A new connection between the volume product and regularization of heat flow

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Volume product

- Let $K \subset \mathbb{R}^n$ be a convex body (i.e., compact and convex set with $\mathrm{int}K \neq \emptyset$) with $0 \in \mathrm{int}K$.
- Polar body of K:

$$K^\circ := \{x \in \mathbb{R}^n \mid \langle x, y \rangle \leq 1, \forall y \in K\}.$$

c.f. $(\mathrm{B}_p^n)^\circ = \mathrm{B}_{p'}^n$ with $p^{-1} + (p')^{-1} = 1$ where

$$\mathrm{B}_p^n := \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid (\sum_{i=1}^n |x_i|^p)^{\frac{1}{p}} \leq 1\}, \quad 1 \leq p \leq \infty.$$

- Volume product of K:

$$v(K) := |K||K^\circ|.$$

  v is linear invariant, i.e., $v(TK) = v(K)$ for any linear isomorphism T on $\mathbb{R}^n$.

Blaschke–Santaló inequality and Mahler conjecture

Theorem 1 (Blaschke 1917, Santaló 1949, Petty 1985)

For any convex body $K \subset \mathbb{R}^n$ with $b_K := \frac{1}{|K|}\int_K x\,dx = 0$, it holds that

$$v(K) \leq v(\mathrm{B}_2^n).$$

Equality holds iff K is a symmetric ellipsoid.

Mahler conjecture

- *Non-symmetric case : For any convex body $K \subset \mathbb{R}^n$ with $b_K = 0$,*

  $$v(K) \geq v(\Delta_0^n),$$

  where Δ_0^n is an n-dimensional simplex with $b_{\Delta_0^n} = 0$.
- *Symmetric case : For any symmetric convex body $K \subset \mathbb{R}^n$ (i.e., $K = -K$),*

  $$v(K) \geq v(\mathrm{B}_\infty^n) = v(\mathrm{B}_1^n).$$

Known results

- Mahler (1938): symmetric and non-symmetric cases for $n = 2$.
- Iriyeh–Shibata (2020): symmetric case for $n = 3$. A short proof by Fradelizi–Hubard–Meyer–Roldán-Pensado–Zvavitch (2022).

Partial answers.

- unconditional convex bodies: Saint-Raymond (1980), Meyer (1986).
- zonoid: Reisner (1986), Gordon–Meyer–Reisner (1988).
- symmetric polytopes in $\mathbb{R}^n$ with $2n+2$ vertices: Lopez and Reisner (1998), Karasev (2021).
- polytopes with not more than $n+3$ vertices in $\mathbb{R}^n$: Meyer–Reisner (2006).
- some bodies with many symmetries: Barthe–Fradelizi (2013), Iriyeh–Shibata (2022).
- Asymptotic estimate: Bourgain–Milman (1986), Kuperberg (2008).

New lower bound for specific volume products

Corollary 1 (Nakamura–T.)

Let $n \geq 2$, $\kappa \in (0,1]$ and $K \subset \mathbb{R}^n$ be a convex body with $0 \in \mathrm{int} K$. Suppose that $\|\cdot\|_K^2$ is C^2 on $\mathbb{R}^n \setminus \{0\}$ and satisfies

$$\nabla^2\left(\frac{1}{2}\|\cdot\|_K^2\right) \geq \kappa \Lambda^{-1}, \quad \nabla^2\left(\frac{1}{2}\|\cdot\|_{K^\circ}^2\right) \geq \kappa \Lambda$$

for some positive definite symmetric matrix $\Lambda \in \mathbb{R}^{n\times n}$. Then it holds that

$$v(K) \geq (\kappa^2 e^{1-\kappa^2})^{\frac{n}{2}} v(\mathrm{B}_2^n).$$

- Our assumptions imply that the principle curvatures on $\partial(\Lambda^{-\frac{1}{2}}K)$ and $\partial(\Lambda^{-\frac{1}{2}}K)^\circ$ are uniformly bounded from below by κ.
- Stancu (2009) and Reisner–Schütt–Werner (2012): The boundary of the local minimizer must be flat, i.e., if there exists a point in either ∂K or ∂K° at which the (generalized) Gauss curvature exists and is not 0 then $v(K)$ is not a local minimum.
- Trivial lower bound:
$$v(K) \geq (\kappa^2)^{\frac{n}{2}} v(\mathrm{B}_2^n).$$
- Mahler's conjecture is true for K satisfying our assumptions with κ close to 1, i.e.,
$$(\kappa^2 e^{1-\kappa^2})^{\frac{n}{2}} v(\mathrm{B}_2^n) \geq v(\Delta_0^n) \quad \text{in non-symmetric case,}$$
$$(\kappa^2 e^{1-\kappa^2})^{\frac{n}{2}} v(\mathrm{B}_2^n) \geq v(\mathrm{B}_\infty^n) \quad \text{in symmetric case.}$$

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$\rightsquigarrow$ Let us enter the world of heat flow!

Hypercontractivity

- For $s > 0$ (time) and $f\colon \mathbb{R}^n \to [0,\infty)$ (initial data), the Ornstein–Uhlenbeck semigroup is given by

$$P_s f(x) := \int_{\mathbb{R}^n} f(e^{-s}x + \sqrt{1-e^{-2s}}y)\,d\gamma(y),$$

  which is a solution of $\partial_s u = \Delta u - \langle x, \nabla u\rangle$ with $u(s,x) = P_s f(x)$.
- Mass-preservation: $\|P_s f\|_{L^1(\gamma)} = \|f\|_{L^1(\gamma)}$ and $\lim_{s\to\infty} P_s f \equiv$ const.
- Contraction: $\|P_s f\|_{L^p(\gamma)} \le \|f\|_{L^p(\gamma)}$ for $p \ge 1$. In particular,

$$1 \le q \le p \quad \Rightarrow \quad \|P_s f\|_{L^q(\gamma)} \le \|P_s f\|_{L^p(\gamma)} \le \|f\|_{L^p(\gamma)}.$$

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$\rightsquigarrow$ **Yes!** This is Hypercontractiviy.

Hypercontractivity

Theorem 2 (Nelson's forward HC, 1973)

Suppose $s > 0$ *and* $p, q \in \mathbb{R} \setminus \{0\}$. *Then*

$$1 < p, q \text{ with } \frac{q-1}{p-1} \le e^{2s} \quad \Rightarrow \quad \|P_s f\|_{L^q(\gamma)} \le \|f\|_{L^p(\gamma)}.$$

Moreover

$$1 < p, q \text{ with } \frac{q-1}{p-1} > e^{2s} \quad \Rightarrow \quad \sup_{0 \le f \in L^p(\gamma)} \frac{\|P_s f\|_{L^q(\gamma)}}{\|f\|_{L^p(\gamma)}} = +\infty.$$

- In below, we reformulate forward HC as $\|P_s[f^{\frac{1}{p}}]\|_{L^q(\gamma)} \le (\int_{\mathbb{R}^n} f \, d\gamma)^{\frac{1}{p}}$.

Forward hypercontractivity $\Rightarrow$ inverse Santaló inequality

Proposition 1 (Nakamura–T.)

Suppose that for small $s>0$, there exists some $q_s=-2s+o(s)<0$, $p_s=2s+o(s)>0$ and $C_{\mathrm{IS}}(s)>0$ such that

$$\|P_s[f^{\frac{1}{p_s}}]\|_{L^{q_s}(\gamma)} \le C_{\mathrm{IS}}(s)^{\frac{1}{p_s}} \Big(\int_{\mathbb{R}^n} f\, d\gamma\Big)^{\frac{1}{p_s}}$$

for all nonnegative log-concave function f. Then

$$v(K) \ge v(\mathrm{B}_2^n) \limsup_{s\downarrow 0} (C_{\mathrm{IS}}(s)^{-1})$$

for all convex body $K\subset\mathbb{R}^n$.

Sketch of proof. Taking the power of p_s,

$$\|P_s[f^{\frac{1}{p_s}}]\|_{L^{q_s}(\gamma)}^{p_s} \le C_{\mathrm{IS}}(s)\Big(\int_{\mathbb{R}^n} f\, d\gamma\Big).$$

Insert $f(x)=e^{-\frac{1}{2}\|x\|_K^2}$ and let $s\downarrow 0$.

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- In our settings, $-\infty < q_s < 0 < p_s < 1$. $\rightsquigarrow$ Nobody knows forward HC...

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Question 2. Does forward HC hold for $-\infty < q < p < 1$?.

$$-\infty < q < p < 1 \quad \Rightarrow \quad \|P_s[f^{\frac{1}{p}}]\|_{L^q(\gamma)} \leq \Big(\int_{\mathbb{R}^n} f \, d\gamma\Big)^{\frac{1}{p}}.$$

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$\rightsquigarrow$ **Yes!** if f has certain strong log-concavity and convexity.

Answer to Question 2: Forward HC for $p, q < 1$

Theorem 3 (Nakamura–T.)

Let $s > 0$, $0 < p < 1$, $q \in (-\infty, 1) \setminus \{0\}$ *satisfy* $\frac{q-1}{p-1} = e^{2s}$, *and* $\beta \geq 1$. *Then for any* $f \colon \mathbb{R}^n \to (0, \infty)$ *satisfying*

$$0 \leq \nabla^2 \log f \leq (1 - \frac{1}{\beta})\mathrm{id}_{\mathbb{R}^n},$$

it holds that

$$\|P_t[f^{\frac{1}{p}}]\|_{L^q(\gamma)} \leq \|P_t[(\frac{\gamma_\beta}{\gamma})^{\frac{1}{p}}]\|_{L^q(\gamma)} (\int_{\mathbb{R}^n} f\, d\gamma)^{\frac{1}{p}}.$$

Here

$$\gamma_\beta(x) := \frac{1}{(2\pi\beta)^{\frac{n}{2}}} e^{-\frac{1}{2\beta}|x|^2}.$$

- The proof is accomplished by the flow monotonicity of the Fokker–Planck flow combined with the Poincaré inequality.

New lower bound for specific volume products again

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for some positive definite symmetric matrix $\Lambda \in \mathbb{R}^{n\times n}$. Then it holds that

$$v(K) \geq (\kappa^2 e^{1-\kappa^2})^{\frac{n}{2}} v(\mathrm{B}_2^n).$$

- To obtain this corollary, we apply the same argument in Proposition 1 to Theorem 3.

Open question toward Mahler conjecture

Conjecture

Let $s > 0$, $p_s = 1 - e^{-2s}$ *and* $q_s = 1 - e^{2s}$.

Non-symmetric case:

$$\sup_{\substack{0 \le f \in L^1(\gamma) \\ \textit{log-concave}}} \frac{\|P_s[f^{\frac{1}{p_s}}]\|_{L^{q_s}(\gamma)}}{\left(\int_{\mathbb{R}^n} f\, d\gamma\right)^{\frac{1}{p_s}}} = \frac{\|P_s[f_*^{\frac{1}{p_s}}]\|_{L^{q_s}(\gamma)}}{\left(\int_{\mathbb{R}^n} f_*\, d\gamma\right)^{\frac{1}{p_s}}}$$

where $f_*(x) := \mathbf{1}_{[-1,\infty)^n} e^{-(x_1+\cdots+x_n)}/\gamma(x)$.

Symmetric case:

$$\sup_{\substack{0 \le f \in L^1(\gamma) \\ \textit{sym. log-concave}}} \frac{\|P_s[f^{\frac{1}{p_s}}]\|_{L^{q_s}(\gamma)}}{\left(\int_{\mathbb{R}^n} f\, d\gamma\right)^{\frac{1}{p_s}}} = \frac{\|P_s[f_{**}^{\frac{1}{p_s}}]\|_{L^{q_s}(\gamma)}}{\left(\int_{\mathbb{R}^n} f_{**}\, d\gamma\right)^{\frac{1}{p_s}}}$$

where $f_{**}(x) := e^{-(|x_1|+\cdots+|x_n|)}/\gamma(x)$.

- $f_*\gamma, f_{**}\gamma$ are conjectured as minimizers of functional version of Mahler conjecture formulated by Fradelizi–Meyer (2008).

Thank you for your attention!