How does the sectional Poisson-Voronoi tessellation look like?

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(joint work with Zakhar Kabluchko and Christoph Thäle)

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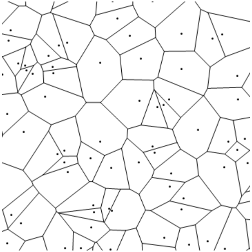
Tessellations:

- A tessellation T in $\mathbb{R}^d$ is a system of convex polytopes (cells), covering the space and having disjoint interiors.



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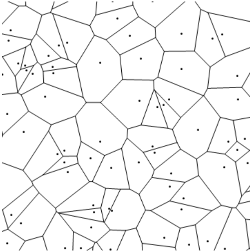

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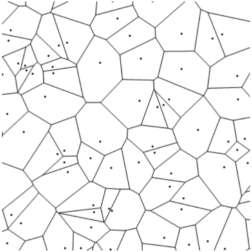

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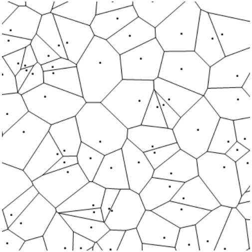

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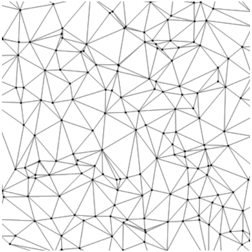

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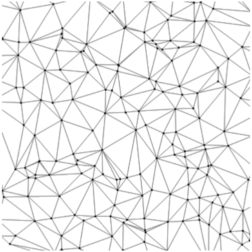

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Poisson point process η in $\mathbb{R}^d$ with intensity measure μ:

- $\forall A \in \mathcal{B}(\mathbb{R}^d)$, $\eta(A) \sim \mathrm{Poi}(\mu(A))$;
- $\forall$ mutually disjoint subsets $A_1, \dots, A_m \in \mathcal{B}(\mathbb{R}^d)$, $\eta(A_1), \dots, \eta(A_m)$ are independent.

Let η be a PPP in $\mathbb{R}^d$ with intensity measure $\gamma\,\mathrm{Leb}_d$, $\gamma > 0$.

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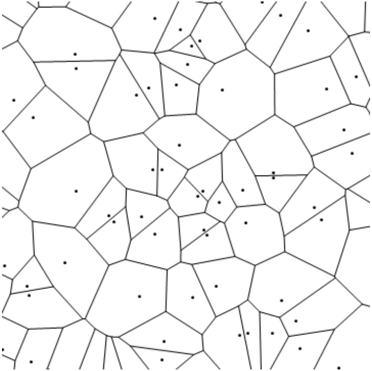

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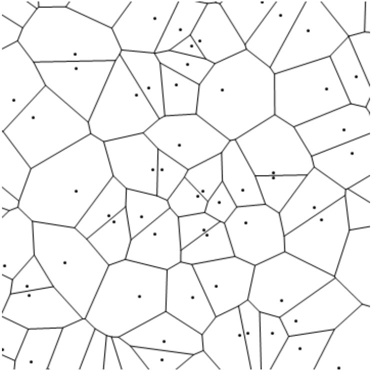

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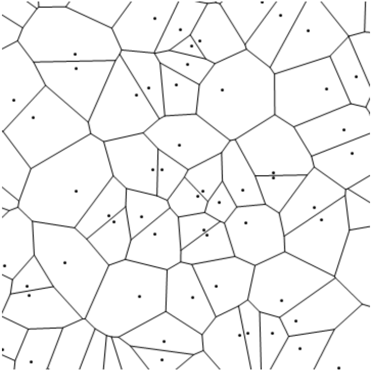
Poisson-Voronoi tessellation: $\mathcal{V}_\gamma^d := \{V(v,\eta) : v \in \eta\}$.

Fact: $\mathcal{V}_\gamma^d$ is almost surely face-to-face, normal random tessellation and $\mathcal{V}_\gamma^d$ is stationary.



Characteristics of stationary random tessellation

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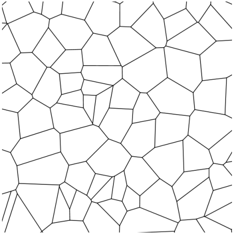

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where $z : \{\text{polytopes}\} \times \{\text{tessellations}\} \mapsto \mathbb{R}^d$ be s.t.
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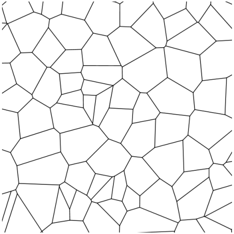

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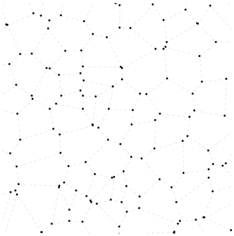

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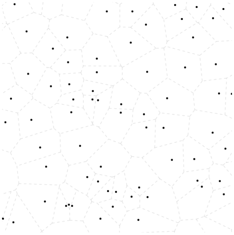

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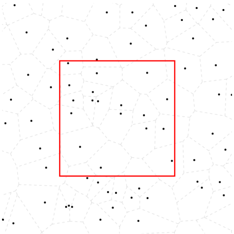

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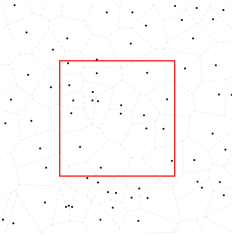

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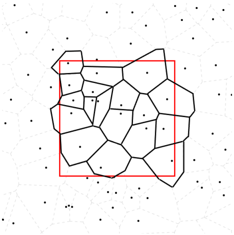

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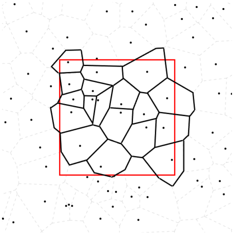

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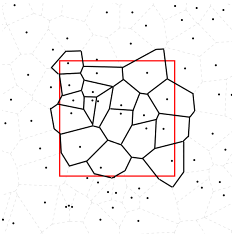

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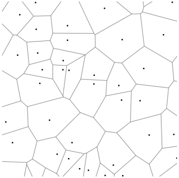

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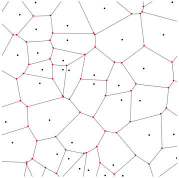

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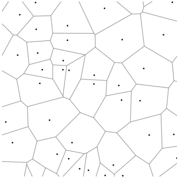

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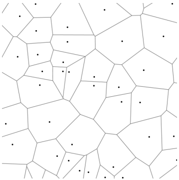
$$\mathbb{E} f_0(Z_d) = 2^d \pi^{\frac{d-1}{2}} \frac{\Gamma(\frac{d^2}{2}+\frac{1}{2})}{\Gamma(\frac{d^2}{2}+1)} \frac{\Gamma(\frac{d}{2}+1)^d}{\Gamma(\frac{d+1}{2})^d}$$

$$\mathbb{E} V_{d-1}(Z_d) = 2^{d-1} \frac{\Gamma(\frac{d}{2})\Gamma(2-\frac{1}{d})}{\Gamma(d-\frac{1}{2})} \Gamma(\tfrac{d}{2}+1)^{1-\frac{1}{d}} \gamma^{-1+\frac{1}{d}}.$$



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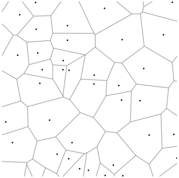

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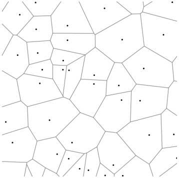

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Define the function:

$$J_{n,k}\left(\beta + \tfrac{1}{2}\right) = c_{n(n+2\beta)} \int_{-\infty}^{\infty} (\cosh u)^{-n(n+2\beta)-2} \left[\frac{1}{2} + \mathrm{i} \int_0^u c_{n+2\beta-1} (\cosh v)^{n+2\beta} \mathrm{d}v\right]^{n-k} \mathrm{d}u,$$

where $c_\alpha = \pi^{-\frac{1}{2}} \Gamma(\frac{\alpha+3}{2}) \Gamma(\frac{\alpha+2}{2})^{-1}$.

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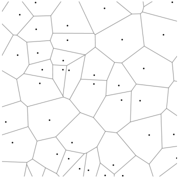

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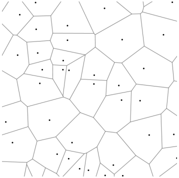
Define the function:

$$J_{n,k}\left(\beta+\tfrac{1}{2}\right) = c_{n(n+2\beta)}\int_{-\infty}^{\infty}(\cosh u)^{-n(n+2\beta)-2}\left[\tfrac{1}{2} + \mathrm{i}\int_0^u c_{n+2\beta-1}(\cosh v)^{n+2\beta}\mathrm{d}v\right]^{n-k}\mathrm{d}u,$$

where $c_\alpha = \pi^{-\frac{1}{2}}\Gamma(\frac{\alpha+3}{2})\Gamma(\frac{\alpha+2}{2})^{-1}$.

- Kabluchko, Thäle, Zaporozhets, 2020; Kabluchko, 2021:

$$\mathbb{E}f_k(Z_d) = C_1(d)\binom{d}{k}J_{d+1,d-k+1}\left(-\tfrac{1}{2}\right)$$



Use simplified notation: $Z_d := Z(\mathcal{V}_\gamma^d)$, $\gamma_k := \gamma_k(\mathcal{V}_\gamma^d)$.

- Trivial cases: $\gamma_d = \gamma$, $\mathbb{E} V_d(Z_d) = \gamma^{-1}$.
- $\mathcal{V}_\gamma^d$ is normal: $(d-k+1)\gamma_k = \gamma \mathbb{E} f_k(Z_d)$ and $2\gamma_1 = (d+1)\gamma_0$.
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Idea of the proof:

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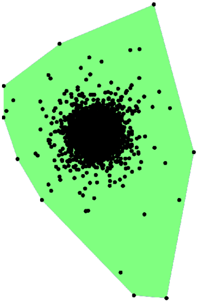
Idea of the proof:

- Suitable stochastic representation of Z_d

$$(Z_d)^o \overset{d}{=} \mathrm{conv}(\Pi_{d,d})$$

where $\Pi_{d,d}$ is a PPP with intensity measure $\|x\|^{-d-1}\mathrm{d}x$.



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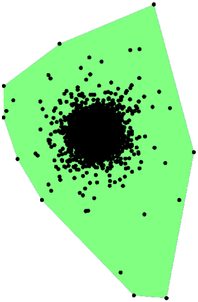

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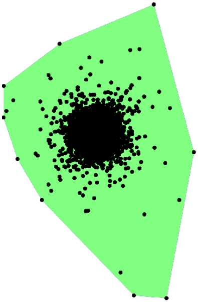
$$x \mapsto \pi^{-\frac{d}{2}}\Gamma(d)\Gamma(\tfrac{d}{2})^{-1}(1+\|x\|^2)^{-d},$$

and denote $P_{n,d} := \mathrm{conv}(X_1, \dots, X_n)$ – β'-polytope.



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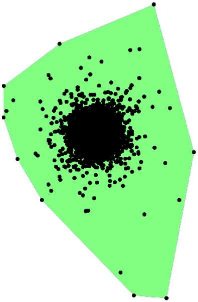

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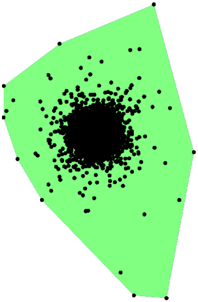

- $\mathbb{E}f_{d-1-k}(\mathrm{conv}(\Pi_{d,d})) = \lim_{n\to\infty} \mathbb{E}f_{d-1-k}(P_{n,d})$ + exact formulas for $\mathbb{E}f_{d-1-k}(P_{n,d})$.

Poisson-Voronoi tessellation $\mathcal{V}_\gamma^d$: f-vector

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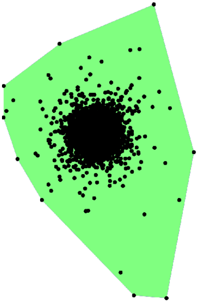

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This approach doesn't help to study expected intrinsic volumes of the typical cell.

Sectional Poisson-Voronoi tessellation

Given a ℓ-dimensional affine subspace $L_\ell \subset \mathbb{R}^d$ define the intersection tessellation

$$\mathcal{V}_\gamma^d \cap L_\ell := \{t \cap L_\ell \colon t \in \mathcal{V}_\gamma^d\}.$$

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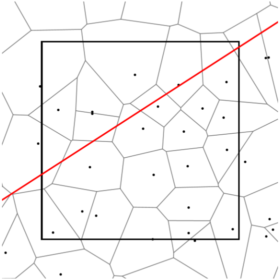

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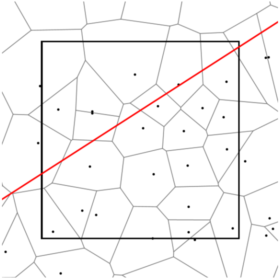

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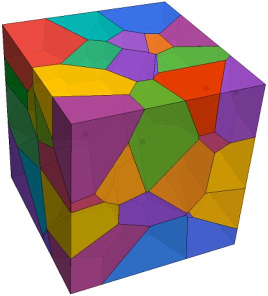
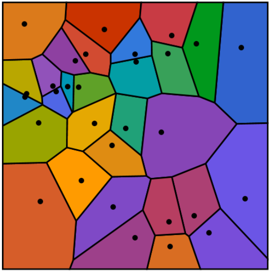


3D Poisson-Voronoi tessellation



2D Poisson-Voronoi tessellation

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Chiu, Van De Weygaert, Stoyan, 1996: "The sectional Poisson-Voronoi tessellation is not a Voronoi tessellation".

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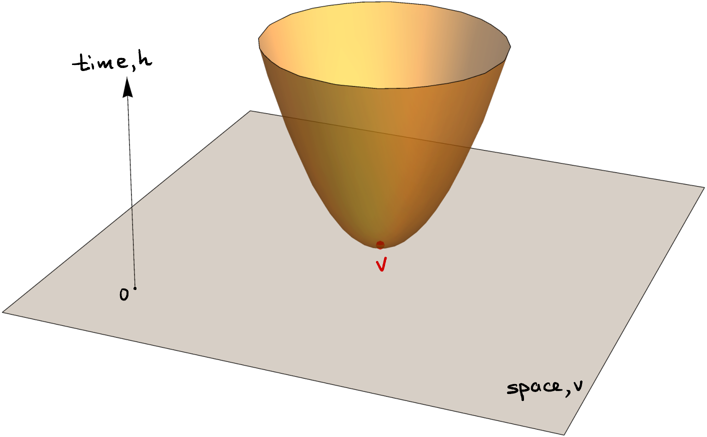
Next question: How does the sectional Poisson-Voronoi tessellation look like?

Poisson-Voronoi tessellation: graphical interpretation

Voronoi cell of (nuclei) $v \in \eta$:

$$V(v,\eta) := \{z \in \mathbb{R}^d : \|z - v\|^2 \le \|z - v'\|^2 \text{ for all } v' \in \eta\}.$$



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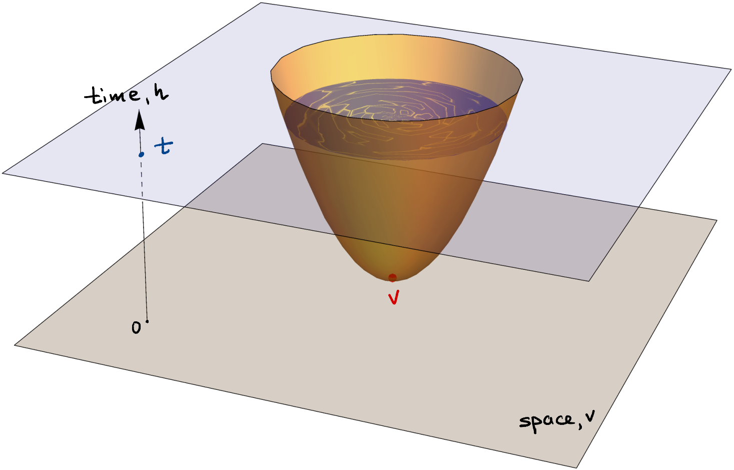

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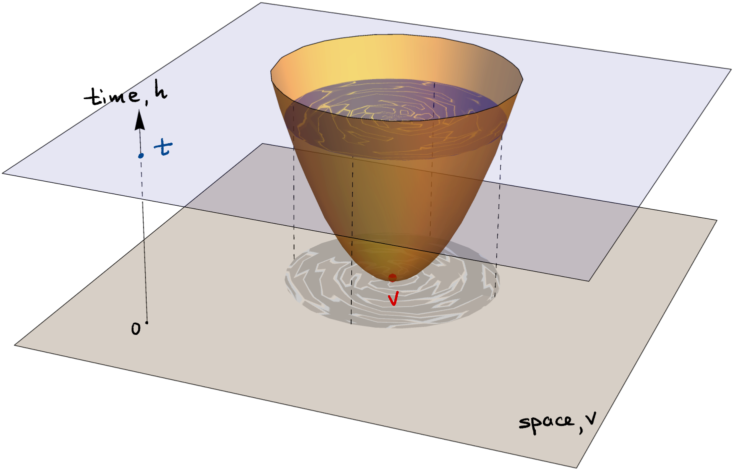

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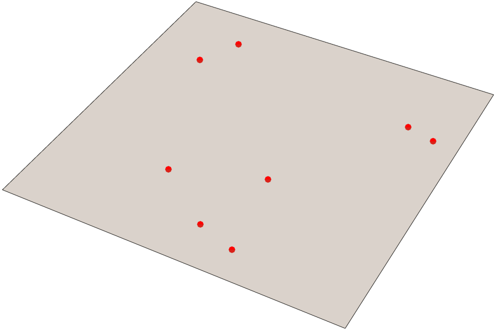

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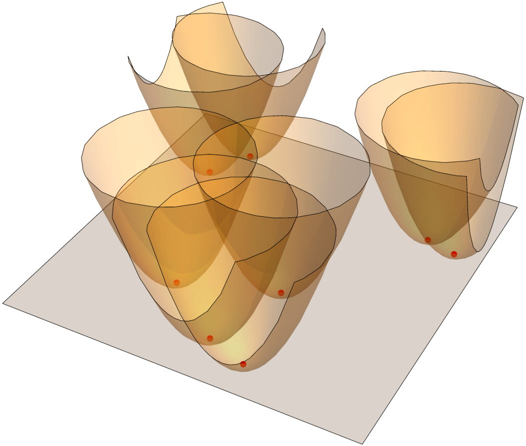

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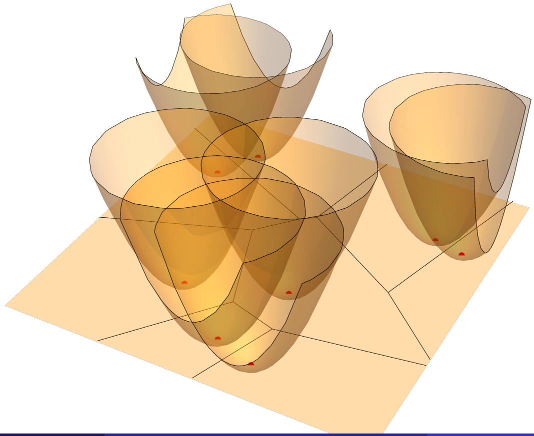
- Cell $V(v,\eta) \leftrightarrow$ paraboloid $\Pi^d_{(v,0)} := \{(w,t) \in \mathbb{R}^{d+1} \colon t = \|v - w\|^2\}$.



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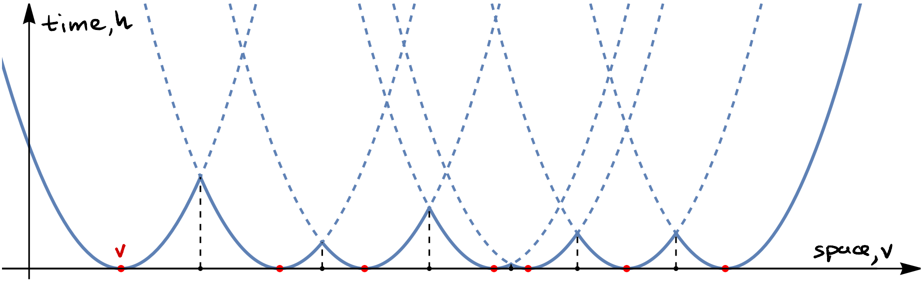

Poisson-Voronoi tessellation: graphical interpretation

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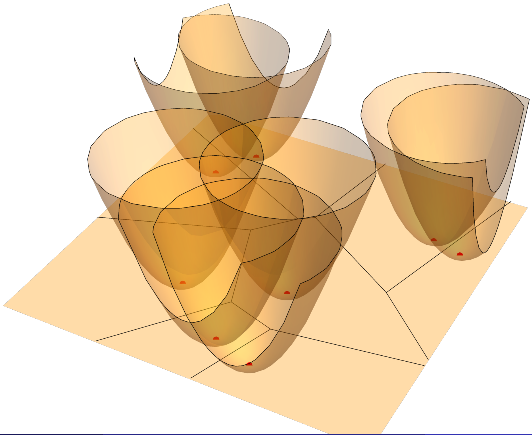

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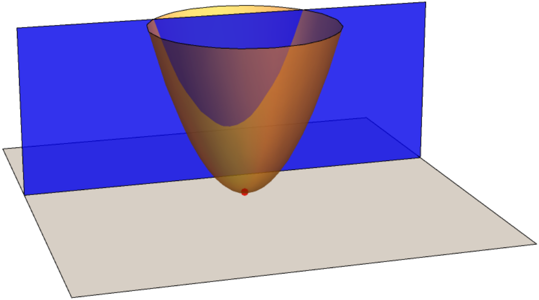
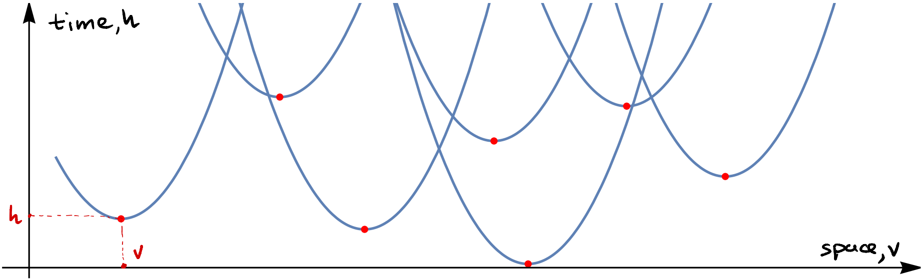
- $\Pi^d_{(v,0)} \cap (L \times \mathbb{R}) \leftrightarrow \Pi^\ell_{(\bar{v},h)}$ in $L \cong \mathbb{R}^\ell$.



Generalized Voronoi cell of $(v, h) \in \xi$:

$$V((v,h),\xi) := \{z \in \mathbb{R}^d \colon \|z - v\|^2 + h \le \|z - v'\|^2 + h' \text{ for all } (v', h') \in \xi\}.$$



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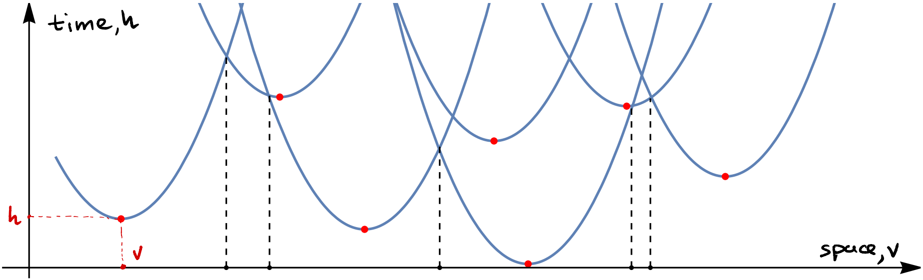

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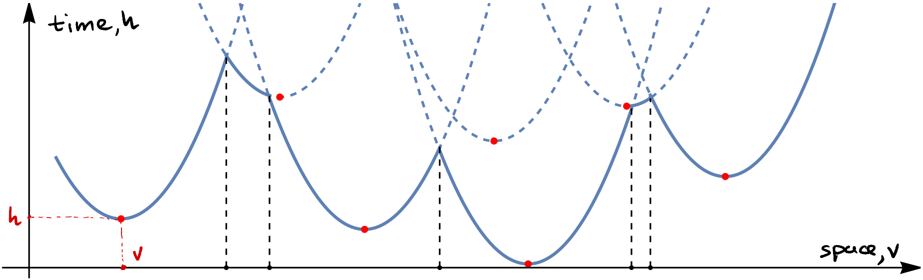
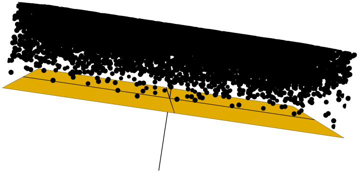


Generalized Voronoi tessellation (Laguerre tessellation)

Let ξ be a PPP in $\mathbb{R}^d \times E$, $E \subset \mathbb{R}$ with intensity measure having density of the form

$$(v, h) \mapsto \gamma f(h), \qquad \gamma > 0.$$



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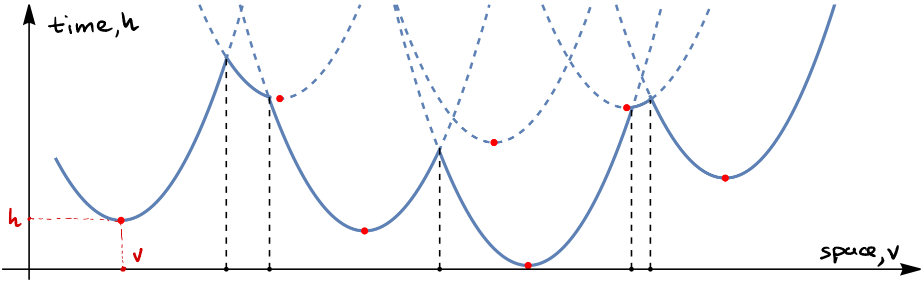

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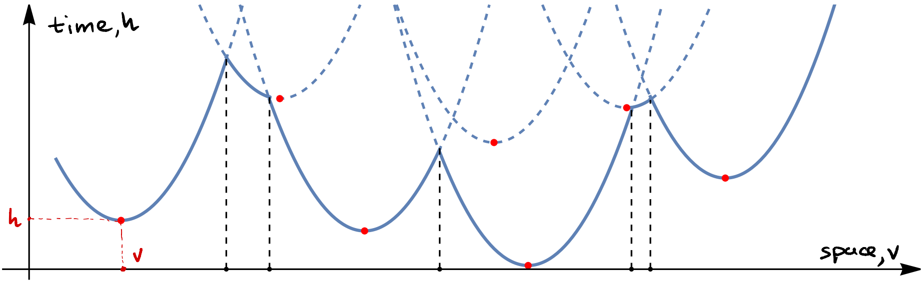

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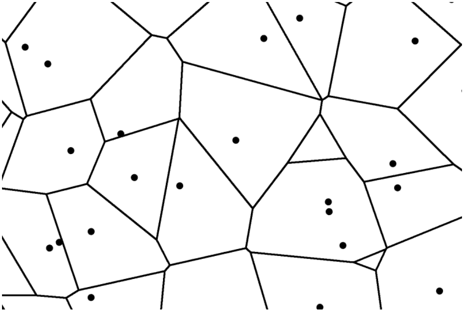

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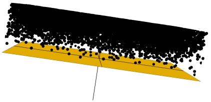

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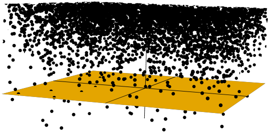

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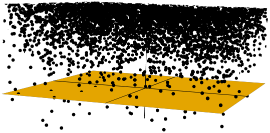

A.G., Z. Kabluchko and C. Thäle, 2022: The collections $\mathcal{V}^d_{\gamma,\beta}$ and $\widetilde{\mathcal{V}}^d_\lambda$ are normal random tessellations and we call it β- and Gaussian-Voronoi tessellation, respectively.

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For any ℓ-dimensional affine subspace L_ℓ we have

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Sectional Poisson-Voronoi tessellation

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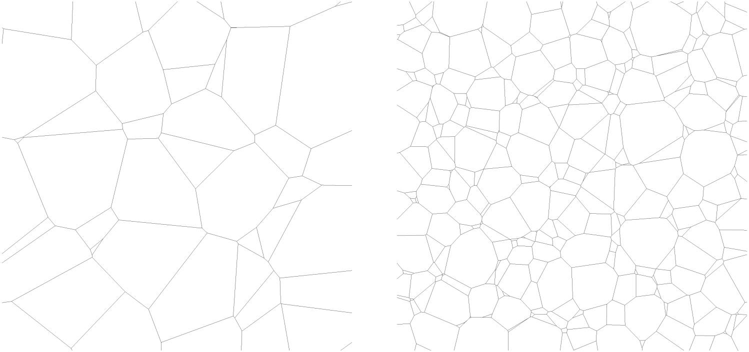

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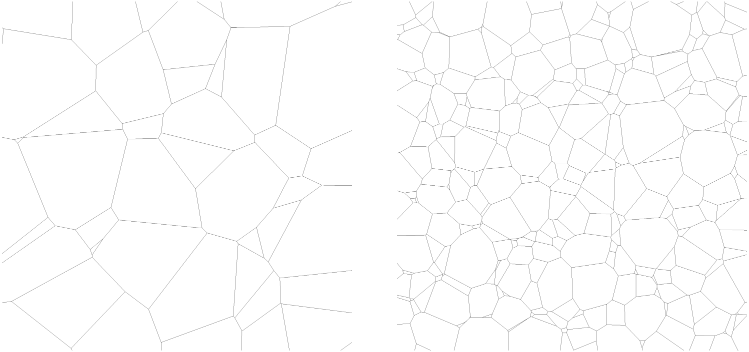
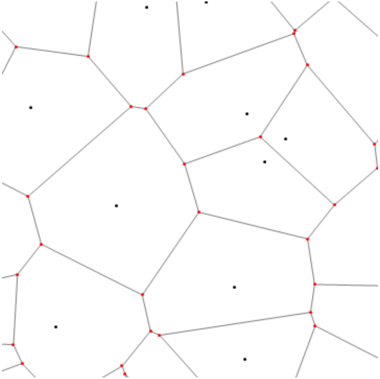


β-Delaunay tessellation (definition via duality)



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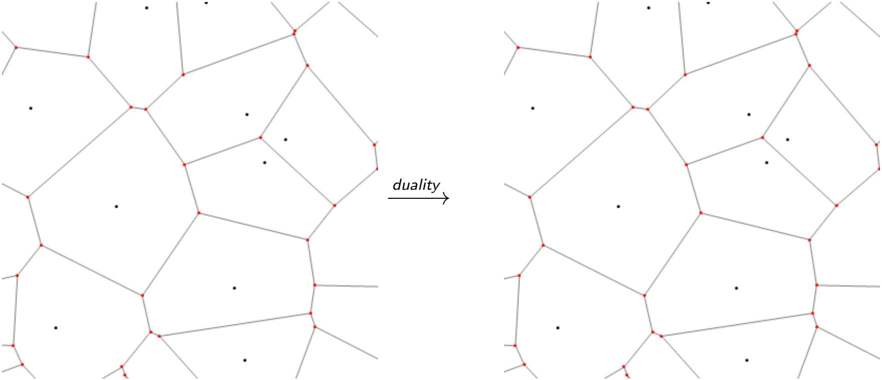

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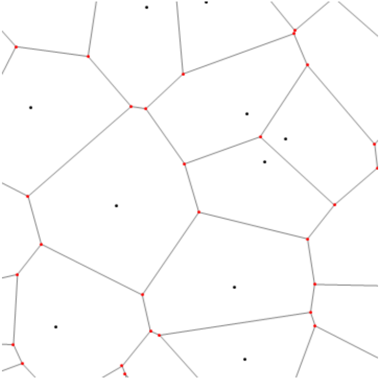

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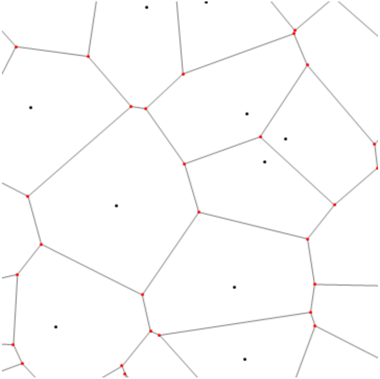

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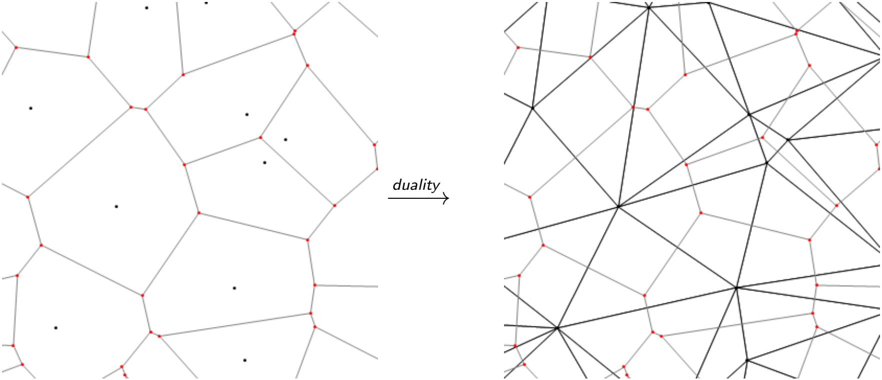

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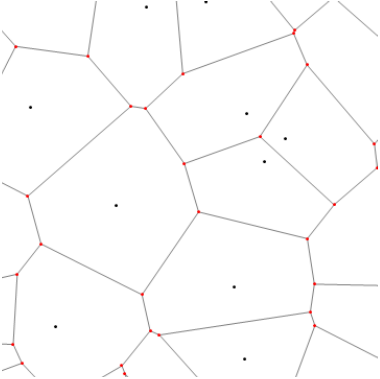

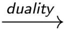

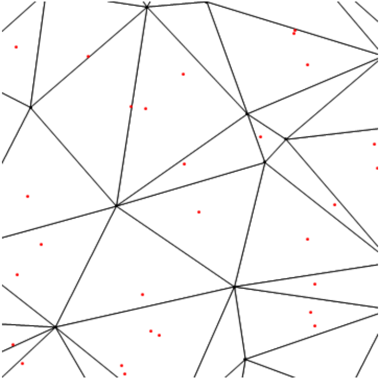

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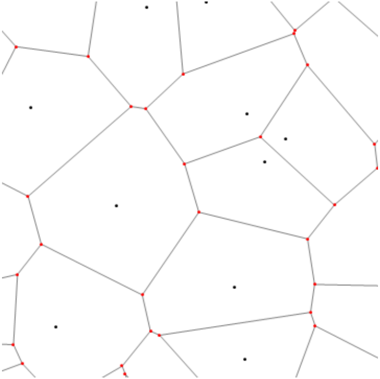

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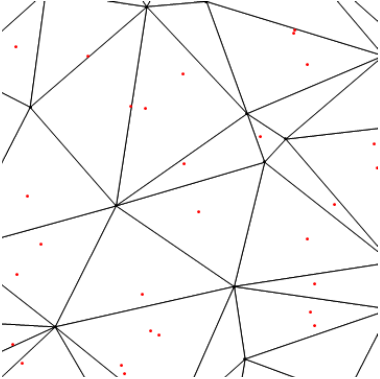

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Fact: $\mathcal{D}_\beta$ is almost surely a simplicial random tessellations.

Stochastic representation of the typical cell of $\mathcal{D}^d_{\gamma,\beta}$

Theorem (A.G., Z. Kabluchko and C.Thäle, 2022)

We have $Z(\mathcal{D}^d_{\gamma,\beta}) \overset{d}{=} R \cdot \mathrm{conv}(Y_1, \ldots, Y_{d+1})$, *where*

(a) $(Y_1, \ldots, Y_{d+1})$ *are random points, whose joint density is*

$$(y_1, \ldots, y_{d+1}) \mapsto \mathit{const} \cdot \mathrm{Vol}(\mathrm{conv}(y_1, \ldots, y_{d+1})) \prod_{i=1}^{d+1} (1 - \|y_i\|^2)^{\beta} \mathbf{1}_{\mathbb{B}^d}(\|y_i\|);$$

(b) R *is a random variable on* $(0, \infty)$ *independent of* $(Y_1, \ldots, Y_{d+1})$ *with density:*

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Remark: $Z(\mathcal{D}^d_{\gamma,\beta})$ is randomly rescaled volume-weighted beta simples;

Alternative definition of $\mathcal{D}^d_{\gamma,\beta}$

- Recall: $\mathcal{D}^d_{\gamma,\beta} = \{D(y)\colon y \in \mathcal{F}_0(\mathcal{V}^d_{\gamma,\beta})\}$

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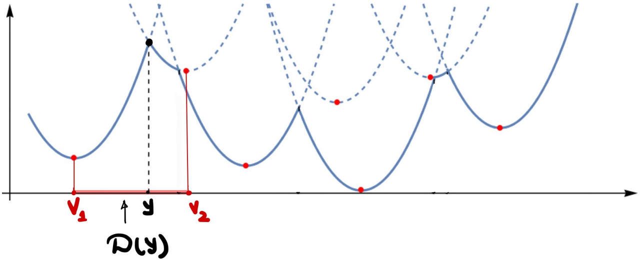

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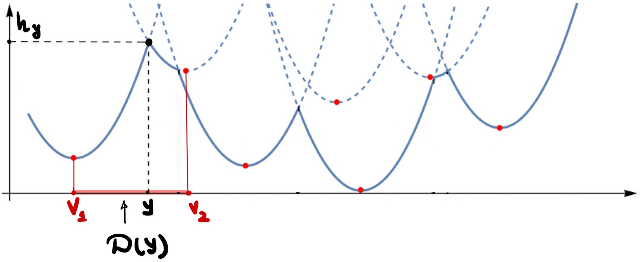

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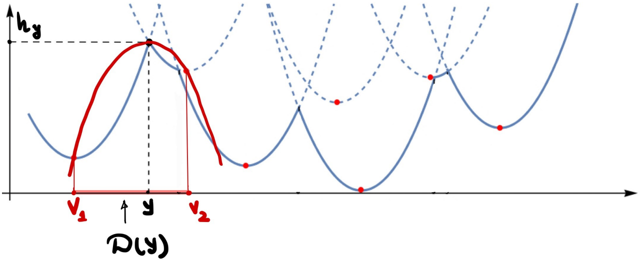

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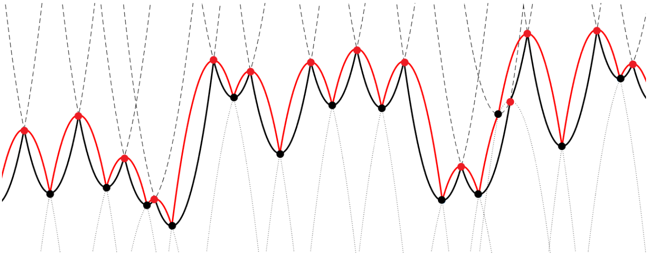

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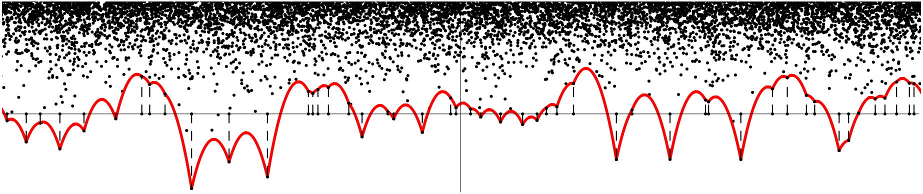

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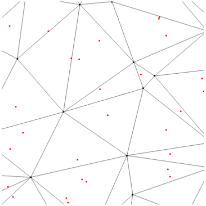

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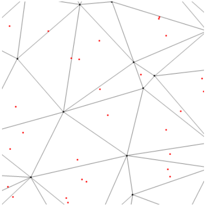

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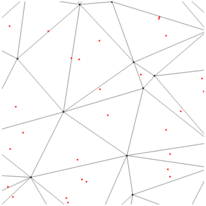

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$$\mathbb{E}\sigma_0(Z(\mathcal{D}^d_{\gamma,\beta})) = \binom{d+1}{k+1} J_{d+1,1}\left(\beta + \tfrac{1}{2}\right).$$

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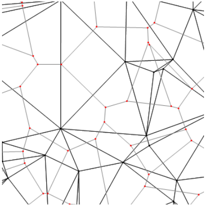
Expected intrinsic volumes of the typical cell of Poisson-Voronoi tessellation

As a consequence of exact stochastic representation we have:



- Explicit formulas for $\mathbb{E}V_d(Z(\mathcal{D}^d_{\gamma,\beta}))$ (via Gamma functions);
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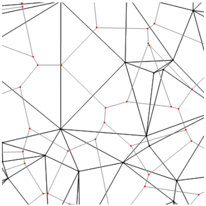

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Corollary (A.G., Z. Kabluchko and C. Thäle, 2023)

For any $0 \le j \le d$ we have

$$\mathbb{E}V_j(Z(\mathcal{V}^d_\gamma)) = \gamma^{-\frac{j}{d}} J_{d-j+1,1}\big(\tfrac{j-1}{2}\big) \frac{2^{d-j+1}\pi^{\frac{d-j}{2}}}{d(d-j)!} \frac{\Gamma(\frac{(d-j+1)(d-1)}{2}+1)}{\Gamma(\frac{(d-j+1)(d-1)+1}{2})} \frac{\Gamma(\frac{d}{2}+1)^{d-j+\frac{j}{d}}}{\Gamma(\frac{d+1}{2})^{d-j}} \frac{\Gamma(d-j+\frac{j}{d})}{\Gamma(\frac{j+1}{2})}.$$

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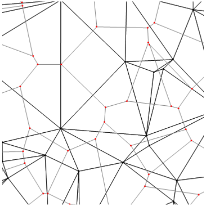

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$$\mathbb{E}V_j(Z(\mathcal{V}^d_\gamma)) \Leftarrow \gamma_{d-j}(\mathcal{V}^d_\gamma \cap L_{d-j}) \Leftarrow \gamma_{d-j}(\mathcal{V}^{d-j}_{\gamma_d, j/2-1})$$

High-dimensional limits

Corollary

For any $d \geq 2$, $1 \leq \ell \leq d-1$ and $L_\ell \in A(d,\ell)$ we have

$$\mathbb{E} V_\ell(Z(\mathcal{V}_\gamma^d \cap L_\ell)) = \frac{d\gamma^{-\frac{\ell}{d}}}{2J_{\ell+1,1}(\frac{d-\ell-1}{2})\pi^{\frac{\ell}{2}}} \frac{\Gamma(\frac{(\ell+1)(d-1)+1}{2})}{\Gamma(\frac{(\ell+1)(d-1)}{2}+1)} \frac{\Gamma(\frac{\ell+2}{2})}{\Gamma(\ell+1-\frac{\ell}{d})} \frac{\Gamma(\frac{d+1}{2})^{\ell+1}}{\Gamma(\frac{d}{2}+1)^{\ell+1-\frac{\ell}{d}}}.$$

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Corollary

For any $\ell \in \mathbb{N}$ and $L_\ell \in A(d,\ell)$ we have

$$\lim_{d\to\infty} \mathbb{E} V_\ell(Z(\mathcal{V}_\gamma^d \cap L_\ell)) = (\pi e)^{-\frac{\ell}{2}} J_{\ell+1,1}(\infty)^{-1} \frac{\Gamma(\frac{\ell}{2})}{2(\ell-1)!\sqrt{\ell+1}},$$

where $J_{\ell+1,1}(\infty)$ is a solid angle of a regular ℓ-dimensional simplex.

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Remark: Case $\ell = 1, 2$ was obtained by Miles, 1984.

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For any $\ell \in \mathbb{N}$ and $L_\ell \in A(d,\ell)$ we have

$$\lim_{d\to\infty} \mathbb{E} V_1(Z(\mathcal{V}_\gamma^d \cap L_\ell)) = (2e)^{-\frac{1}{2}},$$

where $J_{\ell+1,1}(\infty)$ is a solid angle of a regular ℓ-dimensional simplex.

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High-dimensional limits

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For any $d \geq 2$, $1 \leq \ell \leq d-1$ and $L_\ell \in A(d,\ell)$ we have

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Corollary

For any $\ell \in \mathbb{N}$ and $L_\ell \in A(d,\ell)$ we have

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where $J_{\ell+1,1}(\infty)$ is a solid angle of a regular ℓ-dimensional simplex.

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High-dimensional limits

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Observation: $\lim_{d\to\infty} \mathbb{E} V_\ell(Z(\mathcal{V}_\gamma^d \cap L_\ell)) = \mathbb{E} V_\ell(Z(\widetilde{\mathcal{V}}_{\pi e}^\ell))$.

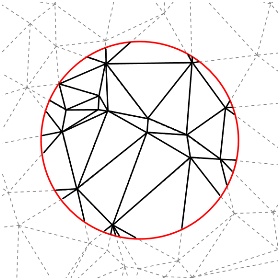
Connection between the models

Theorem (A.G., Z. Kabluchko and C.Thäle, 2023)

For any $\ell \in \mathbb{N}$, $L_\ell \in A(d,\ell)$ and every ball $B_R \subset \mathbb{R}^d$ of radius $R > 0$ centered at the origin we have

$$\lim_{d\to\infty} \mathbb{P}\Big[\bigcup_{t \in (\mathcal{V}_\gamma^d \cap L_\ell)} (\mathrm{bd}\, t \cap B_R) = \bigcup_{t \in \widetilde{\mathcal{V}}_{\pi e}^\ell} (\mathrm{bd}\, t \cap B_R) \Big] = 1.$$

And $Z(\mathcal{V}_\gamma^d \cap L_\ell) \xrightarrow{d} Z(\widetilde{\mathcal{V}}_{\pi e}^\ell)$ as $d \to \infty$.



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- A. Gusakova, Z. Kabluchko and C. Thäle. **The β-Delaunay tessellation I: Description of the model and geometry of typical cells** - Journal of Applied Probability, 2022.
- A. Gusakova, Z. Kabluchko and C. Thäle. **The β-Delaunay tessellation II: The Gaussian limit tessellation** - Electronic Journal of Probability, 2022.
- A. Gusakova, Z. Kabluchko and C. Thäle. **Sectional Voronoi tessellations: Characterization and high-dimensional limits** - To appear in Bernoulli, 2023.

Thank you for attention!