GEOMETRIC FUNCTIONALS OF POLYCONVEX EXCURSION SETS OF POISSON SHOT NOISE PROCESSES

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Poisson shot noise process

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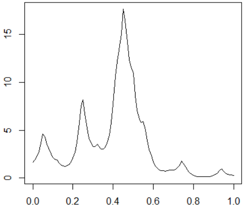

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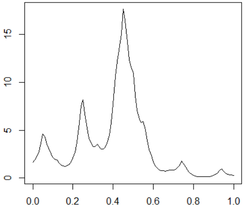

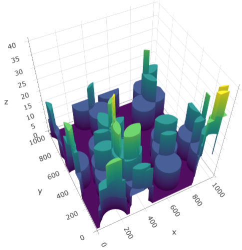

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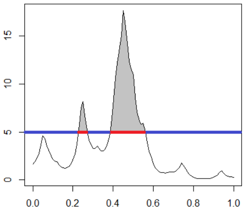

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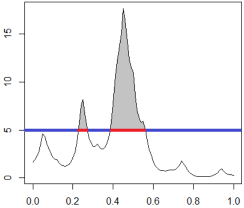

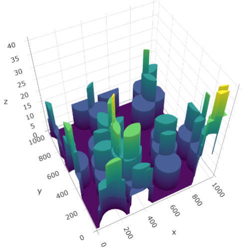

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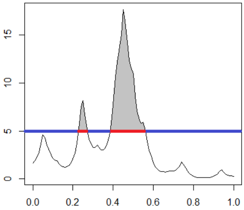

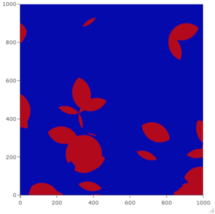

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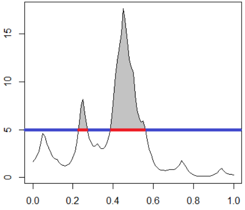

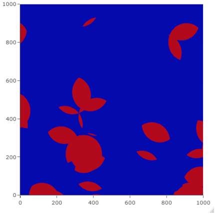

- Consider a geometric functional $\varphi(Z_u \cap W)$ of the excursion set on a compact convex observation window W with $\lambda_d(W) > 0$.

Model with polyconvex excursion sets

- We assume that $g_m : \mathbb{R}^d \to \mathbb{R}_{\geq 0}$ is concave on its compact convex support K_m with $\lambda_d(K_m) > 0$ for $m \in \mathbb{M}$ and $m \mapsto K_m$ is measurable.

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- For $k \in \mathbb{N}$ we denote by $(\mathbf{M}_k)$ the moment condition

$$\int_{\mathbb{M}} V_i(K_m)^k \, \mathbb{Q}(\mathrm{d}m) < \infty, \quad i \in \{1, \dots, d\}.$$

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- Then, for $S_W := \{(x, m) \in \mathbb{R}^d \times \mathbb{M} : K_m + x \cap W \neq \emptyset\}$, $\eta(S_W) < \infty$ a.s.

Model with polyconvex excursion sets

- Let $(x_1, m_1), \dots, (x_n, m_n)$ be the points of the Poisson process in S_W. For $\hat{K}_i = K_{m_i} + x_i$, $i \in \{1, \dots, n\}$,

$$Z_u \cap W = \bigcup_{I \subseteq \{1,\dots,n\}} \Big\{ y \in \bigcap_{j \in I} \hat{K}_j \cap W : \sum_{j \in I} g_{m_j}(y - x_j) \geq u \Big\}.$$

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- The excursion sets are almost surely polyconvex.
- Boolean model arises as special case for $g_m(x) = u$ for $x \in K_m$ and $m \in \mathbb{M}$.

- We consider measurable functions $\varphi : \mathcal{R}^d \mapsto \mathbb{R}$, which are

Geometric functionals

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  - additive, i.e. $\varphi(\emptyset) = 0$ and

$$\varphi(A_1 \cup A_2) = \varphi(A_1) + \varphi(A_2) - \varphi(A_1 \cap A_2)$$

for $A_1, A_2 \in \mathcal{R}^d$.

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  - locally bounded, i.e. $|\varphi(A)| \leq M_\varphi$ for all $\mathcal{K}^d \ni A \subseteq [0,1]^d$.
- Most prominent examples are the intrinsic volumes.

Theorem (T. 2023+)

Let $W_r = rW$ for $r \geq 1$. Under the assumption $(\mathbf{M}_1)$, we have

$$\lim_{r\to\infty} \frac{\mathbb{E}[\varphi(Z_u \cap W_r)]}{V_d(W_r)} = \mathbb{E}[\varphi(Z_u \cap C_0^d)],$$

where $C_0^d := [0,1)^d$ denotes the half-open d-dimensional unit cube.

Limiting behaviour of the expectation

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where $C_0^d := [0,1)^d$ *denotes the half-open* d*-dimensional unit cube.*

- A similar result was shown for $\mathbb{E}[\varphi(Z \cap W_r)]$, where Z is a so called standard random set in Schneider/Weil (2008).

- For a polyconvex set A let $N(A)$ denote the smallest possible number m with $A = K_1 \cup \cdots \cup K_m$ for convex sets K_i, $i = 1, \ldots, m$.

Standard random sets

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- A standard random Z set satisfies $\mathbb{E}[2^{N(Z \cap [0,1]^d)}] < \infty$.
- $Z_u \cap W$ can be written as the number of 2^n convex sets, where $n = \eta(S_W)$, i.e. $\mathbb{E}[2^{N(Z_u \cap [0,1]^d)}]$ is not necessarily finite and therefore not a standard random set.

Example

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- Consider the example, where $g_m(x) = \frac{u}{2}$ for $x \in K_m$.

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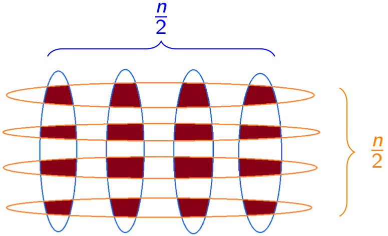
- In a situation as in the following picture



$N(Z_u \cap [0,1]^d) = \frac{n^2}{4}$ and if the probability for such configurations is large enough, $\mathbb{E}[2^{N(Z_u \cap [0,1]^d)}] = \infty$.

Proof idea

- For standard sets we use for $W \subset [0,1]^d$ that $|\varphi(A)| \leq \bar{M}_\varphi$ for $A \subset W$ and hence,

$$\mathbb{E}[|\varphi(Z \cap W)|] \leq \bar{M}_\varphi \mathbb{E}[2^{N(Z \cap W)}] < \infty.$$

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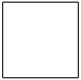
- For $Z_u \cap W$ we use a dynamic decomposition of W into cubes $Q_{n,z}$, which stops when $N(Z_u \cap Q_{n,z}) \leq L$ for some constant $L \in \mathbb{N}$.



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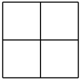

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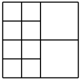

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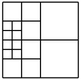

Theorem (T. 2023+)

Under $(\mathbf{M}_2)$ *there exists a constant* $\sigma_0 \geq 0$ *such that*

$$\lim_{r\to\infty} \frac{\mathrm{Var}[\varphi(Z_u \cap W_r)]}{V_d(W_r)} = \sigma_0.$$

Let N *denote a standard Gaussian random variable. If* $\sigma_0 > 0$ *and assumption* $(\mathbf{M}_4)$ *is fulfilled, there exists for* $\diamondsuit \in \{Kol, Was\}$ *a constant* $C > 0$ *satisfying*

$$\mathrm{d}_\diamondsuit\left(\frac{\varphi(Z_u \cap W_r) - \mathbb{E}[\varphi(Z_u \cap W_r)]}{\mathrm{Var}[\varphi(Z_u \cap W_r)]}, N\right) \leq \frac{C}{\sqrt{V_d(W_r)}}$$

for r *large enough.*

- Similar results for the Boolean model in Hug/Last/Schulte (2016) or Last/Penrose (2018).

Previous work

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- Similar results for the Boolean model in Hug/Last/Schulte (2016) or Last/Penrose (2018).
- For specific geometric functionals as the volume of excursion sets of Poisson shot noise processes, similar results were shown in Bulinski/Spodarev/Timmermann (2012), Lachièze-Rey (2019), Lachièze-Rey/Peccati/Yang (2022), Schulte/T. (2022).

Thank you for your attention!