Minkowski Problem in Integral Geometry

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The classical Minkowski Problem

- In this talk, K is always a convex body (with non-empty interiors) in $\mathbb{R}^n$, and $h = h_K$ means its support function.
- Suppose $h \in C_+^2(S^{n-1})$. Then,

$$x = \nabla h(v) + h \cdot v$$

can be viewed as the reverse Gauss map, and

$$\nabla^2 h + hI$$

can be viewed as the reverse Weingarten map from $v^\perp$ to $T_x \partial K$, and hence,

$$\det \left(\nabla^2 h + hI\right)(v)$$

is the reciprocal Gauss curvature at x.

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  is the reciprocal Gauss curvature at x.
- The Minkowski problem (smooth version) asks to construct a convex body K whose Gauss curvature is the prescribing spherical function f, that is

  $$G(x) = f(\nu_K(x)), \quad x \in \partial K.$$

  What are the both sufficient and necessary conditions on f?

The classical Minkowski Problem

- Monge-Ampère equation on S^{n-1}. Given positive $f \in C(S^{n-1})$, find convex solution $h : S^{n-1} \to \mathbb{R}$ to

$$\det\left(\nabla^2 h + hI\right) = f.$$

  Minkowski, Aleksandrov, Fenchel-Jessen, Cheng-Yau, Nirenberg, Pogorelov, Caffarelli...

- However, the classical solution of the above PDE is only suitable for characterizing regular convex bodies (smooth and strictly convex).

The classical Minkowski Problem (Prescribing measure problem)

- Reverse Gauss image: For a $\mathrm{Borel}\ \ \omega \subset S^{n-1}$

  $\boldsymbol{\nu}_K^*(\omega) := \{x \in \partial K : \ \ \exists v \in \omega$, such that v is an outer normal of ∂K at $x\}$.

- Surface area measure:

$$S(K,\omega) := \mathcal{H}^{n-1}\left(\boldsymbol{\nu}_K^*(\omega)\right), \quad \mathrm{Borel}\ \omega \subset S^{n-1}.$$

- The Minkowski problem (general version). Given a Borel measure μ on the sphere S^{n-1}, is it possible to construct a convex body K such that

$$S(K,\cdot) = \mu?$$

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- Discrete Minkowski problems. Minkowski (1897).

Christoffel-Minkowski problems (and area measures)

- Christoffel-Minkowski problem (prescribing curvature problem)

$$\sigma_k(\nabla^2 h + hI) = f.$$

  See Guan-Ma (2003).

- Note that, the engenvalues of $\nabla^2 h + hI$ are principal radii of ∂K.
- Christoffel-Minkowski problem (prescribing area measure problem)

$$S_k(K, \cdot) = \mu,$$

  where $S_k(K, \cdot)$ means Aleksandrov's k-area measures.

Aleksandrov's variation formula (1938)

- Let K be a convex body and $g \in C(S^{n-1})$. Define K_t to be the Aleksandrov body (Wulff shape) as

$$K_t = \{x \in \mathbb{R}^n : x \cdot v \leq h_K(v) + tg(v), \forall v \in S^{n-1}\}, \quad t \in (-\delta, \delta).$$

  Then

$$\frac{d}{dt}\Big|_{t=0} V(K_t) = \int_{S^{n-1}} g(v) dS(K, v).$$

- One doesn't require any regularity assumptions on K. This means that we don't know the exact number of h_{K_t} even if t is sufficiently small.

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- The Minkowski problem is reduced to finding a critical point of the functional

$$\Phi(K) = \log\left(\frac{1}{|\mu|}\int_{S^{n-1}} h_K d\mu\right) - \frac{1}{n}\log V(K).$$

Summary 1

- One can define surface area measure for each convex body. (Complex counterpart? Variational proof of the Calabi conjecture, Berman-Boucksom-Gued (2013).)
- One has the variation formula of the volume with respect to perturbations in the class of convex bodies (without regularity assumption).
- These allow one to solve the Minkowski problem in a direct way (doing calculus...)
- What can we do along with this line?
- Not everything can be differentiated in this way.

Variation formulas for the area measures?

- One-side derivative of the 'surface area' gives the $(n-2)$-area measure

$$\frac{d}{dt}\Big|_{t=0^+} S(K+tL) = \int_{S^{n-1}} h_L(v)\,dS_{n-2}(K,v).$$

- One-side derivative of the 'quermassintegral' gives the k-area measure

$$\frac{d}{dt}\Big|_{t=0^+} W_{n-k-1}(K+tL) = \int_{S^{n-1}} h_L(v)\,dS_k(K,v).$$

- However, it is easy to construct a convex body K and continuous function g (even if $g=1$), such that

$$\frac{d}{dt}\Big|_{t\to 0^+} S(K_t) \neq \frac{d}{dt}\Big|_{t\to 0^-} S(K_t).$$

Dual Minkowski Problem

- The dual quermassintegral (Lutwak 1970s-80s).
- Variation formula (Huang-Lutwak-Yang-Zhang (2016)).

$$\frac{d}{dt}\Big|_{t=0}\widetilde{W}_{n-q}(K_t) = q\int_{S^{n-1}} g(v)\,d\widetilde{C}_q(K,v),$$

  where $\widetilde{C}_q(K,\cdot)$ means the dual curvature measure, and K_t is the log-Wulff shape.
- Minkowski problems for $\widetilde{C}_q$ is called the dual Minkowski problem.
- $\widetilde{C}_n(K,\cdot)$ coincides with the cone-volume measure V_K, and $\widetilde{C}_0(K^*,\cdot)$ coincides with Aleksandrov's integral curvature.
- This is a essentially different variation formula after Aleksandrov.

Recent development regarding the Minkowski type problems and the variational formulas

Andrews, Bianchi, Böröczky, Brendle, Bryan, Chen, Choi, Chow, Cianchi, Cordero-Erausquin, Colesanti, Daskalopoulos, Dou, Feng, Fimiani, Fodor, Fragalá, Gardner, Gluck, Gong, Goodey, Grinberg, Guan, Guang, Haberl, He, Hegedűs, Henk, Hineman, Hong, Hu, Huang, Hug, Ivaki, Jerison, Jian, Jiang, Klain, Klartag, Kolesnikov, Kryvonos, Langharst, Leng, Lewis, Li, Lin, Linke, Liu, Livshyts, Long, Lu, Lutwak, Ma, Marsiglietti, Milman, Miu, Ni, Nyström, Oliker, Pollehn, Rotem, Saari, Salani,Saroglou, Scheuer, Schuster, Sheng, Schneider, Semenov, Stancu, Sun, Trinh, Trudinger, Ulivelli, Umanskiy, Vogel, Wang, Weil, Wu, Xia, Xie, Xing, Xiong, Xu, Xiao, Yang, Yaskin, Yaskina,Ye, Zhang, Zhao, Zhou, Zhu, ...

A motivation of our work

- The dual curvature measures are not translation invariant in general.
- For example, an elementary sequence of the L_p dual curvature measure (Lutwak-Yang-Zhang 2018) is defined by

$$\widetilde{C}_{1,q}(K,\omega) = \int_{\boldsymbol{\nu}_K^*(\omega)} |x|^{q-n} d\mathcal{H}^{n-1}(x).$$

- One can construct a family of translation invariant geometric measures as follows,

$$F_q(K,\omega) = \int_K \widetilde{C}_{1,q}(K-z,\omega)dz.$$

- Question: What is the 'primitive' of $F_q(K,\cdot)$?

Integral Geometry

- In mid-1930s, Blaschke established a school of Integral Geometry in Hamburg. See the books of Santaló and Ren.
- Kinematic formula. For convex bodies K and L,

$$\int_{g\in G(n)} \chi(K\cap gL)d\mu(g) = \frac{1}{\omega_n}\sum_{i=0}^{n}\binom{n}{i}W_i(K)W_{n-i}(L),$$

where $G(n)$ is the group of rigid motions in $\mathbb{R}^n$, and μ is the Haar measure.
- Extensions of the Kinematic formula. Chern (1942, 1952), (knwon as Chern-Yien formula). See also the books of Blaschke, Santaló, Ren, Schneider.
- Dual Kinematic formula (of dual volumes and 'chord integrals'). Zhang (1999).

Chord (power) integrals in Integral Geometry

- Chord integral (Blaschke, Santaló, Ren, Zhang). For a convex body K,

$$I_q(K) = \int_{A(n,1)} |K \cap \ell|^q \, d\ell, \qquad \text{real } q \geq 0,$$

where $|K \cap \ell|$ denotes the length of the chord $K \cap \ell$, and $d\ell$ denotes the normalized Haar measure on the affine Grassmannian $A(n, 1)$.

Geometric properties of chord integrals

- Property 1.

$$I_1(K) = V(K), \quad I_0(K) = \frac{\omega_{n-1}}{n\omega_n} S(K).$$

- Property 2.

$$\int_{A(n,i)} \mathrm{vol}_i(K \cap \xi_i)^2 \, d\xi_i = \frac{\omega_i}{i+1} I_{i+1}(K), \quad \int_{A(n,i)} \mathrm{vol}_i(K \cap \xi_i) \, d\xi_i = I_1(K),$$

where $d\xi_i$ denotes the normalized Haar measure on the affine Grassmannian $A(n,i)$.

- Property 3 (Zhang (1999)).

$$I_q(K) = \frac{q}{\omega_n} \int_K \widetilde{V}_{q-1}(K,z) \, dz.$$

- Property 4.

$$I_q(K) = \int_{S^{n-1}} \int_{u^\perp} X_K(z,u)^q dz du.$$

Analytic properties of chord integrals

- Property 5. For $q > 1$,

$$I_q(K) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\mathbf{1}_K(x)\mathbf{1}_K(y)}{|x-y|^{n+1-q}} \, dxdy;$$

- Property 6. For $q < 1$,

$$I_q(K) = \frac{q(q-1)}{n\omega_n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|\mathbf{1}_K(x) - \mathbf{1}_K(y)|}{|x-y|^{n+1-q}} \, dxdy.$$

Denote $s = 1 - q$. $P_s = I_{1-q}$ is the fractional perimeter. Almgren-Lieb (1989); Bourgain-Brezis-Mironescu (2001); Ludwig (2014); Haddad-Ludwig (2023).

Summary 2

- The ‘chord integral’ $I_q(\cdot)$ connects the ‘volume’ $V(\cdot)$ and the ‘surface area’ $S(\cdot)$.
- Can we do calculus on the operator $I_q(\cdot)$?
- Recall that $S(\cdot)$ is **not** differentiable.

Variation formula and chord measure

- Variation formula for chord integrals. (Lutwak-X.-Yang-Zhang)

  Let $g \in C(S^{n-1})$ and K_t be the Wulff shape generated by $h_K + tg$, that is

  $$K_t = \{x \in \mathbb{R}^n : x \cdot v \le h_K(v) + tg(v), \forall v \in S^{n-1}\}.$$

  Then, for $q > 0$, we have

  $$\frac{d}{dt}\Big|_{t=0} I_q(K_t) = \int_{S^{n-1}} g(v)\, dF_q(K, v).$$

- The chord measure (Lutwak-X.-Yang-Zhang) is defined by

  $$\begin{aligned} F_q(K,\omega) =& \frac{2q}{\omega_n} \int_{\boldsymbol{\nu}_K^*(\eta)} \widetilde{V}_{q-1}(K,z)\, d\mathcal{H}^{n-1}(z), \\ =& \int_K \widetilde{C}_{1,q}(K-z,\omega) dz. \end{aligned}$$

Chord measure

- In the definition

$$F_q(K,\omega) = \frac{2q}{\omega_n} \int_{\boldsymbol{\nu}_K^*(\eta)} \widetilde{V}_{q-1}(K,z)\, d\mathcal{H}^{n-1}(z),$$

for $z \in \partial K$, $\widetilde{V}_{q-1}(K,z)$ can be defined by

$$\widetilde{V}_{q-1}(K,z) = \frac{1}{2n} \int_{S^{n-1}} X_K(z,u)^{q-1} du.$$

- When $q \in (0,1)$, $\widetilde{V}_{q-1}(K,\cdot)$ is unbounded in K, and may also be unbounded on ∂K. (Singularity!)

Properties of the chord measures

- F_q is a translation invariant family of geometric measures.

$$F_q(K, \cdot) = F_q(K + x_0, \cdot).$$

- The chord measure $F_q(K, \cdot)$ is of centroid 0, that is

$$\int_{S^{n-1}} v \, dF_q(K, v) = 0.$$

- $F_1(K, \cdot) = S(K, \cdot)$.

Convergence property of the chord measure

- Convergence. Suppose ∂K is C^2. As $q \to 0$, we have, up to a constant,

$$q\widetilde{V}_{q-1}(K,z) = \frac{q}{2n}\int_{S^{n-1}} X_u(K,z)^{q-1}du \longrightarrow \kappa_1 + \cdots + \kappa_{n-1}.$$

- As a result, in smooth case,

$$F_q(K,\cdot) \to S_{n-2}(K,\cdot) \quad \text{as} \quad q \to 0.$$

  This makes it reasonable to define

$$F_0(K,\cdot) = S_{n-2}(K,\cdot).$$

- In summary, not only $I_q(\cdot) \to S(\cdot)$, but also their derivatives.

  (A little bit suprising because $f_k \to f$ does not imply $f_k' \to f'$.)

The Chord Minkowski problem

- Chord Minkowski problems: Given $q \geq 0$ and a Borel measure μ on the sphere S^{n-1}, try to construct the convex body K such that

$$F_q(K, \cdot) = \mu(\cdot).$$

- We completely solved the chord Minkowski problem for all $q > 0$.

- Solution (Lutwat-X.-Yang-Zhang). Let $q > 0$. Then, both the necessary and sufficient conditions of a measure μ to be a chord measure are:

  - μ is not concentrated on a closed hemisphere of S^{n-1};
  - μ is of centroid 0, that is

$$\int_{S^{n-1}} v \, d\mu(v) = 0.$$

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is suitable (and respectively easy to deal with) for the case $q > 1$.
- The difficulty lies in the case $q \in (0, 1)$.
  - Singularity of $\widetilde{V}_{q-1}(K, \cdot)$ on the boundary;
  - We were not able to find a dominate function: By Property 4,

$$\frac{I_q(K_t) - I_q(K)}{t} = \int_{S^{n-1}} \int_{u^\perp} \frac{X_{K_t}(z, u)^q - X_K(z, u)^q}{t} dzdu.$$

Divergence type formula for chord integrals

- Calculate the derivative
$$\frac{d}{dt}\Big|_{t=0} I_q(K+tK).$$
- Since I_q is $(n+q-1)$-homogeneous, we have
$$\frac{d}{dt}\Big|_{t=0} I_q(K+tK) = (n+q-1)I_q(K).$$
- On the other hand, if there exists such a desired measure $F_q(K,\cdot)$, we should have
$$(n+q-1)I_q(K) = \frac{d}{dt}\Big|_{t=0} I_q(K+tK) = \int_{S^{n-1}} h_K dF_q(K,\cdot).$$

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$$(n+q-1) I_q(K) = \frac{d}{dt}\Big|_{t=0} I_q(K + tK) = \int_{S^{n-1}} h_K \, dF_q(K,\cdot).$$
- Divergence type formula (Lutwak-X.-Yang-Zhang)
$$I_q(K) = \frac{2q}{\omega_n (n+q-1)} \int_{\partial K} \widetilde{V}_{q-1}(K,z)(z \cdot \nu_K(z))\, d\mathcal{H}^{n-1}(z).$$
- However, it is still not a direct computation to obtain the divergence type formula, because of the singularity in $\widetilde{V}_{q-1}(K,\cdot)$.

About the proof of the variation formula

- The idea is to use the divergence type formula to construct a sequence of dominated functions $\phi_t(z,u)$.
- The construction guarantees

$$\frac{X_{K_t}(z,u)^q - X_K(z,u)^q}{t} \leq \phi_t(z,u),$$

- and the divergence type formula was used to give

$$\int_{S^{n-1}} \int_{u^\perp} \lim_{t\to 0} \phi_t(z,u) dz du = \lim_{t\to 0} \int_{S^{n-1}} \int_{u^\perp} \phi_t(z,u) dz du.$$

Summary 3

- The differentials of chord integrals $I_q(\cdot)$ give the chord measures $F_q(\cdot, \cdot)$,
- and the chord measures $F_q(K, \cdot)$ connects the surface area measure $S(K, \cdot)$ and the $(n-2)$-area measure $S_{n-2}(K, \cdot)$.
- We can completely solve the chord Minkowski problem of $F_q(\cdot)$ for arbitrary $q > 0$, which can be arbitrarily close to the critical case.

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- We can completely solve the chord Minkowski problem of $F_q(\cdot)$ for arbitrary $q > 0$, which can be arbitrarily close to the critical case.
- This still gives a strategy to study the Christoffel-Minkowski problem.

A family of Monge-Ampère operators converging to σ_{n-2}

- Proposed here, for $q > 0$, is the following (Lutwak-X.-Yang-Zhang)

$$q\widetilde{V}_{q-1}([h], \overline{\nabla} h) \det\left(\nabla^2 h + hI\right) = f, \quad \text{on } S^{n-1}.$$

- Here,

$$[h] = \left\{x \in \mathbb{R}^n : x \cdot v \leq h(v), \quad \forall v \in S^{n-1}\right\}$$

  is the convex body whose support function is h, and

$$\overline{\nabla} h = \nabla h(v) + h \cdot v$$

  is the boundary point whose outer normal is v. For the convex body $[h]$ and $z = \overline{\nabla} h(v) \in \partial[h]$, the integral operator $\widetilde{V}_{q-1}([h], \overline{\nabla} h)$ is defined as before.
- If h is C^2 and $\det\left(\nabla^2 h + hI\right) > 0$, then, up to a constant

$$q\widetilde{V}_{q-1}([h], \overline{\nabla} h) \det\left(\nabla^2 h + hI\right) \longrightarrow \sigma_{n-2}(\nabla^2 h + hI)$$

- We are able to find both the necessary and sufficient conditions of the existence of its weak solution.
- Question 1: Fix $q > 0$. Under what condition of f, the solution h belongs to $C^{2,\alpha}$?
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- If Question 2 is answered, then by the compactness, we may derive

$$f = q_j \widetilde{V}_{q_j-1}(h, \overline{\nabla} h_{q_j}) \det \left(\nabla^2 h_{q_j} + h_{q_j} I\right) \to \sigma_{n-2}((\nabla^2 h_0 + h_0 I),$$

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- a solution to the $(n-2)$-Christoffel-Minkowski problem.

General operators converging to σ_m

- The section-power integral $W_{m,q}(K)$ of convex body K is defined by

$$W_{m,q}(K) := \int_{A(n,m)} \mathrm{vol}_m(K \cap \xi)^q d\xi,$$

where $d\xi$ denotes the Haar measure on affine Grassmann space $A(n,m)$. By Cauthy-Kubota formula, $W_{m,q}(K) \to W_m(K)$ as $q \to 0^+$.
- Define the section-power measure by

$$F_{m,q}(K,\eta) := q \int_{\nu_K^{-1}(\eta)} \int_{G(n,m)} \mathrm{vol}_m(K_z \cap E)^{q-1} dE d\mathcal{H}^{n-1}(z), \quad \text{Borel } \eta \subset S^{n-1}.$$

- If ∂K is C^2 and has positive curvatures, then, up to a constant,

$$F_{m,q}(K,\cdot) \longrightarrow S_{n-1-m}(K,\cdot)$$

weakly as $q \to 0^+$. (Note that $f_k \to f$ does not imply $f_k' \to f$). In fact,

$$q \int_{G(n,m)} \mathrm{vol}_m(K_z \cap E)^{q-1} dE \longrightarrow C_m(z).$$

The case $m = 2$ (scalar curvature case)

- Take $m = 2$ for example. We can compute that

$$q \int_{G(n,m)} \mathrm{vol}_m(K_z \cap E)^{q-1} dE \to \frac{\pi}{4(n-1)} C_2(z).$$

- Denote e_n to be the normal of ∂K at z. Using a integral geometry formula

$$\begin{aligned} & q \int_{G(n,2)} \mathrm{vol}_2(K_z \cap E)^{q-1} dE \\ = & \frac{q(n-2)}{2|S^{n-2}||S^{n-3}|} \int_{e_n^\perp \cap S^{n-1}} \int_{v_1^\perp \cap S^{n-1}} \mathrm{vol}_2(K_z \cap E)^{q-1} |w \cdot e_n| dw dv_1. \end{aligned}$$

By an estimate of $\mathrm{vol}_2(K_z \cap E)$ and the other integral geometry formula,

$$q \int_{G(n,2)} \mathrm{vol}_2(K_z \cap E)^{q-1} dE \to \int_{G(e_n^\perp,2)} C(E_2) dE_2,$$

where $C(E_2)$ denote the sectional curvature and E_2 is a 2-dim subspace of $e_n^\perp$.

Thank you!