Equality cases in the Alexandrov–Fenchel inequality and mixed area measures of convex bodies

based on joint work with Paul A. Reichert

Daniel Hug | Salzburg, September 2023

Mixed volumes

Let $K_1, \ldots, K_m \subset \mathbb{R}^n$ be convex bodies and $\alpha_1, \ldots, \alpha_m \geq 0$. Then

$$\mathrm{V}(\alpha_1 K_1 + \cdots + \alpha_m K_m) = \sum_{i_1,\ldots,i_n=1}^{m} \mathrm{V}(K_{i_1}, \ldots, K_{i_n})\alpha_{i_1} \cdots \alpha_{i_n}.$$

The functionals $\mathrm{V}(\cdot, \ldots, \cdot) : (\mathcal{K}^n)^n \to [0, \infty)$ are symmetric, hence uniquely determined. Existence of the expansion is proved for polytopes or smooth bodies first.

- Recursive description (**polytopal case**):

$$\mathrm{V}(P_1, \ldots, P_n) = \frac{1}{n} \sum_{(*)} h_{P_n}(u)\, \mathrm{V}(F(P_1, u), \ldots, F(P_{n-1}, u)).$$

- Analytic description (**smooth case**):

$$\mathrm{V}(K_1, \ldots, K_n) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h_{K_n}(u) \det\left(d^2 h_{K_1}(u), \ldots, d^2 h_{K_{n-1}}(u)\right) \mathcal{H}^{n-1}(\mathrm{d}u).$$

There are Borel measures $\mathrm{S}(K_1, \dots, K_{n-1}, \cdot)$ on $\mathbb{S}^{n-1}$ such that

$$\mathrm{V}(K_1, \dots, K_n) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h_{K_n}(u)\ \mathrm{S}(K_1, \dots, K_{n-1}, \mathrm{d}u),$$

where $h_M(u) = \max\{\langle x, u\rangle : x \in M\}$ is the support function of $M \in \mathcal{K}^n$.

- Existence of these measures follows from the Riesz representation theorem (e.g.).
- Explicit descriptions are available if all bodies are polytopes or all are smooth.
- **Polytopal case**:

$$\mathrm{S}(P_1, \dots, P_{n-1}, \bullet) = \sum_{(*)} \mathrm{V}(F(P_1, u), \dots, F(P_{n-1}, u))\, \delta_u(\bullet).$$

- **Smooth case**:

$$\mathrm{S}(K_1, \dots, K_{n-1}, \bullet) = \int_\bullet \det\left(d^2 h_{K_1}(u), \dots, d^2 h_{K_{n-1}}(u)\right)\ \mathcal{H}^{n-1}(\mathrm{d}u).$$

Special case (interpretation as relative or anisotropic surface area):

$$nV(K[n-1],L)=\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\left(\mathrm{V}(K+\varepsilon L)-\mathrm{V}(K)\right).$$

Theorem (Minkowski's first inequality, 1903)

If $K,L\in\mathcal{K}^n$, then

$$\mathrm{V}(K[n-1],L)^n\geq \mathrm{V}(K)^{n-1}\,\mathrm{V}(L)$$

with equality **iff** $\dim K\leq n-2$ *or* K and L lie in parallel hyperplanes *or* K,L are homothetic.

The result follows from the **Brunn–Minkowski inequality** (1887, 1910) together with its equality cases:

$$\mathrm{V}(K+L)^{\frac{1}{n}}\geq \mathrm{V}(K)^{\frac{1}{n}}+(L)^{\frac{1}{n}}.$$

Another consequence of the **Brunn–Minkowski inequality**:

Therem (Minkowski's second inequality)

If $K, L \in \mathcal{K}^n$, then

$$V(K[n-1], L)^2 \geq V(K)\, V(L[2], K[n-2]).$$

The equality cases are known (Bol 1943) but more involved.

Ignoring lower-dim. cases and assuming (wlog) that $L \subseteq K$, equality holds **iff** each support plane of K that is *not* a support plane of L contains only boundary points $x \in \partial K$ where $\dim N(K, x) \geq 3$.

This description is helpful, but ...

Better: each $(n-(n-2)-1) = 1$-extreme support plane $H(K, u)$ of K is a support plane of L.

Consider the unique face of a normal cone of K containing u in its rel. int. – the **touching cone** $T(K, u)$.

Then u is **1-extreme** if $\dim T(K, u) \leq 2$, i.e. there do not exist 3 lin. indep. normal vectors u_1, u_2, u_3 at the same boundary point of K with $u = u_1 + u_2 + u_3$.

Minkowski's general quadratic inequality

Theorem (Minkowski/Alexandrov)

If $K, L, M \in \mathcal{K}^n$ are full-dimensional, then

$$V(K, L, M[n-2])^2 \geq V(K[2], M[n-2])\, V(L[2], M[n-2]).$$

This is no longer a consequence of the Brunn–Minkowski inequality, but requires a much deeper result.

A complete characterization of the equality cases was obtained by Shenfeld & van Handel '22:

Equality holds **iff** there are $x \in \mathbb{R}^n$, $a > 0$ such that $h_K(u) = h_{aL+x}(u)$ for all 1-extreme directions u of M.

Schneider '79: The closure of the set of 1-extreme directions of M is the support of the (mixed) measure

$$S(M[n-2], B^n, \cdot) = S_{n-2}(M, \cdot).$$

Alexandrov–Fenchel inequality

Theorem (Alexandrov–Fenchel inequality, 1937)

Let $K, L \in \mathcal{K}^n$. Let $\boldsymbol{\mathcal{C}} = (C_1, \dots, C_{n-2})$ be an $(n-2)$-tuple in $\mathcal{K}^n$. Then

$$V(K, L, \boldsymbol{\mathcal{C}})^2 \geq V(K, K, \boldsymbol{\mathcal{C}})\, V(L, L, \boldsymbol{\mathcal{C}}), \tag{AFI}$$

where $V(K, L, \boldsymbol{\mathcal{C}}) := V(K, L, C_1, \dots, C_{n-2})$.

- For $m \in \{2, \dots, n\}$ and $\mathcal{C}_{n-m} := (C_{m+1}, \dots, C_n) \in (\mathcal{K}^n)^{n-m}$, consider

$$f_m : \mathcal{K}^n \ni L \mapsto V(L[m], \mathcal{C}_{n-m})^{\frac{1}{m}}.$$

Then f_m is concave. AFI is equivalent to f_2 being concave. The same is true for f_3.

- Equality cases? All known (involved) proofs work by approximation via special polytopes or smooth bodies.

Zonoids are a special class of convex bodies, which often serve as in important test case. They also naturally arise in various contexts, ranging from functional analysis to stochastic geometry.

- A **zonotope** is a finite Minkowski sum of segments (hence a very special centrally symmetric polytope).
- A **zonoid** is a limit of zonotopes. The unit ball B^n is a special zonoid.
- A convex body $K \in \mathcal{K}^n$ is a zonoid **iff** there is an even and finite Borel measure μ on $\mathbb{S}^{n-1}$ such that

$$h_K(u) = \int_{\mathbb{S}^{n-1}} |\langle u, x\rangle| \, \mu(\mathrm{d}x), \quad u \in \mathbb{R}^n.$$

The (generating) measure μ is uniquely determined.

- Projections, affine transformations and faces of zonoids are again zonoids.
- Zonoids are closed – but for $n \geq 3$ nowhere dense – in the space of centrally symmetric convex bodies.

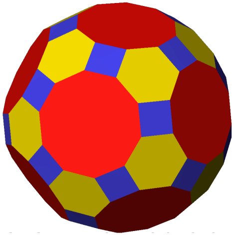
Illustrations of zonoids

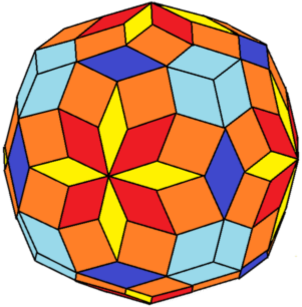
1) https://polytope.miraheze.org/wiki/Zonotope or https://polytope.miraheze.org/wiki/File:Uniform_polyhedron-53-t012.png

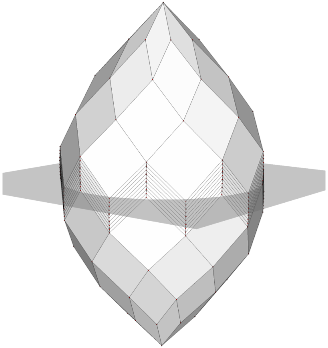
2) A 3-dimensional zonotope composed of 132 rhombi. https://polytope.miraheze.org/wiki/Zonotope

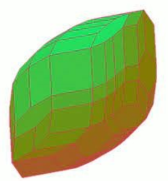
3) Rörig, Witte, Ziegler: Zonotopes with large 2-D cuts. Discrete and Computational Geometry 42 (2009), 527–541.

4) https://www.decatur.de/personal/zono/index.html









Theorem (Equality cases AFI, Shenfeld & van Handel '23+)

Let $K, L \in \mathcal{K}^n$. Let $\mathcal{C} = (C_1, \dots, C_{n-2})$ be a supercritical $(n-2)$-tuple of polytopes, zonoids or smooth convex bodies in $\mathbb{R}^n$ such that $\mathrm{V}(K, K, \mathcal{C}), \mathrm{V}(L, L, \mathcal{C}) > 0$.

Equality holds in (AFI) **iff** there are $a > 0$ and $x \in \mathbb{R}^n$ such that $h_K = h_{aL+x}$ on $\operatorname{supp} \mathrm{S}(B^n, \mathcal{C}, \cdot)$.

Moreover, $\operatorname{supp} \mathrm{S}(B^n, \mathcal{C}, \cdot) = \operatorname{cl} \operatorname{ext}(B^n, \mathcal{C})$ if $\mathcal{C}$ are polytopes or $\mathcal{C}$ are zonoids/smooth bodies.

Some special cases known previously:

- $(C_1, \dots, C_{n-2})$ strongly isomorphic, simple polytopes. [Sch 93]
- $(C_1, \dots, C_{n-2})$ smooth (unique supporting hyperplanes). [Sch 90]
- K, L centrally symmetric, $(C_1, \dots, C_{n-2})$ zonoids, full dimenions. [Sch 88]
- $M = C_1 = \dots = C_{n-2}$ [SvH 22].

Partial confirmations of general conjectures

Aims

- Extend and unify the class of bodies for which the equality cases in AFI can be characterized.
- Introduce a suitable class of bodies (which might be studied further).
- Describe geometrically the support of mixed area measures (for this class of bodies).

Definition

Let $k \in \mathbb{N}$.

- *k*-**topes** $\mathcal{P}_k^n$: polytopes in $\mathbb{R}^n$ with at most k vertices.
- *k*-**polyotope** : finite Minkowski sum of k-topes.
- *k*-**polyoid** : limit of k-polyotopes.
- $K \in \mathcal{K}^n$ is a polyoid (a polyotope), if it is a k-polyoid (a k-polyotope) for some $k \in \mathbb{N}$.

- The class of k-polyoids in $\mathbb{R}^n$ equals the Minkowski class $\mathfrak{M}(\mathcal{P}_k^n)$ of k-topes.
- A 1-polyoid is just a singleton, a 2-polyoid is a zonoid and a 3-polyoid is a *triangle body*. Each polytope P is a k-polyotope and hence a k-polyoid (for some k).
- $\mathcal{P}_k^n \subset \mathcal{P}_\ell^n$ for $k < \ell$. Any k-polyoid is an ℓ-polyoid for $k < \ell$.
- There is a 3-polyoid which is **not** a zonoid, **not** a polytope, and **neither** smooth **nor** strictly convex (the Minkowski sum of a triangle in $\mathbb{R}^2 \times \{0\}$ and a 2-dimensional ball in $\{0\} \times \mathbb{R}^2$).

Corollary

Let $K \in \mathcal{K}^n$, $n \in \mathbb{N}_0$ and $k \in \mathbb{N}$. Then the following are equivalent.

- (a) K is a k-polyoid.
- (b) There is a probability measure μ on $\mathcal{P}_k^n$ with compact support such that

$$h_K = \int h_P \, \mu(\mathrm{d}P). \tag{1}$$

If (b) holds, then K is the limit of a sequence in $\operatorname{pos} \mu := \operatorname{pos} \operatorname{supp} \mu$ (finite pos. comb. of sets from $\operatorname{supp} \mu$).

Special instance of a statement about Minkowski classes of homothety invariant closed classes of bodies.

The generating measure μ is not uniquely determined.

Lemma

Let $\mathcal{K}_* \subseteq \mathcal{K}^n$ be a Borel set, $n \in \mathbb{N}_0$. Suppose that μ is a probability measure on $\mathcal{K}_*$ with bounded support. Let $K \in \mathcal{K}^n$ be defined by

$$h_K = \int h_P \, \mu(\mathrm{d}P). \tag{2}$$

Then K is the limit of a sequence in $\operatorname{pos} \mu$.

Definition

Each $K \in \mathcal{K}^n$ defined via (2), is called a $\mathcal{K}_*$-**macroid** with generating measure μ.
If $\mathcal{K}_* = \mathcal{P}^n$, then K is called a **macroid** with generating measure μ.

Comments

- Macroids have some nice properties (projections, faces, ...).
- Mean section / projection bodies, or SO_n-convolution of h_K, K fixed. Each polyoid is a macroid.

How large is the class of macroids?

Theorem

The class of macroids is strictly larger than the class of a polyoids.

An explicit example of a convex body that is **not** a macroid is provided by a circular cone.

Theorem

Let $K \in \mathcal{K}^n$ be an indecomposable macroid. Then K is a polytope.

Theorem

Let $K, L \in \mathcal{K}^n$. Let $\mathcal{C} = (C_1, \dots, C_{n-2})$ be a supercritical $(n-2)$-tuple of macroids or smooth bodies in $\mathbb{R}^n$. Let $\mathrm{V}(K, L, \mathcal{C}) > 0$. Then AFI holds with equality **iff** there are $a > 0$ and $x \in \mathbb{R}^n$ such that

$$h_K = h_{aL+x} \quad \text{on } \operatorname{supp} \mathrm{S}(B^n, \mathcal{C}, \cdot).$$

Equivalently:

Theorem

Let $n \geq 2$. Let $\mathcal{C} = (C_1, \dots, C_{n-2})$ be a supercritical $(n-2)$-tuple of macroids or smooth bodies in $\mathbb{R}^n$. Let f be a difference of support functions. Then $\mathrm{S}_{f,\mathcal{C}} = 0$ **iff** f is linear on $\operatorname{supp} \mathrm{S}(B^n, \mathcal{C}, \cdot)$.

Here: $\mathrm{S}_{f,\mathcal{C}} := \mathrm{S}(f, \mathcal{C}, \cdot) = S(A, \mathcal{C}, \cdot) - S(B, \mathcal{C}, \cdot)$ if $f = h_A - h_B$.

Lemma (Decomposition)

Assume that $n \geq 3$. Let $\mathcal{C}$ be an $(n-3)$-tuple in $\mathcal{K}^n$, and let $K \in \mathcal{K}^n$ be a $\mathcal{K}_*$-macroid with generating measure μ. Suppose $(K, \mathcal{C})$ is supercritical and f is a difference of support functions. Then

$$\mathrm{S}_{f,K,\mathcal{C}} = 0 \implies \mathrm{S}_{f,P,\mathcal{C}} = 0 \quad \text{for all } P \in \operatorname{pos} \mu.$$

Lemma [SvH 23+]

Equality cases in AFI are understood for $(n-2)$-tuples of supercritical polytopes.

Lemma (Linear gluing)

Let $n \geq 3$. Let $\mathcal{C} = (C_1, \ldots, C_{n-2})$ be an $(n-2)$-tuple of $\mathcal{K}_*$-macroids in $\mathbb{R}^n$ with generating measures $\mu_1, \ldots, \mu_{n-2}$. Let f be a difference of support functions.

Assume that f is linear on $\operatorname{supp} \mathrm{S}(B^n, \mathcal{Q}, \cdot)$ whenever $\mathcal{Q} = (Q_1, \ldots, Q_{n-2}) \in \operatorname{pos}(\mu_1, \ldots, \mu_{n-2})$ with $\overline{\operatorname{span}}\, Q_i = \overline{\operatorname{span}}\, C_i$ for $i \in [n-2]$.

Then f is also linear on $\operatorname{supp} \mathrm{S}(B^n, \mathcal{C}, \cdot)$.

Support of mixed area measures of polyotopes

Definition

Let $K \in \mathcal{K}^n$. If $u \in \mathbb{R}^n \setminus \{0\}$, then $N(K, F(K, u))$ is a closed convex cone containing u.
It has a unique face $T(K, u)$ such that $u \in \operatorname{relint} T(K, u)$: the **touching cone of K in direction** u.

$\mathrm{TS}(K, u) := T(K, u)^\perp \subseteq u^\perp$ is the **touching space of K in direction** u.

The following is a version of the definition of extreme normal vectors.

Definition

If $n \geq 1$ and $\boldsymbol{\mathcal{C}} = (C_1, \ldots, C_{n-1})$ is an $(n-1)$-tuple in $\mathcal{K}^n$, then $u \in \mathbb{S}^{n-1}$ is **$\boldsymbol{\mathcal{C}}$-extreme** if there are 1-dimensional linear subspaces of $\mathrm{TS}(C_i, u)$, for $i \in [n-1]$, with linearly independent directions.

The set of all $\boldsymbol{\mathcal{C}}$-extreme normal vectors is denoted by $\operatorname{ext} \boldsymbol{\mathcal{C}}$.

Theorem [Support characterization]

Let $\mathcal{C} = (C_1, \ldots, C_{n-1})$ be an $(n-1)$-tuple of polyoids or smooth bodies (provided at least one of the bodies is also strictly convex) in $\mathbb{R}^n$. Then

$$\operatorname{supp} \mathrm{S}(\mathcal{C}, \cdot) = \operatorname{cl} \operatorname{ext} \mathcal{C}.$$

Theorem [Equality cases in AFI, geometric form]

Let $K, L \in \mathcal{K}^n$. Let $\mathcal{C} = (C_1, \ldots, C_{n-2})$ be a supercritical $(n-2)$-tuple of polyoids or smooth bodies in $\mathbb{R}^n$. Assume that $\mathrm{V}(K, L, \mathcal{C}) > 0$. Then AFI holds with equality **iff** there are $a > 0$ and $x \in \mathbb{R}^n$ such that

$$h_K = h_{aL+x} \quad \text{on } \operatorname{ext}(B^n, \mathcal{C}).$$

Theorem [Monotonicity of mixed volumes]

Let $K, L \in \mathcal{K}^n$ satisfy $K \subseteq L$. Let $\boldsymbol{\mathcal{C}} = (C_1, \ldots, C_{n-1})$ be an $(n-1)$-tuple of polyoids or smooth bodies (provided at least one of the smooth bodies is also strictly convex) in $\mathbb{R}^n$. Then equality holds in

$$\mathrm{V}(K, \boldsymbol{\mathcal{C}}) \leq \mathrm{V}(L, \boldsymbol{\mathcal{C}})$$

if and only if

$$h_K = h_L \quad \text{on } \operatorname{ext} \boldsymbol{\mathcal{C}}.$$