1982).
  Survey on halfspace depth: **S. Nagy, C. Schütt & E. Werner** (2019)
- hD_K is affine invariant: $\mathrm{hD}_{\alpha K}(\alpha \mathbf{x}) = \mathrm{hD}_K(\mathbf{x})$.
- If $\mathbf{x} \in \mathrm{bd}\, K$, then $\mathrm{hD}_K(\mathbf{x}) = 0$. Moreover, hD_K is log-concave and has a unique maximizer $\mathbf{x}_0$ in the interior of K.

- Let $K \subset \mathbb{R}^d$ be a convex body and $\mathbf{x} \in K$ a fixed point. Among all half-spaces H^+ that contain $\mathbf{x}$ there is at least one $H^+(\mathbf{x})$ that gives minimal volume relative to K, i.e.,

$$\mathrm{Vol}_d(H^+(\mathbf{x}) \cap K) = \min\{\mathrm{Vol}_d(H^+ \cap K) : \mathbf{x} \in H^+\}.$$

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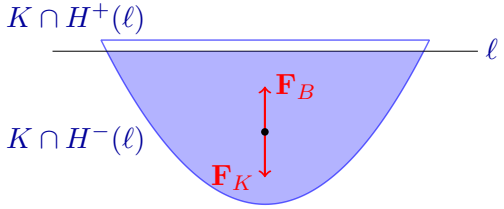
- The super-level set of hD_K is a convex body (for δ small enough) and is exactly the floating body of K: $\mathcal{F}(K,\delta) := [\mathrm{hD}_K \geq \delta] := \{\mathbf{x} \in K : \mathrm{hD}_K(\mathbf{x}) \geq \delta\}$

2-Dimensional Case: the Envelope Curve of Floating Lines

- $K \subset \mathbb{R}^2$ convex body (compact, convex with non-empty interior)
- $\mathbb{A} = \{\ell(h,\theta) : h \in \mathbb{R}, \theta \in [0, 2\pi)\}$ space of all (oriented) affine lines $\ell(h,\theta)$:
  $\ell(h,\theta) : \quad x\cos\theta + y\sin\theta = h.$
- Achimedes' Principle $\mathbf{F}_K = \mathbf{F}_B$ gives the following constrain:
  - weight of the body K: $\mathbf{F}_K = g \cdot m_K = g \cdot \rho_K \cdot \mathrm{Vol}_2(K)$
  - weight of the displaced fluid: $\mathbf{F}_B = g \cdot m_F = g \cdot \rho_F \cdot \mathrm{Vol}_2(K \cap H^-(\ell))$



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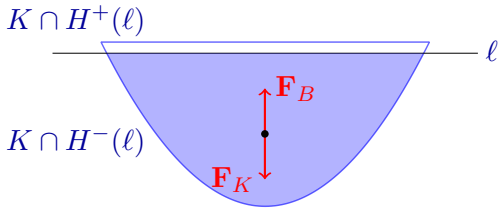
Thus $\dfrac{\mathrm{Vol}_2(K \cap H^+(\ell))}{\mathrm{Vol}_2(K)} = 1 - \dfrac{\mathrm{Vol}_2(K \cap H^-(\ell))}{\mathrm{Vol}_2(K)} = 1 - \dfrac{\rho_K}{\rho_F} = \delta \in [0,1].$



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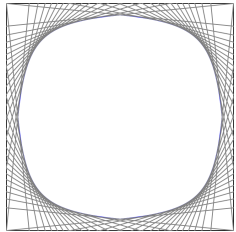
We consider the family of affine lines $\{\ell(h,\theta)\} \subset \mathbb{A}$ such that K is δ-floating w.r.t. $\ell(h,\theta)$, that is, $h = h_\delta^K(\theta)$ is uniquely determined by

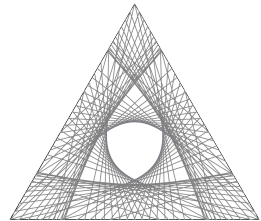
$$\mathrm{Vol}_2(K \cap H^+(\ell(h,\theta))) = \delta\,\mathrm{Vol}_2(K).$$

2-Dimensional Case: the Envelope Curve of Floating Lines

- The family of δ-floating lines $\{\ell(h_\delta^K, \theta) : \theta \in [0, 2\pi)\}$ gives rise to the envelope curve $\mathcal{C}_\delta K$ which is the locus of all points $(x, y) =: \mathbf{x}$ s.t.

$$\underbrace{0 = x\cos\theta + y\sin\theta - h_\delta^K(\theta)}_{(x,y)\in\ell(h_\delta^K,\theta)} =: F(\mathbf{x}, \theta), \quad \text{and} \quad 0 = \frac{\partial}{\partial\theta}F(\mathbf{x}, \theta).$$





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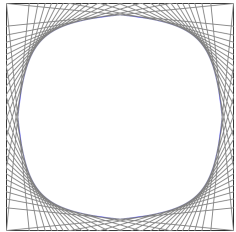
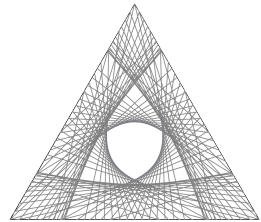
- Solving this system of equation shows that $\mathbf{x}$ is just the centroid (midpoint) of the line segment $K \cap \ell(h_\delta^K, \theta)$. Hence, the δ-floating curve $\mathcal{C}_\delta K$ is parametrized by: $\theta \in [0, 2\pi) \mapsto \mathbf{x}(\theta) = \mathrm{cent}(K \cap \ell(h_\delta^K, \theta))$





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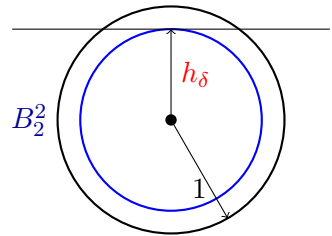
- The δ-floating curve construction $\mathcal{C}_\delta$ is affine covarint: Let $A \in \mathrm{GL}(2)$ and $\mathbf{y} \in \mathbb{R}^2$. Then for the affine transformation $\alpha(\mathbf{x}) = A\mathbf{x} + \mathbf{y}$ we have $(\mathcal{C}_\delta \circ \alpha)K = (\alpha \circ \mathcal{C}_\delta)K$.
- This construction can be traced back to C. Dupin.

[Application de Géométrie et de Méchanique, 1822]

The δ-Floating Curve: Examples

- For the Euclidean unit disk B_2^2 we have that: $\mathcal{C}_\delta B_2^2 = h_\delta \operatorname{bd} B_2^2$.
  Here $h_\delta \in [-1, 1]$ is the unique solution of

$$\delta = \frac{\operatorname{Vol}_2(B_2^2 \cap H^+(\ell(h_\delta)))}{\operatorname{Vol}_2(B_2^2)} = \frac{2}{\pi} \int_{h_\delta}^{1} \sqrt{1 - t^2}\, dt$$



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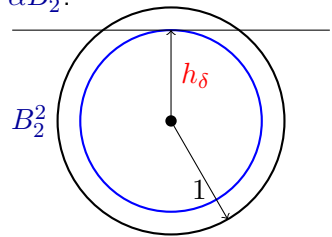
$$= \frac{1}{2} - \frac{1}{\pi} \left(h_\delta \sqrt{1 - h_\delta^2} + \arcsin h_\delta \right).$$

  In particular $h_\delta \sim 1 - C\delta^{2/3}$ for $\delta \to 0^+$, and we find that
  $\operatorname{Vol}_2(B_2^2) - \operatorname{Vol}_2(h_\delta B_2^2) = O(\delta^{2/3})$.
  This also determines the behavior for all ellipsoids $E = \alpha B_2^2$.



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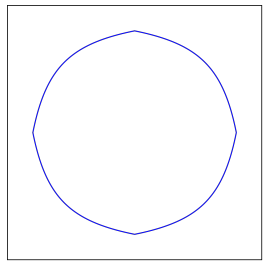
- For the square $B_\infty^2 = [-1,1]^2$ we have that $\mathcal{C}_\delta B_\infty^2$ is the union of 4 congruent hyperbola arcs and is boundary to a convex body $(B_\infty^2)_\delta$.
  We may calculate: $\operatorname{Vol}_2(B_\infty^2) - \operatorname{Vol}_2((B_\infty^2)_\delta) = O(\delta \ln(1/\delta))$ for $\delta \to 0^+$.



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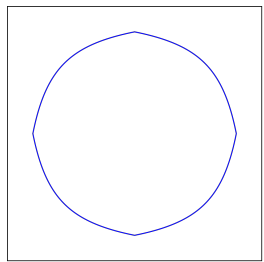
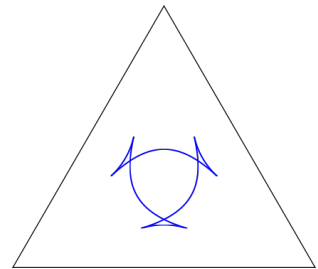
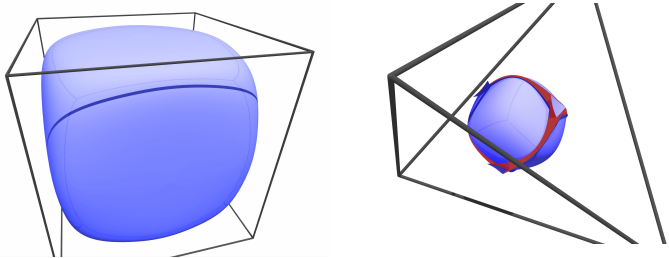
  We may calculate: $\operatorname{Vol}_2(B_\infty^2) - \operatorname{Vol}_2((B_\infty^2)_\delta) = O(\delta \ln(1/\delta))$ for $\delta \to 0^+$.
- For any triangle T we have that $\mathcal{C}_\delta(T)$ is the union of up to 6 hyperbola arcs. In particular $\mathcal{C}_\delta T$ is never a convex curve for $\delta \in (0,1)$.





The Floating Surface in d-Dimensions

The δ-floating surface is the envelope of all δ-floating hyperplanes and may **not** be convex.



However, at least for δ small enough, there will be a "wet-part" of K that always stays below the δ-floating hyperplane H. This "wet-part" is the intersection of all half-spaces H^- and thus a convex body.

- Using the minimal volume cap $C_K(\mathbf{x})$ we estimate

$$\mathrm{Prob}(\mathbf{x} \notin \mathcal{P}(K,n)) \geq \mathrm{Prob}(C_K(\mathbf{x}) \cap \mathcal{P}(K,n) = \emptyset)$$

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$$\begin{aligned}\mathrm{Prob}(\mathbf{x} \notin \mathcal{P}(K,n)) &\geq \mathrm{Prob}(C_K(\mathbf{x}) \cap \mathcal{P}(K,n) = \emptyset) \\ &= \left(1 - \frac{\mathrm{Vol}_d(C_K(\mathbf{x}))}{\mathrm{Vol}_d(K)}\right)^n = (1 - \mathrm{hD}_K(\mathbf{x}))^n \,.\end{aligned}$$

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- Thus

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For the volume of the (convex) floating body of K,

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Theorem (C. Schütt & E. Werner, 1990). Let $K \subset \mathbb{R}^d$ be a convex body. Then

$$\lim_{\delta \to 0^+} \frac{\mathrm{Vol}_d(K \setminus \mathcal{F}(K,\delta))}{(\delta\,\mathrm{Vol}_d(K))^{\frac{2}{d+1}}} = \alpha_d\,\Omega(K).$$

Also building on results of **K. Leichtweiss** (1986).

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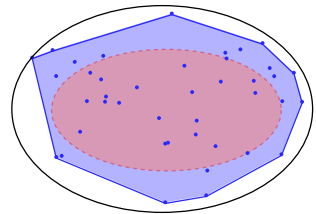
Theorem (C. Schütt, 1991). Let $P \subset \mathbb{R}^d$ be a d-polytope. Then

$$\lim_{\delta \to 0^+} \frac{\mathrm{Vol}_d(P \setminus \mathcal{F}(P,\delta))}{\mathrm{Vol}_d(P)\delta(\log 1/\delta)^{d-1}} = \beta_d \mathrm{flag}(P).$$

- For a convex body $K \subset \mathbb{R}^d$ of class C^2_+ we find again

$$\mathbb{E}\,\mathrm{Vol}(K \setminus \mathcal{P}(K,n)) \geq C\,\mathrm{Vol}_d\left(K \setminus \mathcal{F}\left(K, \frac{1}{n}\right)\right) \sim \Omega(K)\left(\frac{\mathrm{Vol}_d(K)}{n}\right)^{\frac{2}{d+1}}.$$



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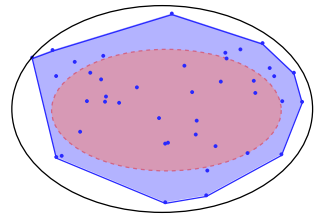
for sufficiently large n.



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Theorem (I. Bárány & L. Dalla, 1997; V. Vu, 2005). Let $K \subset \mathbb{R}^d$ be a convex body. Given some $c > 0$ and sufficiently large n, we have with high probability we have

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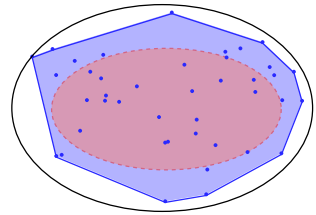

- For a convex body $K \subset \mathbb{R}^d$ of class C^2_+ we find again

$$\mathbb{E}\,\mathrm{Vol}(K \setminus \mathcal{P}(K,n)) \geq C\,\mathrm{Vol}_d\left(K \setminus \mathcal{F}\left(K, \frac{1}{n}\right)\right) \sim \Omega(K)\left(\frac{\mathrm{Vol}_d(K)}{n}\right)^{\frac{2}{d+1}}.$$

Theorem (I. Bárány & Larman, 1988). Let $K \subset \mathbb{R}^d$ be a convex body. Then

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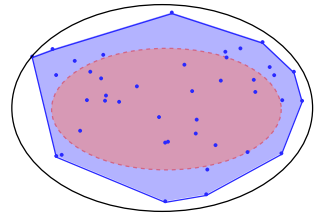
$$\mathcal{F}\left(K, \frac{c\log n}{n}\right) \subset \mathcal{P}(K,n).$$

To be precise, there exist $\alpha = \alpha(d) > 0$ (large) such that

$$\mathrm{Prob}\left(\mathcal{F}\left(K, \frac{c\log n}{n}\right) \subset \mathcal{P}(K,n)\right) \geq 1 - n^{-c\alpha}.$$



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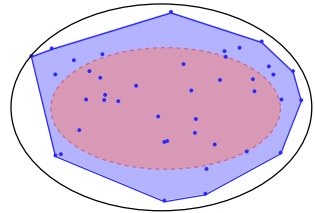

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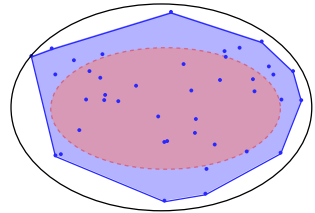
- Then for a convex body $K \subset \mathbb{R}^d$ of class C^2_+ the concentration results also yields for $p \geq 0$:



$$\mathbb{E}\,\mathrm{Vol}_d(K \setminus \mathcal{P}(K,n))^p$$

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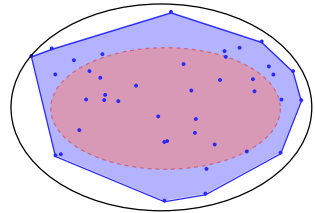

$$\begin{aligned}&\mathbb{E}\,\mathrm{Vol}_d(K \setminus \mathcal{P}(K,n))^p \\ &\quad = \mathbb{E}\,\mathrm{Vol}_d(K \setminus \mathcal{P}(K,n))^p \mathbf{1}(\mathcal{E}_n) + \mathbb{E}\,\mathrm{Vol}_d(K \setminus \mathcal{P}(K,n))^p [1 - \mathbf{1}(\mathcal{E}_n)]\end{aligned}$$

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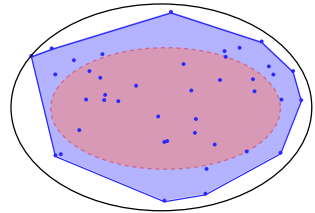

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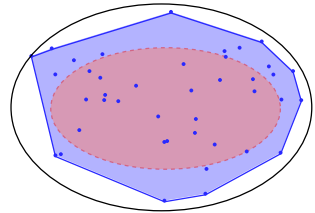

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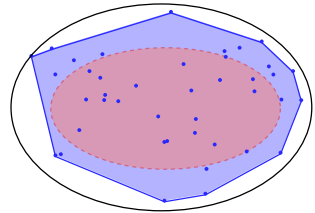

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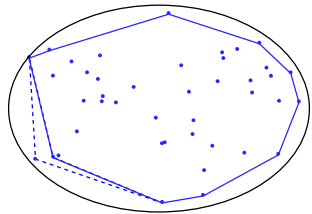
- While this bound is **not optimal**, it is often good enough!

- Concentration can also be used to upper bound the variance by using the Efron–Stein Jackknife inequality (as presented by **M. Reitzner** (2003)):

$$\begin{aligned}\operatorname{Var}\operatorname{Vol}_d(K \setminus \mathcal{P}(K,n)) &= \operatorname{Var}\operatorname{Vol}_d(\mathcal{P}(K,n)) \\ &\leq (n+1)\mathbb{E}\operatorname{Vol}_d(\mathcal{P}(K,n+1) \setminus \mathcal{P}(K,n))^2\end{aligned}$$



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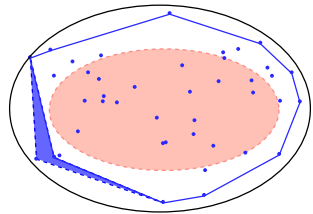

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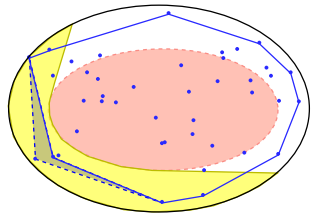

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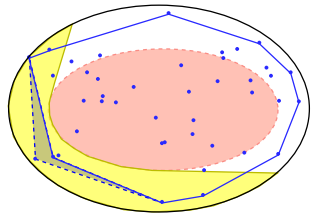

Theorem (M. Reitzner, 2003/05). Let K be a convex body of class C^2_+. Then for sufficiently large n:

$$cn^{-1-\frac{2}{d+1}} \leq \operatorname{Var} \operatorname{Vol}_d(\mathcal{P}(K,n)) \leq Cn^{-1-\frac{2}{d+1}},$$

and, for $1 \leq k \leq d-1$:

$$\tilde{c}n^{1-\frac{2}{d+1}} \leq \operatorname{Var} f_k(\mathcal{P}(K,n)) \leq \tilde{C}n^{1-\frac{2}{d+1}}$$

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I. Bárány & M. Reitzner (2010) also show for a d-polytope $P \subset \mathbb{R}^d$ that

$$cn^{-2}(\log n)^{d-1} \leq \operatorname{Var} \operatorname{Vol}_d(\mathcal{P}(P,n)) \leq Cn^{-2}(\log n)^{d-1},$$

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M. Reitzner (2005), see also **V. Vu** (2006), used the bounds on the variance to derive the following Central Limit Theorem:

Theorem (M. Reitzner, 2005). Let $K \subset \mathbb{R}^d$ be a convex body of class C^2_+ and consider the normalized random variable

$$W_n := \frac{\mathrm{Vol}_d(\mathcal{P}(K,n)) - \mathbb{E}\,\mathrm{Vol}_d(\mathcal{P}(K,n))}{\sqrt{\mathrm{Var}\,\mathrm{Vol}_d(\mathcal{P}(K,n))}}.$$

Then

$$W_n \xrightarrow{d} Z \quad \text{for } n \to \infty,$$

where $Z \sim N(0,1)$ is a standard Gaussian distributed random variable.
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- Then, for example, the Wasserstein distance between W_n and a standard Gaussian random variable $Z \sim N(0,1)$ can be bounded from above by expressions that essentially only involve moments of the first and second order difference operator.
- For example, if $f(\mathbb{X}) = \mathrm{Vol}_d(\mathcal{P}(K,n))$, then

$$D_1 f(\mathbb{X}) = f(\mathbf{X}_1, \dots, \mathbf{X}_n) - f(\mathbf{X}_2, \dots, \mathbf{X}_n) = \mathrm{Vol}_d(\mathcal{P}(K,n) \setminus \mathcal{P}(K,n-1))$$

and using again concentration with respect to the floating body we derive the bound (as before)

$$\mathbb{E}\,|D_1 f(\mathbb{X})|^p = \mathbb{E}\,|\,\mathrm{Vol}_d(\mathcal{P}(K,n) \setminus \mathcal{P}(K,n-1))|^p \lesssim \left(\frac{\log n}{n}\right)^{p+\frac{2}{d+1}}.$$

This approach was in particular very successfully to prove CLT's also for non-uniform random polytopes:

- **C. Thäle, N. Turchi & F. Wespi** (2018): CLT for $f(\mathbb{X}) = V_j(\mathcal{P}(K, n))$ for the intrinsic volumes V_j.

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- **FB, & C. Thäle** (2020) and **FB, C. Thäle & D. Rosen** (2021): CLT for $f(\mathbb{X}) = \mathrm{Vol}_d(\mathcal{P}(K,n))$ where K is a convex body in spherical or hyperbolic space (also dual volumes).