Brunn-Minkowski inequalities for variational functionals in Gauss space

Andrea Colesanti

“Convex geometry and geometric probability”

Paris Lodron Universität - Salzburg

September 25-29, 2023

It was twenty years ago today...

Cortona, summer 2003

The original results presented in this talk were obtained in collaboration with: **Elisa Francini**, **Galyna Livshyts** and **Paolo Salani**.

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- V_n = volume (Lebesgue measure);
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Equivalently: the functional $V_n^{1/n}$ is concave in the class of convex bodies of R^n, equipped with the Minkowski addition.

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- Much more about this topic can be found in: R. J. Gardner, *The Brunn-Minkowski inequality*, 2002.

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Hence, as for the volume, V_i raised to the reciprocal of its homogeneity order, is concave in $\mathcal{K}^n$.

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Here

$$\Delta \bar{v} = \mathrm{trace}(D^2 \bar{v}) = \sum_{i=1}^{n} \bar{v}_{ii},$$

is the **Laplace operator**.

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Behind this result there is the **Polya-Szegö principle**, which implies in particular that the Rayleigh quotient **is monotone decreasing with respect to Steiner symmetryzations**.

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- The **Ehrhard inequality** (Ehrhard, 1983):

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$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-s^2/2} ds.$$

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Theorem (Borell, '71, Sudakov & Tsirelson, '75). *Among all convex bodies with fixed Gaussian measure, the half spaces minimizes the Gaussian perimeter.*

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- The exponent $1/n$ is the best possible.

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The Gaussian principal frequency

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Let $K \in \mathcal{K}^n$; we set, for $v \in C^1(\mathrm{int}(K)) \cap C(K)$,

$$R_\gamma[v] = \frac{\displaystyle\int_K |\nabla v|^2 d\gamma_n}{\displaystyle\int_K v^2 d\gamma_n},$$

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- λ_γ is rotation invariant, but it is **neither translation invariant**, **nor homogeneous**.

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- Due to the lack of homogeneity, (*) does not imply the stronger inequality:

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E. Milman suggested to us an additional argument based on the original proof by Brascamp & Lieb.

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$$v_0 = e^{-u_0},\ v_1 = e^{-u_1},\ u_0, u_1 \text{ convex.}$$

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$$M \;\mapsto\; \text{trace}(M^{-1}), \quad (M > 0),$$

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Multiply both sides by v_t, integrate on K_t with respect to γ_n,

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$\square$

Salzburg, 2004