Pseudo-cones, copolarity, and Minkowski type problems

Rolf Schneider

U Freiburg

Conference on Convex Geometry and Geometric Probability
Salzburg, September 25 – 29, 2023

Some recent "history":

Everybody knows the ordinary polarity of sets $K \subseteq \mathbb{R}^n$,

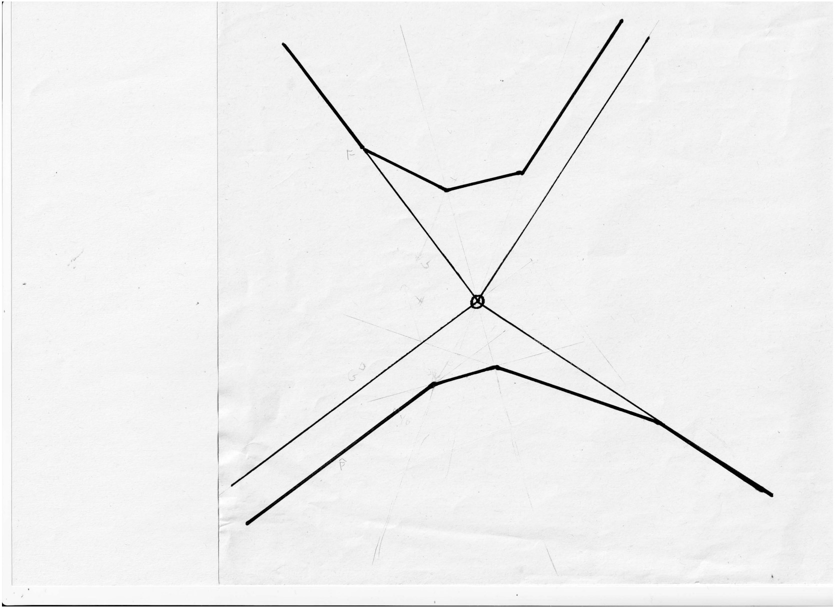
$$K^\circ := \{x \in \mathbb{R}^n : \langle x, y \rangle \leq 1 \text{ for all } y \in K\}.$$

Restricted to the convex bodies K with $o \in \operatorname{int} K$, this is an involution, $K^{\circ\circ} = K$.

Artstein–Avidan, Sadovsky and Wyczesany, "A zoo of dualities" (2023), suggested a **dual** polarity: (up to a reflection)

$$K^* := \{x \in \mathbb{R}^n : \langle x, y \rangle \leq -1 \text{ for all } y \in K\}.$$

It becomes an involution, $K^{**} = K$, when restriced to the "cone-like" sets K, satisfying $o \notin K$ and $\lambda K \subseteq K$ for $\lambda \geq 1$.



The name **copolar set** for K^* was introduced by Rashkovskii (2017).

He used copolarity to define a "copolar addition" by $K \oplus L := (K^* + L^*)^*$ and proved a corresponding (reverse) Brunn–Minkowski inequality for covolumes.

The name **pseudo-cone** was introduced by Y. Xu, J. Li, G. Leng (2023).

They studied copolarity in detail (though not under this name) and characterized it.

They also proved: A nonempty closed convex set K not containing o is a pseudo-cone if and only if K is contained in its recession cone.

Pseudo-cones

Definition. A pseudo-cone in $\mathbb{R}^n$ is a nonempty closed convex set K satisfying

$$o \notin K, \qquad \lambda K \subseteq K \text{ for } \lambda \geq 1.$$

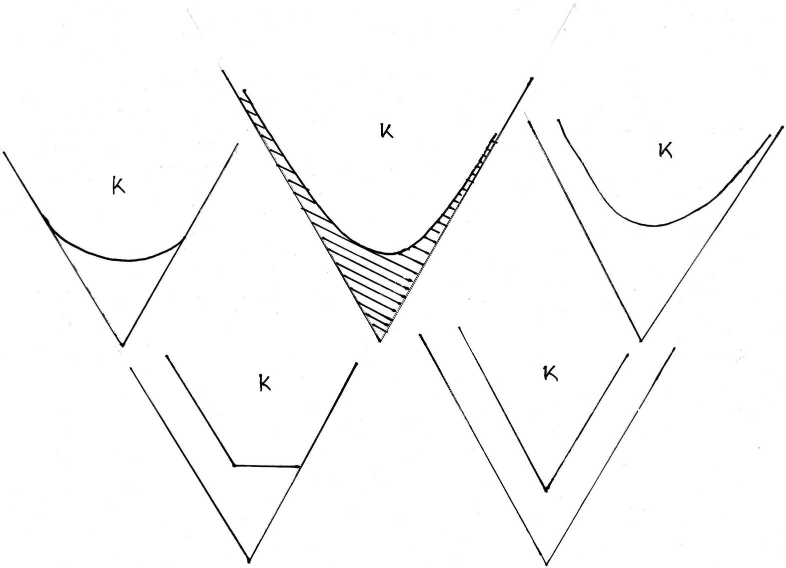
Definition. The recession cone of a closed convex set K is the set

$$\operatorname{rec} K := \{z \in \mathbb{R}^n : K + z \subseteq K\}.$$

We recall that a pseudo-cone K satisfies

$$K \subset \operatorname{rec} K$$

and that among closed convex sets not containing o this characterizes pseudo-cones.


K
K
K
K
K

We fix a closed convex cone $C \subset \mathbb{R}^n$, pointed and with nonempty interior, and denote by

$$ps(C) \quad \text{the set of } C\text{-pseudo-cones},$$

that is, of pseudo-cones K with $\operatorname{rec} K = C$.

On $ps(C)$, the Hausdorff metric d_H can be defined as for convex bodies.

There is a counterpart to the Blaschke selection theorem:

Theorem. Every sequence of C-pseudo-cones with bounded distances from the origin contains a subsequence that converges to some C-pseudo-cone.

Copolarity

Definition. For an arbitrary set $\emptyset \neq A \subseteq \mathbb{R}^n$ we define the **copolar set** by

$$A^* := \{x \in \mathbb{R}^n : \langle x, y \rangle \leq -1 \text{ for all } y \in A\}$$

and the **shadow** of A by

$$\operatorname{shad} A := \{\lambda x : x \in A,\ \lambda \geq 1\}.$$

Lemma. *Let $\emptyset \neq A \subseteq \mathbb{R}^n$. Then $A^* \neq \emptyset$ if and only if $o \notin \operatorname{cl}\operatorname{conv} A$.*

Suppose that $o \notin \operatorname{cl}\operatorname{conv} A$. Then A^* is a pseudo-cone, and $A^{**} = \operatorname{shad}\operatorname{cl}\operatorname{conv} A$.

For a pseudo-cone K we have

$$\mathrm{rec} K^* = (\mathrm{rec}\, K)^\circ,$$

where the dual cone of C is defined by

$$C^\circ := \{x \in \mathbb{R}^n : \langle x, y\rangle \leq 0 \text{ for all } x \in C\}.$$

Copolarity of pseudo-cones can be described as follows.

Definition. Let K be a pseudo-cone. A **crucial pair** of K is a pair $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$ such that $x \in \partial K$ and y is an outer normal vector of K at x, normalized so that $\langle x, y\rangle = -1$.

Lemma. If (x, y) is a crucial pair of the pseudo-cone K, then (y, x) is a crucial pair of K^*.

So far the definitions.

And here is the program:

Minor:

(I) In how far has copolarity similar properties as the ordinary polarity?

Major:

(II) Minkowski type problems: C-pseudo-cones with given surface area measure.

I. Copolarity vs ordinary polarity

(a) **Linearization**

Theorem. Let $\mathcal{CC}^n$ be the set of nonempty closed convex sets on $\mathbb{R}^n$ and $V(\mathcal{CC}^n)$ be the real vector space spanned by the characteristic functions of sets in $\mathcal{CC}^n$.

There is a linear mapping $\Phi : V(\mathcal{CC}^n) \to V(\mathcal{CC}^n)$ such that

$$\Phi(\mathbb{1}_K) = \mathbb{1}_{K^*} \quad \text{for } K \in \mathcal{CC}^n.$$

(For ordinary polarity, such a result can be found in the book of Barvinok (2002)).

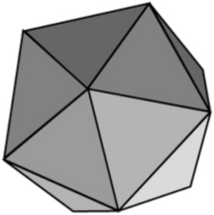
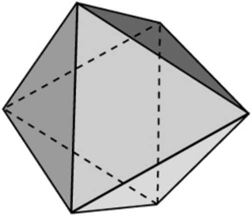
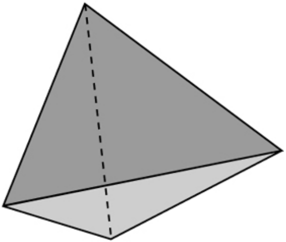
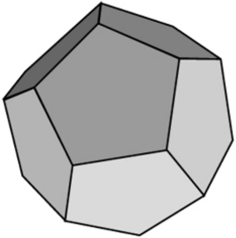
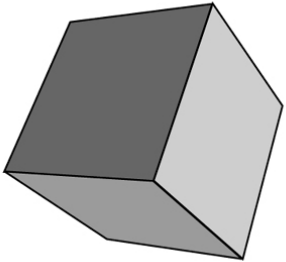
(b) **Characterization**

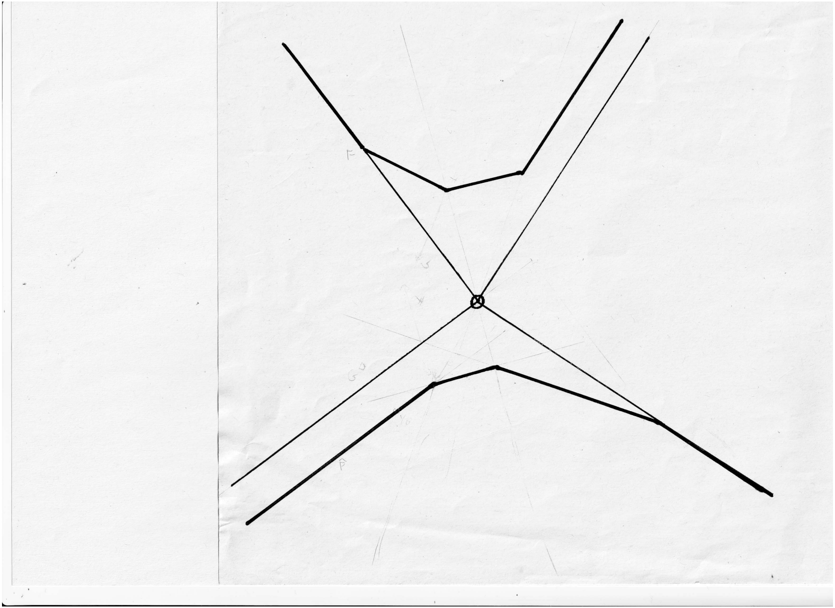
A characterization of the ordinary polarity, due to Böröczky and Sch (2008) has a counterpart, proved by Xu, Li, Leng (2023):

Theorem. A mapping τ from the set of pseudo-cones in $\mathbb{R}^n$ into itself satisfies $\tau(\tau(K)) = K$ and $K \subset L \Rightarrow \tau(K) \supset \tau(L)$ for all pseudo-cones K if and ony if $\tau(K) = g(K^*)$ for some self-adjoint $g \in \mathrm{GL}(n)$.

(c) **Conjugate faces**

For ordinary polarity, conjugate faces of polar polytopes are well known.



Let the pseudo-cone K be polyhedral, that is, intersection of finitely man closed halfspaces.

For a face F of K, the **conjugate face** is defined by

$$\widehat{F} := \{x \in K^* : \langle x, y \rangle = -1 \text{ for all } y \in F\}.$$

Denote by

$\mathcal{F}_b^{\mathrm{int}}(K)$ the set of bounded faces meeting $\operatorname{int} C$,

$\mathcal{F}_u^{\mathrm{int}}(K)$ the set of unbounded faces meeting $\operatorname{int} C$,

$\mathcal{F}_b^{\partial}(K)$ the set of bounded faces contained in ∂C.

Theorem. $F \to \widehat{F}$ is an inclusion-reversing mapping of $\mathcal{F}_b^{\mathrm{int}}(K) \cup \mathcal{F}_u^{\mathrm{int}}(K) \cup \mathcal{F}_b^{\partial}(K)$ to $\mathcal{F}_b^{\mathrm{int}}(K^*) \cup \mathcal{F}_u^{\mathrm{int}}(K^*) \cup \mathcal{F}_b^{\partial}(K^*)$.

It satisfies $\widehat{\widehat{F}} = F$ and $\dim F + \dim \widehat{F} = n - 1$.

It maps

$\mathcal{F}_b^{\mathrm{int}}(K)$ onto $\mathcal{F}_b^{\mathrm{int}}(K^*)$,

$\mathcal{F}_b^{\partial}(K)$ onto $\mathcal{F}_u^{\mathrm{int}}(K^*)$,

$\mathcal{F}_u^{\mathrm{int}}(K)$ onto $\mathcal{F}_b^{\partial}(K^*)$.

Unbounded faces contained in ∂C do not have conjugate faces.

(d) **Smooth pseudo-cones**

Under suitable differentiability assumptions, observations about polarity in **centro-affine differential geometry** due to Salkowski (1934), Laugwitz (1957), Oliker and Simon (1992) can be carried over to copolarity:

Since copolarity reverses crucial pairs, the hypersurfaces ∂K and ∂K^* are naturally mapped to each other. Under this mapping, they have the following properties:

(1) The centro-affine Riemannian metrics are the same.

(2) The centro-affine cubic fundamental forms differ only by the sign.

(3) The equi-affine support functions are reciprocal.

(4) If ∂K is an improper affine hypersphere, then also ∂K^* is an improper affine hypersphere.

II. Minkowski type problems

Let $K \in ps(C)$ be a C-pseudo-cone. Let $\mathcal{H}^k$ denote the k-dimensional Hausdorff measure.

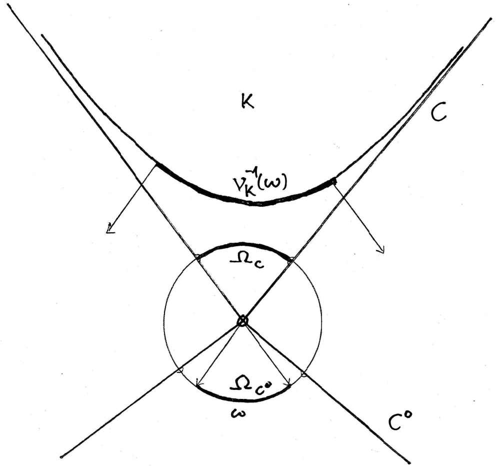
One defines the **surface area measure** of K by

$$S_{n-1}(K,\omega) := \mathcal{H}^{n-1}(\nu_K^{-1}(\omega))$$

for Borel sets $\omega \subset \Omega_{C^\circ}$, where

$$\Omega_{C^\circ} := \mathbb{S}^{n-1} \cap \operatorname{int} C^\circ, \qquad \Omega_C := \mathbb{S}^{n-1} \cap \operatorname{int} C.$$

Thus, $S_{n-1}(K,\cdot)$ is the image measure of $\mathcal{H}^{n-1}$, restricted to ∂K, under the Gauss map.


K
C
$\nu_K^{-1}(\omega)$
Ω_C
Ω_{C°
ω
C°

Minkowski's problem for *C*-pseudo-cones:

What are the necessary and sufficient conditions on a Borel measure φ on Ω_{C° in order that there exist a *C*-pseudo cone K with

$$S_{n-1}(K,\cdot) = \varphi?$$

And what about uniqueness?

Note that, in contrast to the classical case of convex bodies, Ω_{C° is an open subset of a hemisphere, and $S_{n-1}(K,\cdot)$ may be infinite.

For **discrete measures** and polyhedral C, a (more general) existence theorem is already found in the book of Aleksandrov (1950).

No additional assumptions are required.

Aleksandrov writes:

“Proofs of the generalizations of the theorems of Sections 7.3 and 7.4 (still unpublished at present) can be carried out by passing to the limit from polyhedra.”

But this is not true.

Non-discrete measures must satisfy additional assumptions.

Roughly speaking, a surface area measure $S_{n-1}(K,\cdot)$ cannot grow too fast when the boundary of Ω_{C° is approached.

Definition. For compact $\omega \subset \Omega_{C^\circ}$, define

$$\Delta(\omega) := \min\{\angle(u,v) : u \in \omega,\ v \in \partial\Omega_{C^\circ}\},$$

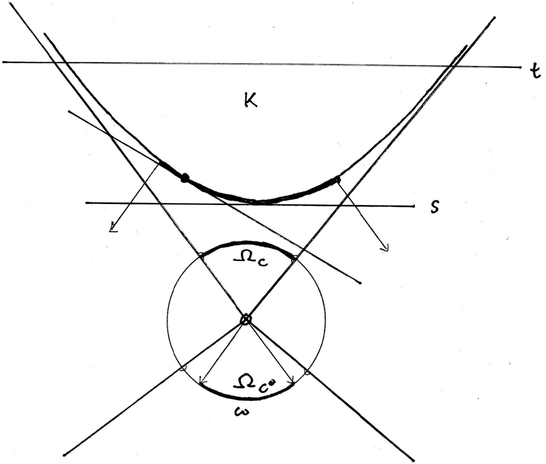
the **distance** of ω from the boundary of Ω_{C°.

We can choose a unit vector $\mathfrak{v} \in \operatorname{int} C \cap \operatorname{int}(-C^\circ)$.
Then (Sch (2020)):

Lemma. For each C-pseudo-cone K, there exists a constant c, depending only on C, $\mathfrak{v}$ and K, such that

$$S_{n-1}(K,\omega) \le \frac{c}{\Delta(\omega)^{n-1}}$$

for each compact set $\omega \subset \Omega_{C^\circ}$.


t
K
s
Ω_c
Ω_{c°
ω

Equivalently:

If we define

$$\delta(u) := \min\{\angle(u, v) : v \in \partial\Omega_{C^\circ}\}, \quad u \in \Omega_{C^\circ},$$

then $\varphi = S_{n-1}(K, \cdot)$ for a C-pseudo-cone K satisfies

$$\int_{\Omega_{C^\circ}} \delta(u)^{n-1} \, \varphi(\mathrm{d}u) < \infty.$$

It is still an **open problem** whether this condition is sufficient.

Three special solutions

The first one is already a bit older.

(1) ***C*-close pseudo-cones and finite measures**

A C-pseudo-cone K is called

C**-full** if $C \setminus K$ is bounded,
C**-close** if $C \setminus K$ has finite volume.

For the covolume (volume of $C \setminus K$) of C-full sets K, Khovanskiĭ and Timorin (2014) obtained versions of the classical inequalities of convex geometry (Brunn–Minkowski, Aleksandrov–Fenchel, Minkowski), reversed.

This was extended to C-close sets (Sch 2018). The theory can now be found in the last chapter of

R. Schneider, *Convex Cones: Geometry and Probability.* Lecture Notes in Math. **2319**, Springer, Cham, 2022.

There one finds:

Theorem. Every nonzero, finite Borel measure on Ω_{C° with compact support (contained in Ω_{C°) is the surface area measure of a C-full pseudo-cone.

Theorem. Every nonzero, finite Borel measure on Ω_{C° is the surface area measure of a C-close pseudo-cone.

Theorem. If K, L are C-close pseudo-cones with the same surface area measure, then $K = L$.

Extension to the L_p case by J. Yang, D. Ye, B. Zhu (2023).

(2) *C*-close pseudo-cones and infinite measures

To reformulate the growth condition, define

$$\omega(\alpha) := \{u \in \Omega_{C^\circ} : \delta(u) > \alpha\}$$

(a kind of inner parallel set of Ω_{C°).

Theorem. Let φ be a nonzero Borel measure on Ω_{C°. If there are constants $c > 0$ and $\kappa \in (0, 1/n)$ such that

$$\varphi(\omega(\alpha)) \leq c\alpha^{-\kappa}$$

for $\alpha > 0$, then φ is the surface area measure of a *C*-close set.

The question for a necessary and sufficient condition remains open.

Since this is our only existence result for infinite measures, we give some ideas. The first step (from Sch (2018)) is to adapt Aleksandrov's approach for convex bodies to pseudo-cones and measures with compact support.

Let $\omega \subset \Omega_{C^\circ}$ be compact, let $C(\omega)$ be the space of continuous functions on ω.

For positive $f \in \mathcal{C}(\omega)$, the **Wulff shape** is defined by

$$[f] := C \cap \bigcap_{u \in \omega} \{x \in \mathbb{R}^n : \langle x, u \rangle \leq -f(u)\}.$$

For $K \in ps(C)$, let $\overline{h}_K$ be the absolute support function of K. Then

$$[\overline{h}_K] = K \quad \text{and} \quad f \leq \overline{h}_K.$$

Note that $[f] \in \mathcal{K}(C, \omega)$, the family of pseudo-cones K that are C-determined by ω, which means that

$$K = C \cap \bigcap_{u \in \omega} \{x \in \mathbb{R}^n : \langle x, u \rangle \le -\overline{h}_K(u)\}.$$

Aleksandrov's approach (adapted) is to maximize the functional

$$\Phi(f) := V_n(C \setminus [f])^{-1/n} \int_\omega f \, \mathrm{d}\varphi, \quad f \in \mathcal{C}(\omega),$$

over $\{\overline{h}_L : L \in \mathcal{K}(C, \omega)\}$.

A maximum exists.

Then one needs Aleksandrov's variational lemma (adapted):

$$\lim_{t\to 0} \frac{V_n(C \setminus [\overline{h}_K + tf]) - V_n(C \setminus K)}{t} = \int_\omega f(u)\, S_{n-1}(K, \mathrm{d}u).$$

Result:

There is a set $M \in \mathcal{K}(C, \omega)$ with

$$V_n(C \setminus M) = 1$$

and such that

$$K := \lambda^{\frac{1}{n-1}} M \quad \text{with} \quad \lambda := \frac{1}{n} \int_\omega \overline{h}_M \, \mathrm{d}\varphi$$

satisfies

$$\varphi = S_{n-1}(K, \cdot).$$

This is now applied to a sequence $(\omega_j)_{j\in\mathbb{N}}$ of compact sets such that

$$\varphi(\omega_1) > 0, \quad \omega_j \subset \operatorname{int}\omega_{j+1}, \quad \bigcup_{j\in\mathbb{N}} \omega_j = \Omega_{C^\circ}.$$

For every $j \in \mathbb{N}$, there is a set $M_j \in \mathcal{K}(C, \omega_j)$ with

$$V_n(C \setminus M_j) = 1$$

and such that

$$K_j := \lambda_j^{\frac{1}{n-1}} M_j \quad \text{with} \quad \lambda_j := \frac{1}{n}\int_{\omega_j} \overline{h}_{M_j}\,\mathrm{d}\varphi$$

satisfies

$$\varphi(\cdot) = S_{n-1}(K_j, \cdot) \quad \text{on } \omega_j.$$

By the selection theorem, a subsequence of $(M_j)_{j\in\mathbb{N}}$ converges to a pseudo-cone M.

Since the covolume is lower-semicontinuous, M is C-close.

Problem: Is the sequence $(\lambda_j)_{j\in\mathbb{N}}$ bounded?

Equivalently: Is

$$\int_{\Omega_{C^\circ}} \overline{h}_M \,\mathrm{d}\varphi < \infty?$$

If "yes", then also a subsequence of $(K_j)_{j\in\mathbb{N}}$ converges to a pseudo-cone K, which is C-close and satisfies

$$\varphi = S_{n-1}(K,\cdot),$$

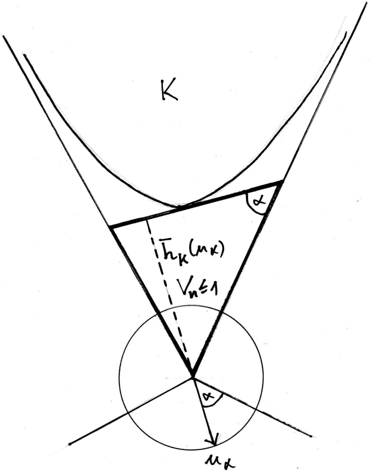
by weak continuity properties.

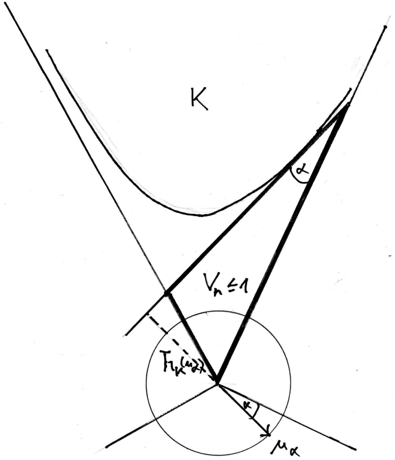
Lemma. There is a constant c_1, depending only on C , such that every C-close set K with $V_n(C \setminus K) \leq 1$ satisfies

$$\overline{h}_K(u) \leq c_1 \delta(u)^{1/n} \quad \text{for } u \in \Omega_{C^\circ}.$$

Therefore, the assumption $\varphi(\omega(\alpha)) \leq c\alpha^{-\kappa}$ with $\kappa \in (0, 1/n)$ yields

$$\begin{aligned} c_1^{-1} \int_{\Omega_{C^\circ}} \overline{h}_M \, \mathrm{d}\varphi &\leq \int_{\Omega_{C^\circ}} \delta^{1/n} \, \mathrm{d}\varphi \\ &= \int_0^\infty \varphi\left(\left\{u \in \Omega_{C^\circ} : \delta^{1/n} > \alpha\right\}\right) \mathrm{d}\alpha \\ &= \int_0^\infty \varphi(\omega(\alpha^n)) \, \mathrm{d}\alpha = \int_0^{(\pi/2)^{1/n}} \varphi(\omega(\alpha^n)) \, \mathrm{d}\alpha \\ &\leq c \int_0^{(\cdot)} \alpha^{-n\kappa} \, \mathrm{d}\alpha < \infty. \end{aligned}$$


K
$\overline{h}_K(u_\alpha)$
$V_n \leq 1$
α
u_α


K
$V_n \leq 1$
α
$\overline{h}_K(u_\alpha)$
u_α

(3) **Weighted surface area measures**

The idea of Minkowski's problem is to determine the shape of a convex body by its surface area measure.

The shape of a pseudo-cone is, far away from the origin, more and more determined by its recession cone.

Therefore, close to the origin, the surface area of a pseudo-cone should be given higher weight.

For convex bodies, weighted Minkowski problems have been treated by Livshyts (2019), Kryvonos and Langharst (2023).

Definition Let $\Theta : C \setminus \{o\} \to (0, \infty)$ be continuous and homogeneous of degree $-p$, where $n-1 < p < n$.

The Θ**-weighted surface area measure** of $K \in ps(C)$ is defined by

$$S_{n-1}^{\Theta}(K,\omega) := \int_{\nu_K^{-1}(\omega)} \Theta(x)\, \mathcal{H}^{n-1}(\mathrm{d}x)$$

for Borel sets $\omega \subset \Omega_{C^\circ}$.

The Θ**-weighted covolume** of $K \in ps(C)$ is defined by

$$V_\Theta(K) := \int_{C\setminus K} \Theta(x)\, \mathcal{H}^n(\mathrm{d}x).$$

Lemma. Under the assumption $n-1 < p < n$, the weighted surface area measure and the weigthed covolume of C-pseudo-cones are finite.

Theorem. For every nonzero, finite Borel measure on Ω_{C° there exists a C-pseudo-cone K with

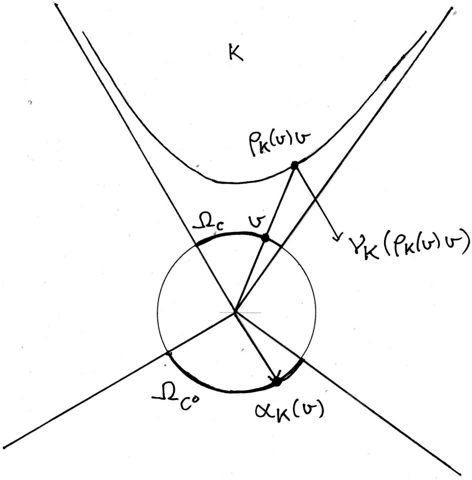
$$S^{\Theta}_{n-1}(K,\cdot)=\varphi.$$

Uniqueness?

Only a preliminary result:

Theorem. If $\Theta(x)=\|x\|^p$ and φ has compact support (contained in Ω_{C°), then a C-pseudo-cone with Θ-weighted surface area measure φ is uniquely determined.

A technical intermezzo:


K
ρK(v)v
Ωc
v
γK(ρK(v)v)
ΩC°
αK(v)

For $K \in ps(C)$, let ϱ_K be its radial function.

For almost all $v \in \Omega_C$ we can define

$$\alpha_K(v) := \nu_K(\varrho_K(v)v)\qquad (\textbf{radial Gauss map}).$$

For a bounded, continuous function $g : \Omega_{C^\circ} \to \mathbb{R}$,

$$\int_{\Omega_{C^\circ}} g(u)\, S^{\Theta}_{n-1}(K, \mathrm{d}u) = \int_{\Omega_C} g(\alpha_K(v))\Theta(\varrho_K(v)v)\frac{\varrho_K^{n-1}(v)}{|\langle v, \alpha_K(v)\rangle|}\,\mathrm{d}v.$$

This can be used, together with the dominated convergence theorem, to show the weak continuity of the Θ-weighted surface area measure on $\mathcal{K}(C,\omega)$, for compact $\omega \subset \Omega_{C^\circ}$.

Proof of the existence theorem: first for compact $\omega \subset \Omega_{C^\circ}$, then using an increasing sequence $\omega_j \uparrow \Omega_{C^\circ}$.

For compact ω, one maximizes

$$\Phi(f) := V_\Theta([f])^{-\frac{1}{n-p}} \int_\omega f \, \mathrm{d}\varphi, \quad f \in \mathcal{C}(\omega).$$

But the required variational lemma is now more difficult, since Aleksandrov's approach (mixed volumes, Minkowski's inequalities) cannot be used.

Fortunately, Huang, Lutwak, Yang, Zhang (2016) have found a different approach. It was extended to the weighted situation by Kryvonos and Langharst (2023). This can be carried over to pseudo-cones:

Lemma. Let $K \in \mathcal{K}(C, \omega)$, for some compact ω, and let $f \in \mathcal{C}(\omega)$. Then

$$\lim_{t \to 0} \frac{V_\Theta([\overline{h}_K + tf]) - V_\Theta(K)}{t} = \int_\omega f \, S_{n-1}^\Theta(K, \mathrm{d}u).$$

For one of the lemmas of Huang, Lutwak, Yang, Zhang (2016), needed to prove the above, we give (adapted to pseudo-cones) a more direct proof, avoiding convexifications (the polars of Wulff shapes).

Thank you for your attention!