Conference on Convex Geometry and Geometric Probability

Floating Bodies and Polarity in Spaces of Constant Curvature

jointly with Elisabeth Werner

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Blaschke's affine surface area

- $K \subset \mathbb{R}^n$ a convex body of class C^2_+ that contains the origin in the interior

Blaschke's affine surface area:

$$\mathrm{as}_1(K) = \int_{\partial K} H_{n-1}(K, \mathbf{x})^{\frac{1}{n+1}} \, \mathcal{H}^{n-1}(\mathrm{d}\mathbf{x}) = \int_{\partial K} \kappa_o(K, \mathbf{x})^{\frac{1}{n+1}} \, C_K(\mathrm{d}\mathbf{x})$$

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- $\mathbf{n}_K : \partial K \to \mathbb{S}^{n-1}$ Gauss map; $H_{n-1}(K, \cdot)$ Gauss–Kronecker curvature.
- $\kappa_o(K, \mathbf{x})$ is the centro-affine invariant curvature that is related to the volume of a centered ellipsoid $\mathcal{E}_o(K, \mathbf{x})$ that osculates the boundary ∂K at $\mathbf{x}$, i.e.,

$$\kappa_o(K, \mathbf{x}) = \left(\frac{\mathrm{Vol}(\mathcal{E}_o(K, \mathbf{x}))}{\mathrm{Vol}(B_2^n)} \right)^{-2} = \frac{H_{n-1}(K, \mathbf{x})}{\langle \mathbf{n}_K(\mathbf{x}), \mathbf{x} \rangle^{n+1}}.$$

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- C_K is the centro-affine invariant cone-volume measure on ∂K that is absolutely continuous with respect to the Hausdorff measure $\mathcal{H}^{n-1}$ with density

$$\frac{\mathrm{d}C_K}{\mathrm{d}\mathcal{H}^{n-1}}(\mathbf{x}) = \langle \mathbf{n}_K(\mathbf{x}), \mathbf{x} \rangle.$$

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- Extension of as_1 to convex bodies K without curvature conditions: Petty (1985), **Leichtweiß** (1988, 1989), **Schütt & Werner** (1990), **Lutwak** (1986, 1991), Schütt (1993), Dolzmann & Hug (1995), Hug (1996).

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- Functional analogs $\mathrm{as}_1^{(s)}(f)$ for s-concave functions f of class C^2 were defined by **Artstein-Avidan, Klartag, Schütt, Werner** (2012) and $\mathrm{as}_\lambda(\varphi)$ for $\log$-concave functions φ by **Caglar & Werner** (2014, 2015) and **Caglar, Fradelizi, Guédon, Lehec, Schütt, Werner** (2016).

General affine surface areas

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- The family of $\mathrm{SL}(n)$-invariant surface area measures as_p were introduced by **Lutwak** (1996), for $p \geq 1$, and subsequently extended and studied by: **Hug** (1996, $p \geq 0$), **Meyer & Werner** (2000, $p < 0$), **Schütt & Werner** (2002), **Werner & Ye** (2008), **Ludwig** (2010), **Ludwig & Reitzner** (2010), **Haberl & Parapatits** (2014), ...

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- as_p is a $\mathrm{SL}(n)$-invariant valuation on all convex bodies that is positively homogeneous of degree $q = n\frac{n-p}{n+p}$. For $p = n$, as_n is $\mathrm{GL}(n)$-invariant and often called the centro-affine surface.

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- For $p > 0$, $\mathrm{as}_p \in [0, +\infty)$ is upper semi-continuous and $q \in (-n, n)$ and for $p < 0$, $p \neq -n$, $\mathrm{as}_p \in (0, +\infty]$ is lower semi-continuous and $|q| > n$.

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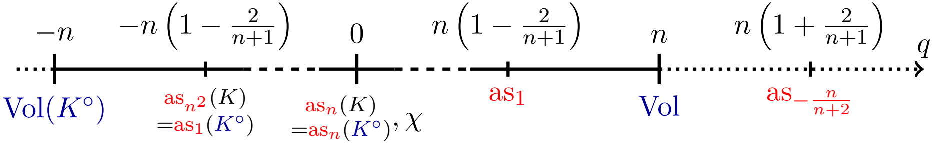
Theorem (Ludwig & Reitzner, 2010). (Ann. of Math.)

Φ is a upper semi-continuous and $\mathrm{SL}(n)$-invariant valuation of homogeneous degree $q \in [-n, n]$ if and only if

$$\begin{cases} \text{for } q = 0: & \Phi = C_0 + C_1\,\mathrm{as}_n \\ \text{for } |q| < n: & \Phi = C_1\,\mathrm{as}_p \end{cases} \quad \Bigg| \quad \begin{array}{ll} \text{for } q = n: & \Phi = C_0 \mathrm{Vol} \\ \text{for } q = -n: & \Phi(K) = C_0 \mathrm{Vol}(K^\circ) \end{array}$$

for some $C_0, C_1 \in \mathbb{R}$, $C_1 > 0$ and $p > 0$ s.t. $q = n\frac{n-p}{n+p}$.



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- Further generalizations to Orlicz affine surface areas were introduced and characterized as natural $\mathrm{SL}(n)$-invariant semi-continuous valuations by **Ludwig** (Adv. Math., 2010) and **Ludwig & Reitzner** (Ann. of Math., 2010). A Hadwiger type characterization of upper semi-continuous centro-affine invariant valuations was established **Haberl & Parapatits** (JAMS, 2014).

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- The affine surface area and its relatives also appear naturally in the asymptotic optimal and random polytopal approximation of convex bodies (volume, number of vertices, edges, ...): **Bárány, Böröczky, Buchta, Calka, Chatterjee, Fodor, Gusakova, Hoehner, Hug, Kabluchko, Lachièze-Rey, Larman, Ludwig, Peccati, Reitzner, Rosen, Schneider, Schulte, Schütt, Thäle, Vu, Werner, Yukich**, ... and many more!

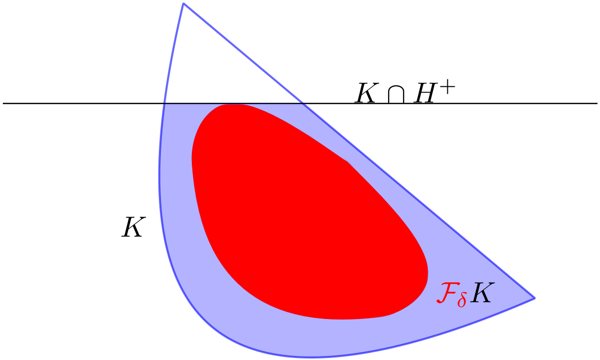
The Floating Body

Definition. For $\delta > 0$ the floating body of K is defined by

(Bárány & Larman, 1988)
(Schütt & Werner, 1990)

$$\mathcal{F}_\delta K = \bigcap \left\{ K \cap H^- : \mathrm{Vol}(K \cap H^+) \leq \delta \right\}.$$

- $H^\pm$ ... closed half-spaces
- Vol ... Lebesgue measure



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- $H^\pm$... closed half-spaces
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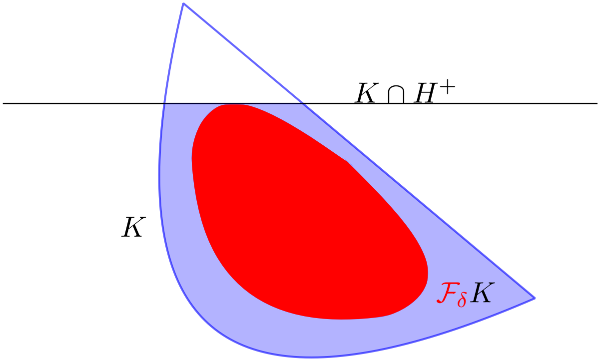
- $\lim_{\delta \to 0^+} \mathcal{F}_\delta K = \mathcal{F}_0 K = K$
- $\mathcal{F}_\delta$ is equi-affine covariant, i.e., $\mathcal{F}_\delta(AK + \mathbf{x}) = A(\mathcal{F}_\delta K) + \mathbf{x}$ for $A \in \mathrm{SL}(n)$ and $\mathbf{x} \in \mathbb{R}^n$



Volume Derivative of the Floating Body

- The floating body construction can be traced back into the 19th century to **Dupin**. **Blaschke** (1920s) used a version of Dupin's floating body to introduce the affine surface area as_1 which was later generalized by **Leichtweiß** (1986).

Theorem (Schütt & Werner, 1990). (Math. Scand.)

$$\lim_{\delta \to 0^+} \frac{\mathrm{Vol}(K) - \mathrm{Vol}(\mathcal{F}_\delta K)}{\delta^{\frac{2}{n+1}}} = c_n\, \mathrm{as}_1(K) = c_n \int_{\partial K} \kappa_o(K, \mathbf{x})^{\frac{1}{n+1}}\, C_K(\mathrm{d}\mathbf{x}).$$

- $c_n := (2\pi)^{-\frac{n-1}{n+1}}\, \Gamma\big(\frac{n+3}{2}\big)^{\frac{2}{n+1}}$

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- $\mathcal{F}_\delta^\circ K \supset K$ and $\lim_{\delta\to 0^+} \mathcal{F}_\delta^\circ K = K$ and $\mathcal{F}_\delta^\circ(AK) = A(\mathcal{F}_\delta^\circ K)$ for all $A\in \mathrm{SL}(n)$.

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Theorem (Meyer & Werner, 2000; Werner & Ye, 2008). (both Adv. Math.)

$$\lim_{\delta \to 0^+} \frac{\mathrm{Vol}(\mathcal{F}_\delta^\circ K) - \mathrm{Vol}(K)}{\delta^{\frac{2}{n+1}}} = c_n \, \mathrm{as}_{-n/(n+2)}(K) = c_n \int_{\partial K} \kappa_o(K, \mathbf{x})^{-\frac{1}{n+1}} \, C_K(\mathrm{d}\mathbf{x}),$$

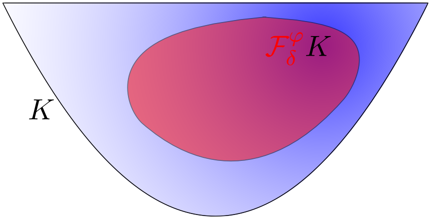
where K is a convex body of class C^2_+ that contains the origin in the interior.

Weighted Floating Bodies

Definition (Werner, 2002). The φ-weighted floating body of a convex body $K \subset \mathbb{R}^n$ is $\mathcal{F}_\delta^\varphi K = \bigcap \{K \cap H^- : \mathrm{Vol}^\varphi(K \cap H^+) \leq \delta\}$.

Lemma. $\mathcal{F}_\delta^\varphi K = \{\mathbf{x} \in K : \mathrm{mcd}_{K,\varphi}(\mathbf{x}) \geq \delta\}$ is the δ-superlevel set of the minimal cap density function $\mathrm{mcd}_{K,\varphi}(\mathbf{x}) = \min\{\mathrm{Vol}^\varphi(K \cap H^+) : \mathbf{x} \in H^+\}$.

- φ a strictly positive continuous function; $\mathrm{Vol}^\varphi(A) = \int_A \varphi(\mathbf{x})\,\lambda(\mathrm{d}\mathbf{x})$.



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Lemma. Let $K \subset \mathbb{R}^n$ be a convex body. Then:

i) If $\psi \leq \varphi$, then $\mathcal{F}_\delta^\psi K \subset \mathcal{F}_\delta^\varphi K$.
If $L \subset K$ is another convex body, then $\mathcal{F}_\delta^\varphi L \subset \mathcal{F}_\delta^\varphi K$.

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ii) If K is of class C^2_+, then $\|h(K,\cdot) - h(\mathcal{F}_\delta^\varphi K, \cdot)\|_\infty \leq C\delta^{\frac{2}{n+1}}$. Moreover,

$$\lim_{\delta \to 0^+} \frac{h(K,\mathbf{u}) - h(\mathcal{F}_\delta^\varphi K, \mathbf{u})}{\delta^{\frac{2}{n+1}}} = c_n \varphi(\mathbf{x})^{-\frac{2}{n+1}} H_{n-1}(K,\mathbf{x})^{\frac{1}{n+1}}, \text{ for all } \mathbf{u} \in \mathbb{S}^{n-1}$$

where $\mathbf{x} \in \partial K$ is the uniquely determined point with outer unit normal $\mathbf{u}$.

Volume Derivative of the Weighted Floating Bodies

Extension of the uniform case, $\varphi = \psi \equiv 1$, established by **Schütt & Werner** (1990):

Theorem(B., Ludwig, & Werner, 2018). $K \subset \mathbb{R}^n$ a convex body. (Trans. Amer. Math. Soc.)

$$\lim_{\delta \to 0^+} \frac{\mathrm{Vol}^{\psi}(K) - \mathrm{Vol}^{\psi}(\mathcal{F}_{\delta}^{\varphi} K)}{\delta^{\frac{2}{n+1}}} = c_n \int_{\partial K} \kappa_o(K, \mathbf{x})^{\frac{1}{n+1}} \varphi(\mathbf{x})^{-\frac{2}{n+1}} \psi(\mathbf{x}) \, C_K(\mathrm{d}\mathbf{x}).$$

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Theorem(B. & Werner, 2023$^+$). If K a convex body of class C_+^2 that contains the origin in the interior, then

$$\lim_{\delta \to 0^+} \frac{\mathrm{Vol}^\psi(\mathcal{F}_\delta^{\varphi,\circ} K) - \mathrm{Vol}^\psi(K)}{\delta^{\frac{2}{n+1}}} = c_n \int_{\partial K} \kappa_o(K, \mathbf{x})^{-\frac{1}{n+1}} \varphi(\mathbf{x}^\circ)^{-\frac{2}{n+1}} \psi(\mathbf{x}) \, C_K(\mathrm{d}\mathbf{x}),$$

where $\mathbf{x}^\circ \in \partial K^\circ$ is the polar point determined by $\mathbf{x} \in \partial K$ such that $\langle \mathbf{x}^\circ, \mathbf{x} \rangle = 1$.

- Extends uniform case by **Meyer & Werner** (2000) and **Werner & Ye** (2008).

Projective Model of Spaces of Constant Curvature

- The classical spaces of constant curvature (e.g. spherical space, hyperbolic space) can be model on $\mathbb{R}^n$ such that geodesics in the space of constant curvature are affine line segments in $\mathbb{R}^n$.
  Beltrami (1900), **Cartan** (1930): a Riemannian metric is projective if and only if it has constant curvature.

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- The gnomonic projection is a diffeomorphism from the half-sphere model $\mathbb{S}^n_+ := \{\mathbf{u} \in \mathbb{R}^{n+1} : \|\mathbf{u}\|_2 = 1$ and $u_{n+1} > 0\}$ or the hyperbolid model $\mathbb{H}^n := \{\mathbf{u} \in \mathbb{R}^{n,1} : \mathbf{u} \circ \mathbf{u} = -1$ and $u_{n+1} > 0\}$ to $\mathbb{R}^n$ given by
  $$g(\mathbf{u}) = \frac{1}{u_{n+1}}(u_1, \ldots, u_n).$$
  It maps geodesics of $\mathbb{S}^n_+$ and $\mathbb{H}^n$ to affine line segments in $\mathbb{R}^n$.

  $\mathbf{u} \circ \mathbf{u} = u_1^2 + \ldots + u_n^2 - u_{n+1}^2$ ... indefinite inner product of the Lorentz–Minkowski space $\mathbb{R}^{n,1} = (\mathbb{R}^{n+1}, \circ)$.

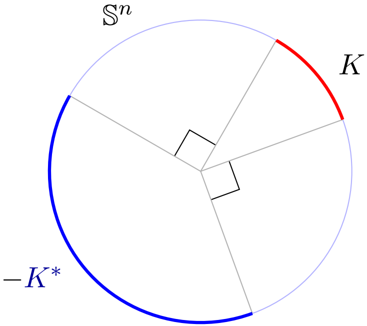
Projective Model of Spaces of Constant Curvature

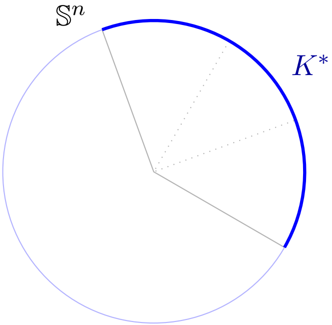
- The classical spaces of constant curvature (e.g. spherical space, hyperbolic space) can be model on $\mathbb{R}^n$ such that geodesics in the space of constant curvature are affine line segments in $\mathbb{R}^n$.
  Beltrami (1900), **Cartan** (1930): a Riemannian metric is projective if and only if it has constant curvature.
- The gnomonic projection is a diffeomorphism from the half-sphere model $\mathbb{S}^n_+ := \{\mathbf{u} \in \mathbb{R}^{n+1} : \|\mathbf{u}\|_2 = 1$ and $u_{n+1} > 0\}$ or the hyperbolid model $\mathbb{H}^n := \{\mathbf{u} \in \mathbb{R}^{n,1} : \mathbf{u} \circ \mathbf{u} = -1$ and $u_{n+1} > 0\}$ to $\mathbb{R}^n$ given by
  $$g(\mathbf{u}) = \frac{1}{u_{n+1}}(u_1, \ldots, u_n).$$
  It maps geodesics of $\mathbb{S}^n_+$ and $\mathbb{H}^n$ to affine line segments in $\mathbb{R}^n$.
- In this way we can identify a spherical or hyperbolic convex body K with the Euclidean convex body $\overline{K} := g(K) \subset \mathbb{R}^n$ and the spherical or hyperbolic Lebesgue measure in $\mathbb{R}^n$ is expressed by $\mathrm{Vol}^\varepsilon = \mathrm{Vol}^{\varphi_\varepsilon}$, where $\varphi_\varepsilon(\mathbf{x}) = (1 + \varepsilon\|\mathbf{x}\|_2^2)^{-\frac{n+1}{2}}$, with $\varepsilon = +1$ in the spherical case and $\varepsilon = -1$ in the hyperbolic case.

Dual Convex Body / De Sitter Convex Bodies

- The dual body of spherical convex body K is defined by

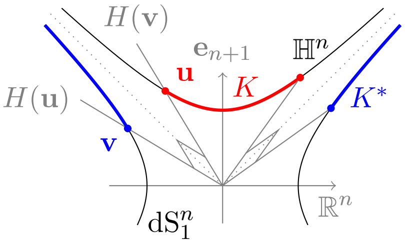
$$K^* := \{\mathbf{v} \in \mathbb{S}^n : \langle \mathbf{u}, \mathbf{v} \rangle \geq 0 \text{ for all } \mathbf{u} \in K\} = \bigcap_{\mathbf{u} \in K} H^+(\mathbf{u}).$$





Dual Convex Body / De Sitter Convex Bodies

- The dual space of $\mathbb{H}^n$ is the de Sitter space $\mathrm{dS}^n_1 := \{\mathbf{u} \in \mathbb{R}^{n+1} : \mathbf{u} \circ \mathbf{u} = 1\}$.
  The dual body of a hyperbolic convex body K or a de Sitter convex body L is defined by
  $$K^* := \{\mathbf{v} \in \mathrm{dS}^n_1 : \mathbf{u} \circ \mathbf{v} \leq 0 \text{ for all } \mathbf{u} \in K\} = \bigcap_{\mathbf{u} \in K} H^+(\mathbf{u}),$$
  $$L^* := \{\mathbf{u} \in \mathbb{H}^n : \mathbf{u} \circ \mathbf{v} \leq 0 \text{ for all } \mathbf{v} \in L\} = \bigcap_{\mathbf{v} \in L} H^+(\mathbf{v}).$$

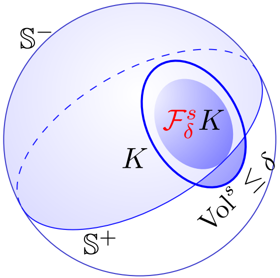


Volume of the Floating Body in Spaces of Constant Curvature

Theorem(B. & Werner, 2016/2018). Let K be a spherical, hyperbolic or de Sitter convex body. (Adv. Math. / JDG)

$$\lim_{\delta\to 0^+} \frac{\mathrm{Vol}^\varepsilon(K) - \mathrm{Vol}^\varepsilon(\mathcal{F}_\delta^\varepsilon K)}{\delta^{\frac{2}{n+1}}} = c_n \int_{\partial K} H_{n-1}^\varepsilon(K,\mathbf{u})^{\frac{1}{n+1}} \,\mathrm{Vol}_{\partial K}^\varepsilon(\mathrm{d}\mathbf{u}) =: c_n\, \Omega_1^\varepsilon(K).$$

- $\mathrm{Vol}_{\partial K}^\varepsilon \ldots (d-1)$-dimensional Hausdorff measure restricted to ∂K
  $\cong$ surface area measure of ∂K
- $H_{n-1}^\varepsilon(K,\cdot)$ generalized Gauss–Kronecker curvature of ∂K as a hypersurface.



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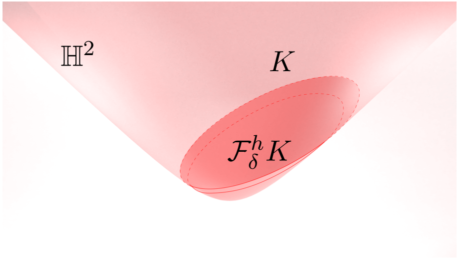

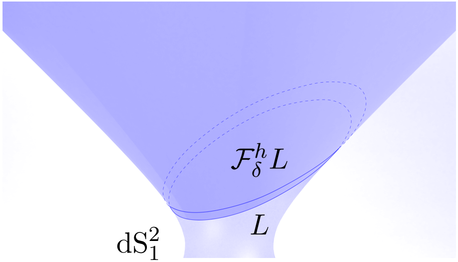

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Theorem(B. & Werner, 2023⁺). Let K be a spherical, hyperbolic or de Sitter convex body of class C_+^2. Then

$$\lim_{\delta\to 0^+} \frac{\mathrm{Vol}^\varepsilon(\mathcal{F}_\delta^{\varepsilon,*} K \setminus K)}{\delta^{\frac{2}{n+1}}} = c_n \int_{\partial K} H_{n-1}^\varepsilon(K,\mathbf{u})^{-\frac{1}{n+1}} \, \mathrm{Vol}_{\partial K}^\varepsilon(\mathrm{d}\mathbf{u}) =: c_n\, \Omega_{-\frac{n}{n+2}}^\varepsilon(K).$$

- $\mathcal{F}_\delta^{\varepsilon,*} K = (\mathcal{F}_\delta^\varepsilon K^*)^*$... floating body conjugated by duality mapping

Space Limits 1/3

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Space Limits 1/3

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- Let $\mathrm{Sp}^n(\lambda)$ be the space form of constant curvature $\lambda \in \mathbb{R}$.

Theorem(B. & Werner, 2018). $K \subset \mathrm{Sp}^n(\lambda)$ a convex body. (JDG)

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- Moreover, if $\overline{K} \subset \mathbb{R}^n$ is fixed, then
$$\lim_{\lambda\to 0} \Omega_1^{\lambda}(\overline{K}) = \mathrm{as}_1(\overline{K}) = \int_{\partial \overline{K}} H_{n-1}(\overline{K}, \mathbf{x})^{\frac{1}{n+1}} \,\mathrm{Vol}_{\partial \overline{K}}(\mathrm{d}\mathbf{x}) = \int_{\partial \overline{K}} \kappa_o(\overline{K}, \mathbf{x})^{\frac{1}{n+1}} \,C_{\overline{K}}(\mathrm{d}\mathbf{x})$$

State of the Art

$$\int_{\partial K} H^{\varepsilon}_{n-1}(K)^{\frac{1}{n+1}} \, \mathrm{dVol}^{\varepsilon}_{\partial K}$$

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& \uparrow \\
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& \downarrow \\
\mathbb{R}^n[\lambda = 0]: & \mathcal{F}_{\delta},\ \mathrm{as}_1
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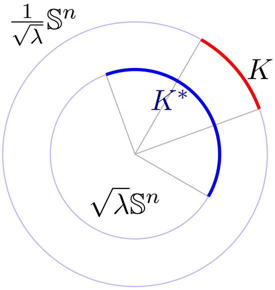
Space Limits 2/3

Theorem(B. & Werner, 2023$^+$). $K \in \mathrm{Sp}^n(\lambda)$ a convex body of class C^2_+.

$$\lim_{\delta\to 0^+} \frac{\mathrm{Vol}^\lambda(\mathcal{F}_\delta^{\lambda,*}K) - \mathrm{Vol}^\lambda(K)}{\delta^{\frac{2}{n+1}}} = \frac{c_n}{\lambda} \int_{\partial K} H_{n-1}^\lambda(K,\mathbf{u})^{-\frac{1}{n+1}} \,\mathrm{Vol}_{\partial K}^\lambda(\mathrm{d}\mathbf{u}).$$

- We see the duality $*$ as a mapping between $\mathrm{Sp}^n(\lambda) \to \mathrm{Sp}^n(1/\lambda)$ for $\lambda > 0$, and between $\mathrm{Sp}^n(\lambda)$ and $\mathrm{Sp}_1^n(1/\lambda)$ for $\lambda < 0$.



Space Limits 2/3

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Corollary. Let $\overline{K} \subset \mathbb{R}^n$ be a convex body of class C^2_+. Then, for $\delta_\lambda = \kappa_{n-1} \lambda^{\frac{n+1}{2}} \delta$,

$$\lim_{\lambda \to 0} \mathcal{F}_{\delta_\lambda}^{\lambda,*} \overline{K} = \mathcal{I}_\delta^{V_1} \overline{K} = \{\mathbf{x} \in \mathbb{R}^n : |V_1(\mathrm{conv}(\overline{K}, \mathbf{x})) - V_1(\overline{K})| \le \delta\}.$$

- $\mathcal{I}_\delta^{V_1}$ is a special case of the separation body introduced by **Schneider** (2020) and can also be seen as a V_1-illumination body introduced by **Werner** (1994).

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Moreover: $$\lim_{\delta\to 0^+} \frac{\mathrm{Vol}(\mathcal{I}_\delta^{V_1}\overline{K}) - \mathrm{Vol}(\overline{K})}{\delta^{\frac{2}{n+1}}} = d_n \int_{\partial\overline{K}} H_{n-1}(\overline{K},\mathbf{x})^{-\frac{1}{n+1}} \,\mathrm{Vol}_{\partial\overline{K}}(\mathrm{d}\mathbf{x}). \qquad (\diamond)$$

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- $(\diamond)$ was communicated to Schneider by **Olaf Mordhorst**.

State of the Art

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A Notion of Polarity Dependent on a Fixed Point

- The duality * as a mapping from $\mathrm{Sp}^n(\lambda)$ to $\mathrm{Sp}^n(1/\lambda)$, respectively $\mathrm{Sp}^n_1(1/\lambda)$, does **not** depend on a fixed point. Thus it is no surprise that in the limit $\lambda \to 0$ we obtain a translation invariant curvature measure.

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Definition. For $\lambda \neq 0$ and $\mathbf{e} \in \mathrm{Sp}^n(\lambda)$ fixed we consider the e-polarity operator from $\mathrm{Sp}^n(\lambda)$ to $\mathrm{Sp}^n(\lambda)$ that is defined for a convex body K that contains $\mathbf{e}$ in the interior (and is contained in the open half space around $\mathbf{e}$ if $\lambda > 0$) by

$$K^{\mathbf{e}} := G^\lambda_{\mathbf{e}}(K^*), \text{ where } G^\lambda_{\mathbf{e}}(\mathbf{u}) = \tfrac{1}{\sqrt{|\lambda|}} \tfrac{\mathbf{v}}{\|\mathbf{v}\|} \text{ for } \mathbf{v} = \begin{cases} \lambda \frac{\mathbf{u}}{\langle \mathbf{u}, \mathbf{e} \rangle} + (1-\lambda)\mathbf{e} & \lambda > 0 \\ \lambda \frac{\mathbf{u}}{\mathbf{u} \circ \mathbf{e}} + (1+\lambda)\mathbf{e} & \lambda < 0 \end{cases}.$$

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- Conjugating the λ-floating body in $\mathrm{Sp}^n(\lambda)$ gives: $\mathcal{F}^{\lambda,\mathbf{e}}_\delta K := (\mathcal{F}^\lambda_\delta K^{\mathbf{e}})^{\mathbf{e}}$.

Space Limits 3/3

Theorem(B. & Werner, 2023$^+$). $K \in \mathrm{Sp}^n(\lambda)$ a convex body of class C^2_+ that contains $\mathbf{e}$ in the interior. Then

$$\lim_{\delta\to 0^+} \frac{\mathrm{Vol}^\lambda(\mathcal{F}_\delta^{\lambda,\mathbf{e}} K) - \mathrm{Vol}^\lambda(K)}{\delta^{\frac{2}{n+1}}} = c_n \underbrace{\int_{\partial K} \left(\frac{H_{n-1}^\lambda(K,\mathbf{u})}{f_{\mathbf{e}}^\lambda(K,\mathbf{u})^{n+1}}\right)^{-\frac{1}{n+1}} f_{\mathbf{e}}^\lambda(K,\mathbf{u})\, \mathrm{Vol}_{\partial K}^\lambda(\mathrm{d}\mathbf{u})}_{=:\Omega^\lambda_{-\frac{n}{n+2}}(K)},$$

where $f_{\mathbf{e}}^\lambda(K,\mathbf{u}) = \sqrt{\left|\frac{\lambda + [\tan_\lambda d_\lambda(\mathbf{e}, H(K,\mathbf{u}))]^2}{1+\lambda[\tan_\lambda d_\lambda(\mathbf{e}, H(K,\mathbf{u}))]^2}\right|}$.

- d_λ geodesic distance on $\mathrm{Sp}^n(\lambda)$; $H(K,\mathbf{u})$ hyperplane tangent to K at $\mathbf{u} \in \partial K$.
- $\tan_\lambda \alpha = \begin{cases} \frac{\tan(\sqrt{\lambda}\alpha)}{\sqrt{\lambda}} & \lambda > 0, \\ \alpha & \lambda = 0, \\ \frac{\tanh(\sqrt{-\lambda}\alpha)}{\sqrt{-\lambda}} & \lambda < 0, \end{cases}$

Space Limits 3/3

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- For $\lambda = \varepsilon = \pm 1$: $f_{\mathbf{e}}^\lambda \equiv 1$ and $\Omega^{\lambda,\mathbf{e}}_{-\frac{n}{n+2}} = \Omega^{\varepsilon}_{-\frac{n}{n+2}}$ is independent of $\mathbf{e}$.
- Moreover, if $\overline{K} \subset \mathbb{R}^n$ is fixed, then
$$\lim_{\lambda\to 0} \Omega^{\lambda,o}_{-\frac{n}{n+2}}(\overline{K}) = \mathrm{as}_{-\frac{n}{n+2}}(\overline{K}) = \int_{\partial\overline{K}} \kappa_o(\overline{K},\mathbf{x})^{-\frac{1}{n+1}}\, C_{\overline{K}}(\mathrm{d}\mathbf{x})$$

State of the Art

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$$\begin{tikzcd}
\mathbb{S}^n, \mathbb{H}^n, \mathrm{dS}^n_1[\lambda = \varepsilon = \pm 1]: & \mathcal{F}^{\varepsilon}_{\delta},\ \Omega^{\varepsilon}_{1} & & \mathcal{F}^{\varepsilon,*}_{\delta},\ \Omega^{\varepsilon}_{-\frac{n}{n+2}} & \\
\mathrm{Sp}^n(\lambda), \mathrm{Sp}^n_1(\lambda)[\lambda \neq 0]: & \mathcal{F}^{\lambda}_{\delta},\ \Omega^{\lambda}_{1} \arrow[u] \arrow[d] & \mathcal{F}^{\lambda,*}_{\delta},\ O^{\lambda}_{-\frac{n}{n+2}} \arrow[ur] \arrow[d] & & \mathcal{F}^{\lambda,\mathbf{e}}_{\delta},\ \Omega^{\lambda,\mathbf{e}}_{-\frac{n}{n+2}} \arrow[ul] \arrow[d] \\
\mathbb{R}^n[\lambda = 0]: & \mathcal{F}_{\delta},\ \mathrm{as}_1 & \mathcal{I}^{V_1}_{\delta},\ O_{-\frac{n}{n+2}} & & \mathcal{F}^{\circ}_{\delta},\ \mathrm{as}_{-\frac{n}{n+2}}
\end{tikzcd}$$

$$\int_{\partial K} H_{n-1}(K)^{\frac{1}{n+1}} \,\mathrm{dVol}_{\partial K} = \int_{\partial K} \kappa_o(K)^{\frac{1}{n+1}} \,\mathrm{d}C_K \qquad \int_{\partial K} H_{n-1}(K)^{-\frac{1}{n+1}} \,\mathrm{dVol}_{\partial K} \qquad \int_{\partial K} \kappa_o(K)^{-\frac{1}{n+1}} \,\mathrm{d}C_K$$

Properties of weighted L_p-affine surface area

Theorem(B. & Werner, 2023^{+}). For the (φ, ψ)-weighted surface area, defined by

$$\Omega_{-\frac{n}{n+2}}(K; \varphi, \psi) = \int_{\partial K} \kappa_o(K, \mathbf{x})^{-\frac{1}{n+1}} \varphi(\mathbf{x}^\circ)^{-\frac{2}{n+1}} \psi(\mathbf{x})\, C_K(\mathrm{d}\mathbf{x}),$$ we have:

- For $A \in \mathrm{SL}(n)$ we have that

$$\Omega_{-\frac{n}{n+2}}(AK; \varphi \circ A^\top, \psi \circ A^{-1}) = \Omega_{-\frac{n}{n+2}}(K; \varphi, \psi)$$

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- It is a valuation and lower semi-continuous on convex bodies of class C_+^2 that contain the origin in the interior. [**Schütt**, 1994 / **Ludwig**, 2001 + 2010]
- For a convex body K of class C_+^2 that contains the origin in the interior we have the polarity formula: [**Hug**, 1996]
$$\Omega_{-\frac{n}{n+2}}(K;\varphi,\psi) = \int_{\partial K^\circ} \kappa_o(K^\circ,\mathbf{y})^{\frac{n+2}{n+1}}\, \psi(\mathbf{y}^\circ)\, \varphi(\mathbf{y})^{-\frac{2}{n+1}}\, C_{K^\circ}(\mathrm{d}\mathbf{y}).$$

Proof: Volume of Floating Body conjugated by Polarity

Proof. Let K be a spherical, hyperbolic or de Sitter convex body of class C^2_+ that contains $\mathbf{e}_{d+1}$ in the interior and/or is contained in the interior of $H^+(\mathbf{e}_{d+1})$.

$$\begin{aligned}
\lim_{\delta\to 0^+} \frac{\mathrm{Vol}^{\varepsilon}(\mathcal{F}_\delta^{\varepsilon,*}K \setminus K)}{\delta^{\frac{2}{n+1}}} &= \lim_{\delta\to 0^+} \frac{\mathrm{Vol}^{\varphi_\varepsilon}(\mathcal{F}_\delta^{\varphi_\varepsilon,\circ}\overline{K}) - \mathrm{Vol}^{\varphi_\varepsilon}(\overline{K})}{\delta^{\frac{2}{n+1}}} \\
&= c_n \int_{\partial \overline{K}} \kappa_o(\overline{K},\mathbf{x})^{-\frac{1}{n+1}} \frac{\varphi_\varepsilon(\mathbf{x})}{\varphi_\varepsilon(\mathbf{x}^\circ)^{\frac{2}{n+1}}}\, C_{\overline{K}}(\mathrm{d}\mathbf{x}) \\
&=
\end{aligned}$$

Lemma. $g(K^*) = -\varepsilon\overline{K}^\circ$. Thus $g(\mathcal{F}_\delta^{\varepsilon,*}K) = \mathcal{F}_\delta^{\varphi_\varepsilon,\circ}\overline{K}$.

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Theorem (B. & Werner, 2018). (JDG)

If $\mathbf{x}\in\partial\overline{K}$ is a normal boundary point, then

$$H_{n-1}^\varepsilon(\overline{K},\mathbf{x})^{\frac{1}{n+1}} = \kappa_o(\overline{K},\mathbf{x})^{\frac{1}{n+1}} \sqrt{\frac{1+\varepsilon\|\mathbf{x}\|_2^2}{1+\varepsilon\langle \mathbf{n}_{\overline{K}}(\mathbf{x}),\mathbf{x}\rangle^{-2}}},$$ and

$$\frac{\mathrm{dVol}^\varepsilon_{\partial\overline{K}}}{\mathrm{d}C_{\overline{K}}}(\mathbf{x}) = \frac{\sqrt{1+\varepsilon\langle \mathbf{n}_{\overline{K}}(\mathbf{x}),\mathbf{x}\rangle^{-2}}}{(1+\varepsilon\|\mathbf{x}\|_2^2)^{n/2}}.$$

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$$= c_n \int_{\partial\overline{K}} \kappa_o(\overline{K},\mathbf{x})^{-\frac{1}{n+1}} \frac{\left|1+\varepsilon\langle \mathbf{n}_{\overline{K}}(\mathbf{x}),\mathbf{x}\rangle^{-2}\right|}{\left|1+\varepsilon\|\mathbf{x}\|_2^2\right|^{\frac{n+1}{2}}}\, C_{\overline{K}}(\mathrm{d}\mathbf{x})$$

$$= c_n \int_{\partial\overline{K}} H^\varepsilon_{n-1}(\overline{K},\mathbf{x})^{-\frac{1}{n+1}}\, \mathrm{Vol}^\varepsilon_{\partial\overline{K}}(\mathrm{d}\mathbf{x}).$$