Random approximation of convex bodies in Hausdorff distance

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joint with

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Random approximation of a convex body

$K \subset \mathbb{R}^d \dots$ convex body (compact with non-empty interior).

Approximate K with a random polytope

$$K_n = [X_1, \dots, X_n], \quad n \in \mathbb{N},$$

where $X_1, X_2, \dots$ are i.i.d. uniformly chosen in K or on the boundary ∂K w.r.t. to a probability measure $\mathbb{P}$.

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How good is this approximation?

There are several ways of quantifying this question ...

The quality of the approximation

For example, we measure the error of approximation with, e.g.,

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or the Hausdorff distance

$$\begin{aligned} \delta_H(K, K_n) &= \max\{\sup_{x \in K} \mathrm{dist}(x, K_n), \sup_{y \in K_n} \mathrm{dist}(y, K)\} \\ &= \sup_{x \in \partial K} \mathrm{dist}(x, K_n) = \sup_{x \in \partial K} \inf_{y \in K_n} \|x - y\| \end{aligned}$$

$$\mathbb{E}\delta_{\Delta}(K, K_n) = \int_{\partial K} \cdots \int_{\partial K} \delta_{\Delta}(K, K_n)\, d\mathbb{P}(X_1) \cdots d\mathbb{P}(X_n)$$

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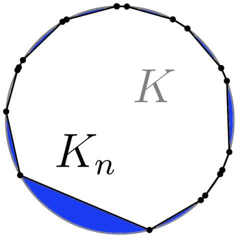

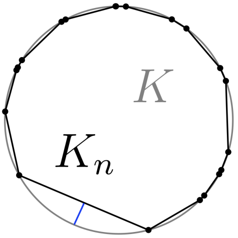

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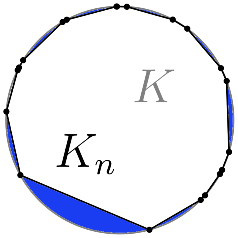

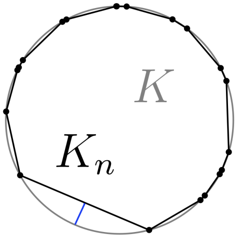

$$\lim_{n\to\infty} \mathbb{E}\delta_{\Delta}(K, K_n) = 0 \quad \text{and} \quad \lim_{n\to\infty} \mathbb{E}\delta_H(K, K_n) = 0.$$

What is the speed of convergence?

Two regimes: Smooth bodies and polytopes

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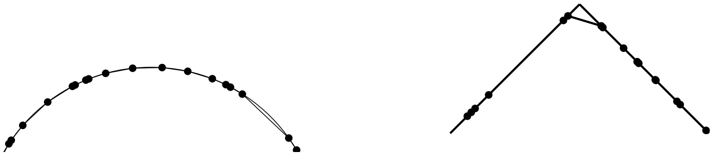
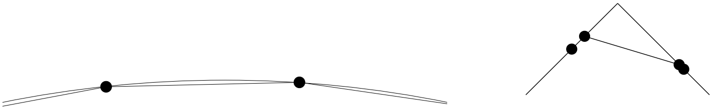
Locally, approximating smooth bodies and polytopes looks like. . .



. . . and at scale $n^{-1/(d-1)}$. . .



Symmetric difference metric δ_Δ

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Theorem (Reitzner; Schütt&W)

If K satisfies mild smoothness assumptions, then

$$\lim_{n\to\infty} n^{2/(d-1)}\mathbb{E}\delta_\Delta(K,K_n) = c_d\, as(K)^{\frac{d+1}{d-1}}$$

- c_d is a constant
- $as(K) = \int_{\partial K} \kappa(x)^{\frac{1}{d+1}}\ d\mu_K(x)$ is the affine surface area of K

Theorem (Reitzner&Schütt&W)

If K is a simple polytope (every vertex meets d facets), then

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Theorem (Bárány&Buchta)

If K is a polytope and K_n^{in} the convex hull of n points sampled uniformly and i.i.d. inside K, then

$$\lim_{n\to\infty} \frac{n}{(\log n)^{d-1}}\,\mathbb{E}\delta_\Delta(K, K_n^{\mathrm{in}}) = \frac{\#\ \mathrm{flags}(K)}{(d+1)^{d+1}(d-1)!}$$

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- points chosen on ∂K: $\mathbb{E}\delta_\Delta(K,K_n) \sim n^{-d/(d-1)}$
- points chosen in K: $\mathbb{E}\delta_\Delta(K,K_n^{\mathrm{in}}) \sim \frac{(\log n)^{d-1}}{n}$

Symmetric difference metric δ_Δ; points chosen on ∂K

- smooth case: $\mathbb{E}\delta_\Delta(K, K_n) \sim n^{-2/(d-1)}$
- polytope: $\mathbb{E}\delta_\Delta(K, K_n) \sim n^{-d/(d-1)}$

Hausdorff distance δ_H

In contrast to δ_Δ, no (?!) result of type

$$\lim_{n\to\infty} a_n \, \mathbb{E}\delta_H(K, K_n) = c(K) \in (0, \infty)$$

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- upper bounds for convergence almost surely by Dümbgen&Walter
- results by Brunel

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- We will again consider the 2 regimes: smooth bodies and polytopes

Hausdorff distance - smooth bodies

Theorem (Glasauer&Schneider)

Let K be a C^3_+ convex body in $\mathbb{R}^d$ and $\mathbb{P} = h\,\mu_K$ a probability measure with density h. Then

$$\lim_{n\to\infty}\left(\frac{n}{\log n}\right)^{2/(d-1)}\delta_H(K,K_n) = \frac{1}{2}\left(\frac{1}{|B_2^{d-1}|}\max_{x\in\partial K}\frac{\kappa(x)^{\frac{1}{2}}}{h(x)}\right)^{\frac{2}{d-1}} \text{ in probability.}$$

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From this, one gets

Under the same assumption as in the Theorem,

$$\lim_{n\to\infty}\left(\frac{n}{\log n}\right)^{2/(d-1)}\mathbb{E}\delta_H(K,K_n) = \frac{1}{2}\left(\frac{1}{|B_2^{d-1}|}\max_{x\in\partial K}\frac{\kappa(x)^{\frac{1}{2}}}{h(x)}\right)^{\frac{2}{d-1}}$$

For polytopes, everything happens at the vertices, i.e.,...

If K is a polytope, then

$$\delta_H(K, K_n) = \sup_{x \in \partial K} \mathrm{dist}(x, K_n) = \max_{i=1,\dots,M} \mathrm{dist}(v_i, K_n),$$

where $v_1, \dots, v_M$ are the vertices of K.

A small detour: Random points inside a polygon

Approximate K with

$$K_n^{\mathrm{in}} := [Y_1, \ldots, Y_n] \quad n \in \mathbb{N},$$

where $Y_1, Y_2, \ldots$ are i.i.d. uniform on K (instead of ∂K).

Theorem (Bräker&Hsing&Bingham)

If $K \subset \mathbb{R}^2$ is a polygon with M vertices, then

$$\lim_{n\to\infty} \mathbb{P}[n^{1/2}\delta_H(K, K_n^{\mathrm{in}}) \leq x] = \prod_{i=1}^{M} \big(1 - p_i(x)\big), \quad x > 0,$$

where p_i depends only on the i-th interior angle of K.

The only result we found on $\mathbb{E}\delta_H(K, K_n)$ is an inside result

Theorem (Bárány)

If K is a polytope, then, for n large enough,

$$c_1(K) \leq n^{1/d}\, \mathbb{E}\delta_H(K, K_n^{\mathrm{in}}) \leq c_2(K).$$

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- smooth case: $\mathbb{E}\delta_H(K, K_n) \sim \left(\frac{\log n}{n}\right)^{2/(d-1)}$
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If K is a polytope, then, for n large enough,

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What is the exact asymptotic constant?

Theorem (Prochno&Schütt&Sonnleitner&W '23)

If $K \subset \mathbb{R}^2$ is a polygon with M vertices, then

$$\lim_{n\to\infty} n\,\mathbb{E}\delta_H(K, K_n) = |\partial K| \int_0^\infty 1 - \prod_{j=1}^{M} \Big(1 - x \int_{\ell_j}^\infty e^{-x(y+h(y,\alpha_j))}\mathrm{d}y\Big)\mathrm{d}x,$$

where α_j is the interior angle at the vertex v_j, $1 \le j \le M$.

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The influence of the interior angles

$$\ell_j = \begin{cases} 1 & \text{if } \alpha_j < \pi/2, \\ \sin(\alpha_j)^{-1} & \text{if } \alpha_j \ge \pi/2. \end{cases}$$

- $\alpha < \pi/2$ and $y \in [1, \cos(\alpha)^{-1}]$

or

- $\alpha \geq \pi/2$ and $y > \sin(\alpha)^{-1}$

$$h(y,\alpha) = y\frac{\sin(\alpha)\sqrt{y^2-1} - \cos(\alpha)}{\sin^2(\alpha)y^2 - 1}$$

- $h(y,\alpha) = 1, \quad$ else

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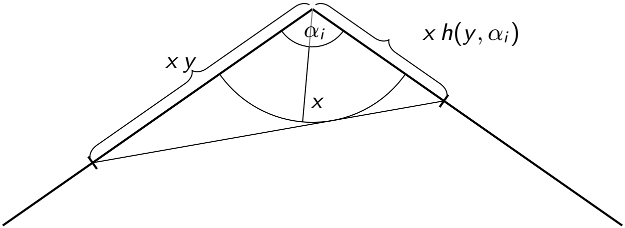

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Corollary 1 (Prochno&Schütt&Sonnleitner&W '23)

If $K \subset \mathbb{R}^2$ is a polygon with M vertices such that $\alpha_i \geq \frac{\pi}{2}$ for all i, then

$$\frac{1}{5} c(K) \leq \lim_{n\to\infty} n\,\mathbb{E}[\delta_H(K,K_n)] \leq c(K)$$

where

$$c(K) := |\partial K| \sum_{k=1}^{M} (-1)^{k+1} \sum_{1\leq i_1<\cdots<i_k\leq M} \frac{1}{\frac{1}{\sin\alpha_{i_1}} + \cdots + \frac{1}{\sin\alpha_{i_k}}}.$$

Proof of Corollary 1

By the theorem

$$\lim_{n\to\infty} n\,\mathbb{E}\delta_H(K,K_n) = |\partial K| \int_0^\infty 1 - \prod_{i=1}^{M} \left(1 - x\int_{\ell_i}^\infty e^{-x(y+h(y,\alpha_i))}\mathrm{d}y\right)\mathrm{d}x$$

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$$h(y,\alpha_i) = y\,\frac{\sin\alpha_i\sqrt{y^2-1} - \cos\alpha_i}{\sin^2\alpha_i\,y^2 - 1} \geq \frac{1}{\sin\alpha_i}\,\frac{\sin\alpha_i\sqrt{y^2-1} - \cos\alpha_i}{\sin^2\alpha_i\,y^2 - 1} \geq 0$$

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$$x\int_{\ell_i}^\infty e^{-x(y+h(y,\alpha_i))}\mathrm{d}y \leq x\int_{\ell_i}^\infty e^{-xy}\mathrm{d}y = e^{-x\ell_i} = e^{-\frac{x}{\sin\alpha_i}}$$

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Corollary 2 (Prochno&Schütt&Sonnleitner&W '23)

If $K^{\mathrm{reg}} \subset \mathbb{R}^2$ is a regular M-gon, then

$$\lim_{n\to\infty} n\,\mathbb{E}\delta_H(K^{\mathrm{reg}},K_n) \asymp |\partial K^{\mathrm{reg}}| \frac{\log M}{M},$$

where $\asymp$ indicates equality up to absolute constants.

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Proof of Corollary 2

If $K^{\mathrm{reg}} \subset \mathbb{R}^2$ is a regular M-gon, then $\alpha_i = \alpha$ for all i:

$$\sum_{k=1}^{M}(-1)^{k+1} \sum_{1\le i_1<\dots i_k\le M} \frac{1}{\frac{1}{\sin\alpha_{i_1}}+\dots+\frac{1}{\sin\alpha_{i_k}}} = \sum_{k=1}^{M}(-1)^{k+1}\frac{\sin\alpha}{k}\binom{M}{k}$$

$$\lim_{n\to\infty} n\,\mathbb{E}\delta_H(K,K_n) \asymp |\partial K| \sum_{k=1}^{M}(-1)^{k+1} \sum_{1\le i_1<\dots i_k\le M} \frac{1}{\frac{1}{\sin\alpha_{i_1}}+\dots+\frac{1}{\sin\alpha_{i_k}}}$$

Corollary 2 (Prochno&Schütt&Sonnleitner&W '23)

If $K^{\mathrm{reg}} \subset \mathbb{R}^2$ is a regular M-gon, then

$$\lim_{n\to\infty} n\,\mathbb{E}\delta_H(K^{\mathrm{reg}},K_n) \asymp |\partial K^{\mathrm{reg}}| \frac{\log M}{M},$$

where $\asymp$ indicates equality up to absolute constants.

Proof of Corollary 2

If $K^{\mathrm{reg}} \subset \mathbb{R}^2$ is a regular M-gon, then $\alpha_i = \alpha$ for all i:

$$\sum_{k=1}^{M}(-1)^{k+1} \sum_{1\le i_1<\dots i_k\le M} \frac{1}{\frac{1}{\sin\alpha_{i_1}}+\dots+\frac{1}{\sin\alpha_{i_k}}} = \sum_{k=1}^{M}(-1)^{k+1}\frac{\sin\alpha}{k}\binom{M}{k}$$

$$\sin\alpha = \sin((\pi(1-\frac{2}{M})) \asymp \frac{2\pi}{M}$$

$$\sum_{k=1}^{M}(-1)^{k+1}\sum_{1\le i_1<\dots i_k\le M}\frac{1}{\frac{1}{\sin\alpha_{i_1}}+\cdots+\frac{1}{\sin\alpha_{i_k}}} \asymp \frac{2\pi}{M}\sum_{k=1}^{M}(-1)^{k+1}\frac{1}{k}\binom{M}{k}$$

$$
\begin{aligned}
\sum_{k=1}^{M}(-1)^{k+1}\sum_{1\leq i_1<\ldots i_k\leq M}\frac{1}{\frac{1}{\sin\alpha_{i_1}}+\cdots+\frac{1}{\sin\alpha_{i_k}}} &\asymp \frac{2\pi}{M}\sum_{k=1}^{M}(-1)^{k+1}\frac{1}{k}\binom{M}{k} \\
&\asymp \frac{2\pi}{M}\sum_{k=1}^{M}\frac{1}{k}
\end{aligned}
$$

$$\sum_{k=1}^{M}(-1)^{k+1}\sum_{1\le i_1<\dots i_k\le M}\frac{1}{\frac{1}{\sin\alpha_{i_1}}+\cdots+\frac{1}{\sin\alpha_{i_k}}} \asymp \frac{2\pi}{M}\sum_{k=1}^{M}(-1)^{k+1}\frac{1}{k}\binom{M}{k}$$

$$\asymp \frac{2\pi}{M}\sum_{k=1}^{M}\frac{1}{k}$$

$$\lim_{n\to\infty} n\,\mathbb{E}\delta_H(K^{\mathrm{reg}},K_n) \asymp |\partial K^{\mathrm{reg}}|\sum_{k=1}^{M}(-1)^{k+1}\sum_{1\le i_1<\dots i_k\le M}\frac{1}{\frac{1}{\sin\alpha_{i_1}}+\cdots+\frac{1}{\sin\alpha_{i_k}}}$$

$$\sum_{k=1}^{M}(-1)^{k+1}\sum_{1\le i_1<\dots i_k\le M}\frac{1}{\frac{1}{\sin\alpha_{i_1}}+\cdots+\frac{1}{\sin\alpha_{i_k}}} \asymp \frac{2\pi}{M}\sum_{k=1}^{M}(-1)^{k+1}\frac{1}{k}\binom{M}{k}$$

$$\asymp \frac{2\pi}{M}\sum_{k=1}^{M}\frac{1}{k}$$

$$\lim_{n\to\infty} n\,\mathbb{E}\delta_H(K^{\mathrm{reg}},K_n) \asymp |\partial K^{\mathrm{reg}}|\sum_{k=1}^{M}(-1)^{k+1}\sum_{1\le i_1<\dots i_k\le M}\frac{1}{\frac{1}{\sin\alpha_{i_1}}+\cdots+\frac{1}{\sin\alpha_{i_k}}}$$

$$\asymp |\partial K^{\mathrm{reg}}|\,\frac{2\pi}{M}\sum_{k=1}^{M}\frac{1}{k} \asymp |\partial K^{\mathrm{reg}}|\,\frac{\log M}{M}$$

Ingredients of the proof of the Theorem

- $n\,\mathbb{E}|\delta_H(K,K_n)| = \int_0^\infty \mathbb{P}\left(\delta_H(K,K_n) \geq \frac{r}{n}\right) dr$

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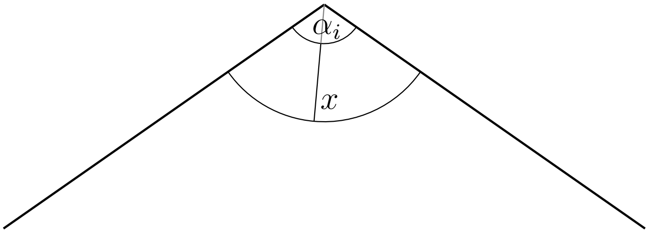
- Show that **uniformly**
$\lim_{n\to\infty} \mathbb{P}\left(\delta_H(K,K_n) \geq \frac{r}{n}\right) = 1 - \prod_{i=1}^{M}\left(1 - r\int_{\ell_i}^{\infty} e^{-r(y+h(y,\alpha_i))}\mathrm{d}y\right)$

- Write $\delta_H(K, K_n) = \max_i \operatorname{dist}(v_i, K_n)$
- Look at a single event and compute $\mathbb{P}[\cdot\operatorname{dist}(v_i, K_n) \geq \frac{r}{n} = x]$



- Write $\delta_H(K, K_n) = \max_i \operatorname{dist}(v_i, K_n)$
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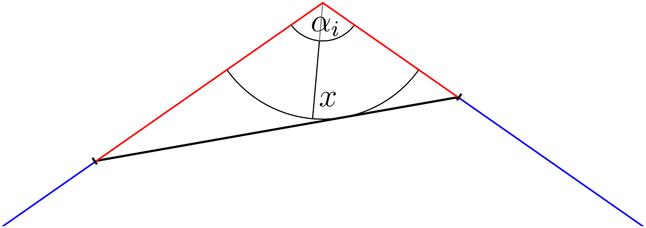
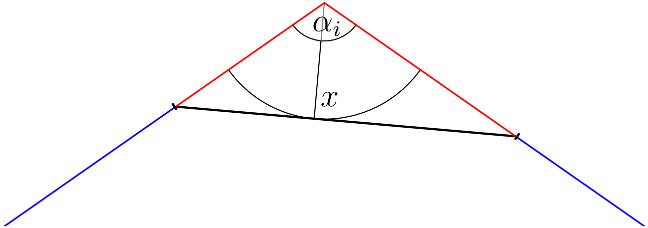


- Write $\delta_H(K, K_n) = \max_i \mathrm{dist}(v_i, K_n)$
- Look at a single event and compute $\mathbb{P}[\cdot \mathrm{dist}(v_i, K_n) \geq \frac{r}{n} = x]$



    - Parametrize tangent lines by intersection with an edge
    - Discretize the edge and compute the probabilities
- Show asymptotic independence of vertices

A crucial lemma: asymptotic independence

A crucial lemma: asymptotic independence

Lemma

If $v_1, \dots, v_M$ are the vertices of K, then

$$\lim_{n\to\infty} \sup_{r\in(0,R]} \left| \mathbb{P}\Big[\bigcap_{\ell=1}^{k} \{\mathrm{dist}(v_{i_\ell}, K_n) \geq r/n\} \Big] - \prod_{\ell=1}^{k} \mathbb{P}[\mathrm{dist}(v_{i_\ell}, K_n) \geq r/n] \right| = 0,$$

where $1 \leq k \leq M$ and the indices $1 \leq i_1 < \cdots < i_k \leq M$ are arbitrary.