BOOLEAN MODELS IN HYPERBOLIC SPACE

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Joint work with Daniel Hug and Günter Last (Karlsruhe)

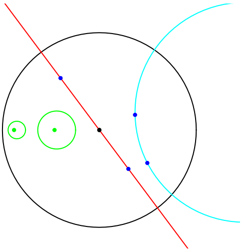
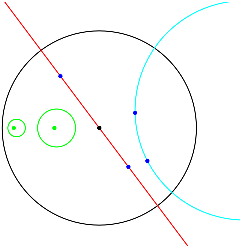


Figure: Circles and geodesics in the Poincaré disk model

- $\mathbb{H}^d$ d-dimensional hyperbolic space



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- $\mathbb{H}^d$ d-dimensional hyperbolic space
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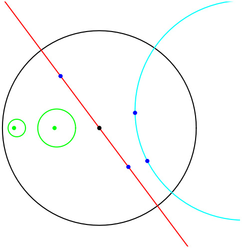

Figure: Circles and geodesics in the Poincaré disk model

- $\mathbb{H}^d$ d-dimensional hyperbolic space
- $\mathcal{K}^d$ compact convex subsets of $\mathbb{H}^d$
- $\mathcal{I}_d$ set of isometries $\varrho : \mathbb{H}^d \to \mathbb{H}^d$, λ Haar measure on $\mathcal{I}_d$ such that for the Hausdorff measure $\mathcal{H}^d$ and $x \in \mathbb{H}^d$,

$$\mathcal{H}^d(\cdot) = \int_{\mathcal{I}_d} 1\{\varrho x \in \cdot\} \, \lambda(\mathrm{d}\varrho).$$

Balls in hyperbolic spaces

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- p fixed point in $\mathbb{H}^d$
- $\mathbb{B}_R$ closed ball with radius R and centre p

Balls in hyperbolic spaces

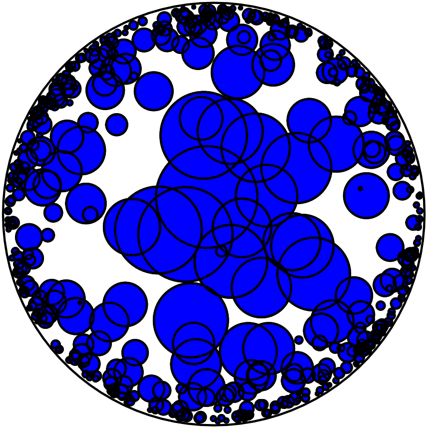
- p fixed point in $\mathbb{H}^d$
- $\mathbb{B}_R$ closed ball with radius R and centre p
- Surface area and volume of $\mathbb{B}_R$ are given by

$$\mathcal{H}^{d-1}(\partial \mathbb{B}_R) = \omega_d \sinh^{d-1}(R)$$

and

$$\mathcal{H}^{d}(\mathbb{B}_R) = \omega_d \int_0^R \sinh^{d-1}(r) \, \mathrm{d}r$$

with ω_d the surface area of the unit ball in $\mathbb{R}^d$.

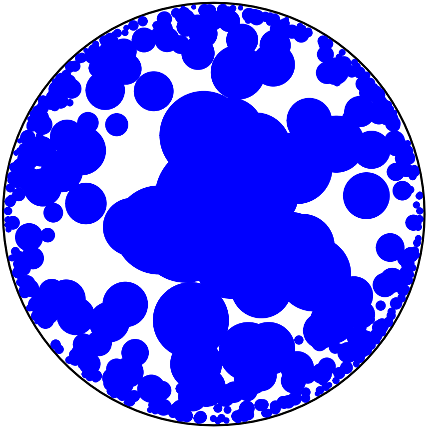


- η stationary Poisson process on $\mathcal{K}^d$, i.e., the intensity measure Λ satisfies

$$\Lambda(\varrho \cdot) = \Lambda(\cdot)$$

for all $\varrho \in \mathcal{I}_d$.

Boolean Model in hyperbolic space



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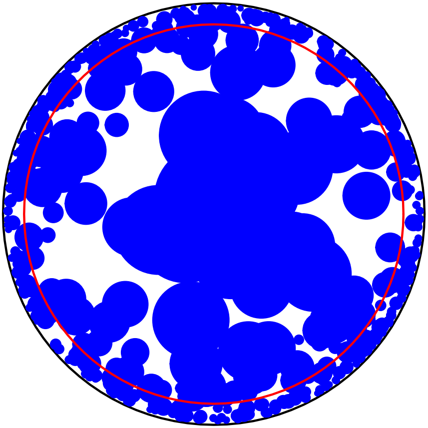
- Boolean model

$$Z = \bigcup_{K \in \eta} K$$



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- Boolean model

$$Z = \bigcup_{K \in \eta} K$$

- Consider $Z \cap \mathbb{B}_R$ as $R \to \infty$

In the following, we assume η to be locally finite, i.e.,

$$\Lambda(\{K \in \mathcal{K}^d : K \cap C\}) < \infty \quad \text{for all compact} \quad C \subset \mathbb{H}^d.$$

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Then, for $f : \mathcal{K}^d \to [0, \infty)$,

$$\int_{\mathcal{K}^d} f(K)\,\Lambda(\mathrm{d}K) = \gamma \int_{\mathcal{K}^d} \int_{\mathcal{I}_d} f(\varrho G)\,\lambda(\mathrm{d}\varrho)\,\mathbb{Q}(\mathrm{d}G)$$

with $\gamma \in [0, \infty)$ and a probability measure $\mathbb{Q}$ invariant under all $\varrho \in \mathcal{I}_d$ with $\varrho(\mathrm{p}) = \mathrm{p}$.

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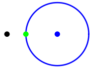
For the 1-parallel set $\mathrm{G}^{(1)}$ of G we have $\mathbb{E}\,\mathrm{Vol}(\mathrm{G}^{(1)}) < \infty$.

Horoballs



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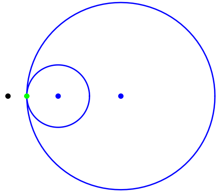

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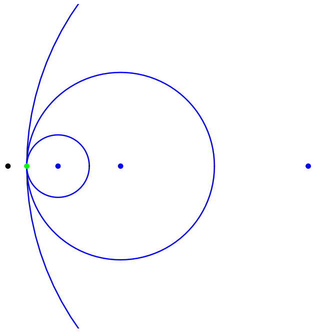

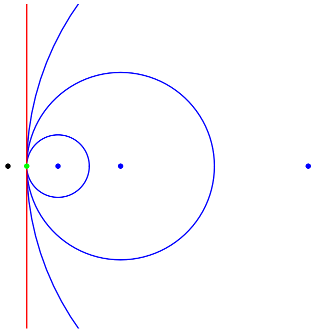

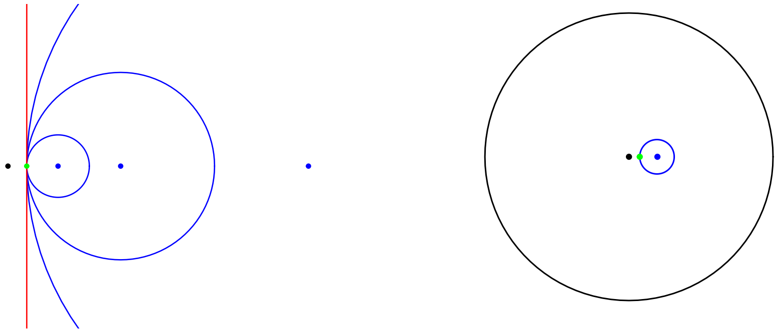

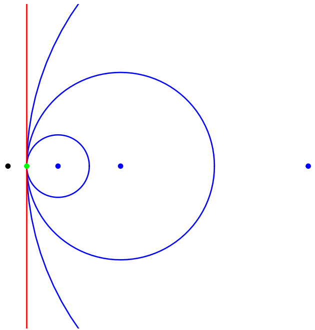

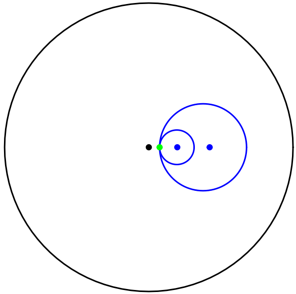

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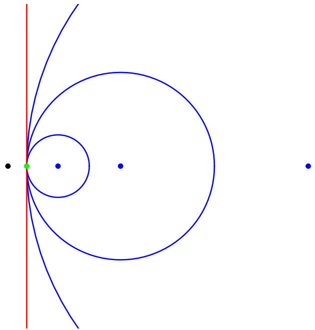

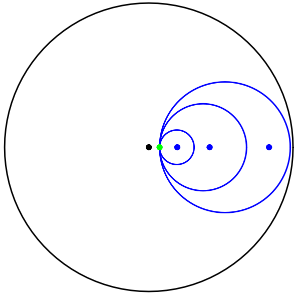

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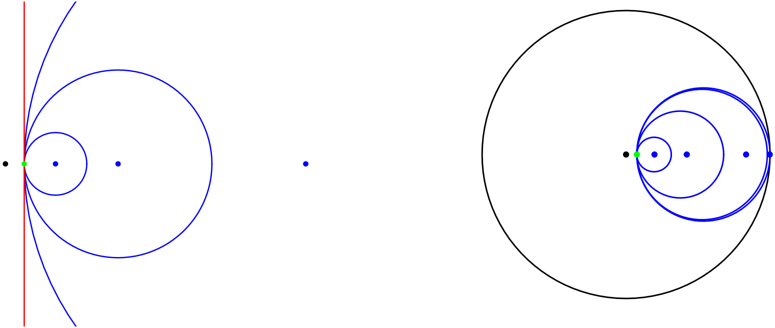

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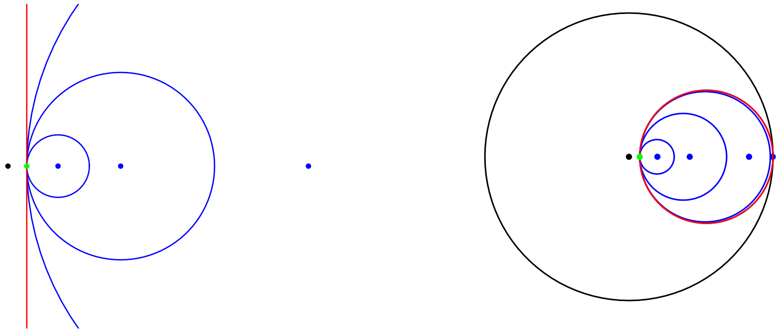

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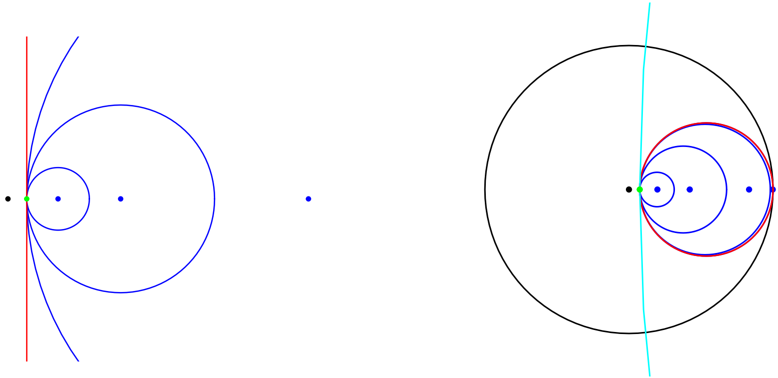

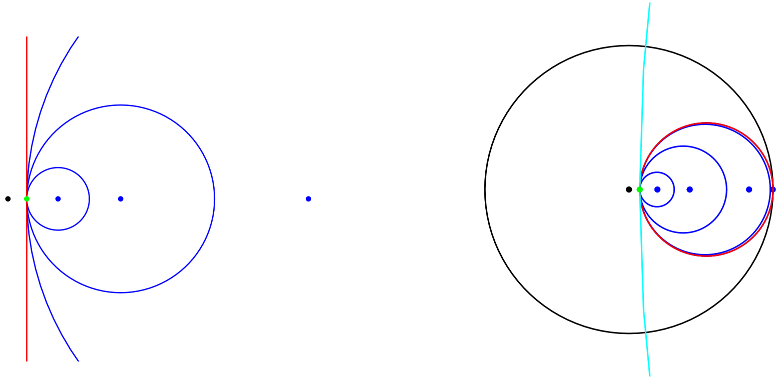

For $u \in \mathbb{S}^{d-1}$ and $t \in \mathbb{R}$, let $\mathbb{B}_{u,t} = \lim_{r\to\infty} \mathbb{B}(\exp_{\mathrm{p}}((t+R)u), R)$. We denote $\mathbb{B}_{u,t}$ as **horoball**.

Volume

Define $v_1 = \mathsf{E}\,\mathrm{Vol}(\mathrm{G})$ and

$$C(x,z) = \mathsf{E}\lambda(\{\varrho \in \mathcal{I}_d : \varrho x, \varrho z \in \mathrm{G}\}) \quad \text{for} \quad x, z \in \mathbb{H}^d.$$

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Theorem: Hug/Last/S. 23+

For $W \in \mathcal{K}^d$,

$$\mathsf{E}\,\mathrm{Vol}(Z \cap W) = \mathrm{Vol}(W)\big(1 - e^{-\gamma v_1}\big)$$

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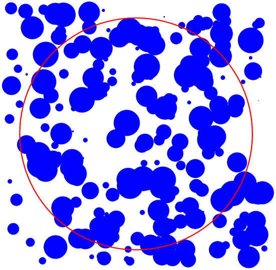
and, if $\mathbb{E}\,\mathrm{Vol}(\mathrm{G}^{(1)})^2 < \infty$,

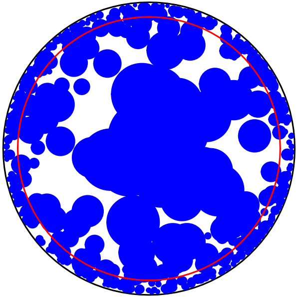
$$\lim_{R\to\infty} \frac{\mathrm{Var}\,\mathrm{Vol}(Z \cap \mathbb{B}_R)}{\mathrm{Vol}(\mathbb{B}_R)} = e^{-2\gamma v_1} \int_{\mathbb{H}^d} \big(e^{\gamma C(\mathrm{p},z)} - 1\big)\mathsf{P}(z \in \mathbb{B}_{U,T})\,\mathcal{H}^d(\mathrm{d}z)$$

with independent $U \sim \mathrm{Uniform}(\mathbb{S}_{\mathrm{p}}^{d-1})$ and $-T \sim \mathrm{Exp}(d-1)$.

Comparison with Euclidean case



$$e^{-2\gamma v_1} \int_{\mathbb{R}^d} \left(e^{\gamma C(z)} - 1\right) \, \mathrm{d}z$$



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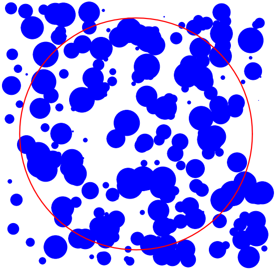
with $C(z) = \mathsf{E}\,\mathrm{Vol}(\mathrm{G} \cap (\mathrm{G} + z))$



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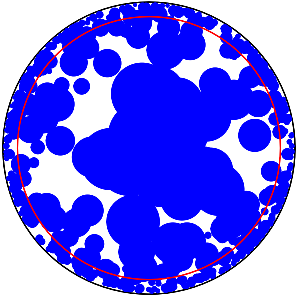

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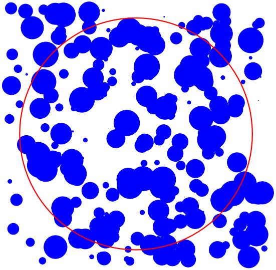

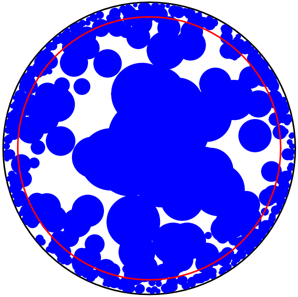

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$$\mathrm{Vol}(\mathbb{B}_R) \approx \frac{\omega_d}{d} R^d$$

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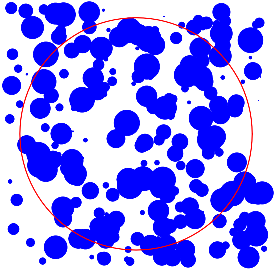

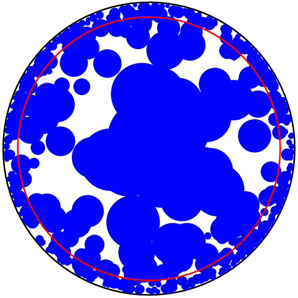

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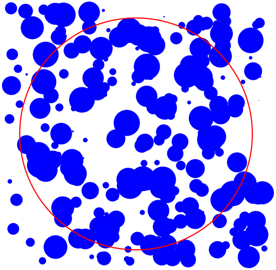

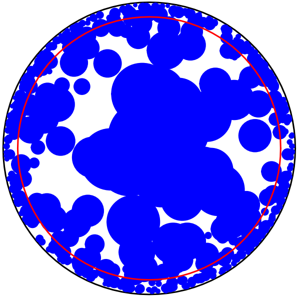

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Geometric functionals

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A map $\phi : \mathcal{R}^d \to \mathbb{R}$ is called a **geometric functional** if it is

- measurable,
- additive, i.e.,

$$\phi(U \cup V) = \phi(U) + \phi(V) - \phi(U \cap V), \quad U, V \in \mathcal{R}^d,$$

- isometry invariant, i.e.,

$$\phi(\varrho U) = \phi(U) \quad \text{for} \quad \varrho \in \mathcal{I}_d \quad \text{and} \quad U \in \mathcal{R}^d,$$

- locally bounded, i.e.,

$$\sup\{|\phi(K)| : K \in \mathcal{K}^d, \varrho \in \mathcal{I}_d, K \subseteq \varrho \mathbb{B}_1\} < \infty.$$

On the set of horoballs $\mathbb{B}_h^d$ we define the measure

$$\mu_{\mathrm{hb}}(\cdot) = \frac{d-1}{\omega_d} \int_{\mathbb{S}_\mathsf{p}^{d-1}} \int_{\mathbb{R}} 1\{\mathbb{B}_{u,t} \in \cdot\} e^{(d-1)t} \, \mathrm{d}t \, \mathcal{H}_\mathsf{p}^{d-1}(\mathrm{d}u).$$

Asymptotic expectations

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Theorem: Hug/Last/Schulte 23+

For a geometric functional $\phi : \mathcal{R}^d \to \mathbb{R}$ that is continuous on $\mathcal{K}^d$,

$$\lim_{R\to\infty} \frac{\mathrm{E}\phi(Z \cap \mathbb{B}_R)}{\mathrm{Vol}(\mathbb{B}_R)} = \gamma \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!} \int_{\mathbb{B}_h^d} \int_{\mathcal{K}^d} \int_{(\mathcal{K}^d)^{n-1}} \phi(G \cap K_2 \cap \ldots \cap K_n \cap B) \times \Lambda^{n-1}(\mathrm{d}(K_2, \ldots, K_n)) \, \mathrm{Q}(\mathrm{d}G) \, \mu_{\mathrm{hb}}(\mathrm{d}B).$$

Asymptotic variances

Theorem: Hug/Last/Schulte 23+

Assume that $\mathrm{E}\,\mathrm{Vol}(\mathrm{G}^{(1)})^2 < \infty$ and let $\phi : \mathcal{R}^d \to \mathbb{R}$ be a geometric functional that is continuous on $\mathcal{K}^d$. Then,

$$\lim_{R\to\infty} \frac{\mathrm{Var}\,\phi(Z \cap \mathbb{B}_R)}{\mathrm{Vol}(\mathbb{B}_R)}$$
$$= \gamma \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\mathbb{B}_h^d} \int_{\mathcal{K}^d} \int_{(\mathcal{K}^d)^{n-1}} \phi^*(G \cap K_2 \cap \ldots \cap K_n \cap B)^2$$
$$\times \Lambda^{n-1}(\mathrm{d}(K_2, \ldots, K_n))\, \mathrm{Q}(\mathrm{d}G)\, \mu_{\mathrm{hb}}(\mathrm{d}B)$$

with $\phi^*(\cdot) = \mathrm{E}\phi(Z \cap \cdot)$.

Theorem: Hug/Last/Schulte 23+

Let $\phi : \mathcal{R}^d \to \mathbb{R}$ be a geometric functional such that

$$\liminf_{R\to\infty} \frac{\operatorname{Var} \phi(Z \cap \mathbb{B}_R)}{\operatorname{Vol}(\mathbb{B}_R)} > 0$$

and let N be a standard Gaussian random variable.

a) If $\mathsf{E}\,\operatorname{Vol}(\mathrm{G}^{(1)})^2 < \infty$, then

$$S_R := \frac{\phi(Z \cap \mathbb{B}_R) - \mathbb{E}\phi(Z \cap \mathbb{B}_R)}{\sqrt{\operatorname{Var}(\phi(Z \cap \mathbb{B}_R))}} \xrightarrow{d} N \quad \text{as} \quad R \to \infty.$$

Central limit theorems

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b) If $\mathrm{E}\operatorname{Vol}(\mathrm{G}^{(1)})^4 < \infty$, there exists a constant $C\in(0,\infty)$ such that

$$\sup_{t\in\mathbb{R}} \left|\mathbb{P}(S_R\le t) - \mathrm{P}(N\le t)\right| \le \frac{C}{\sqrt{\operatorname{Vol}(\mathbb{B}_R)}}, \quad R\ge 1.$$

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Thank you!

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