Borsuk's Partition Problem, Hadwiger's Covering Conjecture, and the Boltyanski-Gohberg Conjecture

Chuanming Zong, Tianjin University

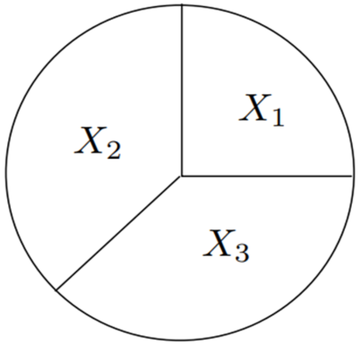
On the Occasion of Prof. DDr. Christian Buchta's Retirement

1、Borsuk's Partition Problem

Let X be a bounded set in E^n. We define its diameter as

$$d(X) = \sup\{\|\mathbf{x}, \mathbf{y}\|: \mathbf{x}, \mathbf{y} \in X\}.$$

In 1933, K. Borsuk proposed the following problem: In E^n, can every bounded set be divided into n+1 subsets of smaller diameters?



Let $\alpha(X)$ denote the smallest number such that X can be divided into $\alpha(X)$ subsets of smaller diameters. $\alpha(X) \leq n+1$?

Main Known Results

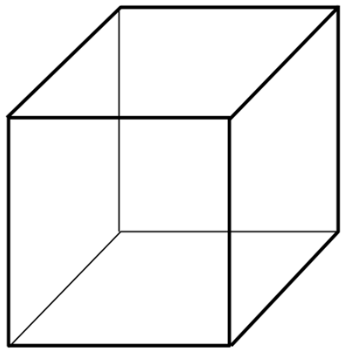
Theorem 1.1 (Bonnesen and Fenchel, 1934). In E^2, every bounded set X can be divided into three subsets of diameters at most $\left(\sqrt{3}/2\right) d(X)$.

Theorem 1.2 (Perkal, 1947; Eggleston …). In E^3, every bounded set X can be divided into four subsets of diameters at most $0.9887d(X)$.

Theorem 1.3 (Kahn and Kalai, 1993; …). When $n \geq 64$, there is a bounded set X in E^n satisfying $\alpha(X) > n + 1$.

2、Hadwiger's Covering Conjecture

In E^n, let K be a convex body and let $\gamma(K)$ denote the smallest number of the translates of $int(K)$ (or λK with $0 < \lambda < 1$) that their union can cover K.



In 1957, H. Hadwiger made the following conjecture: In E^n,

$$\gamma(K) \leq 2^n$$

holds for every convex body K, where the equality holds if and only if K is a parallelopiped.

Main Known Results

Theorem 2.1 (Levi, 1955). In E^2, every convex domain K can be covered by four translates of $int(K)$.

Theorem 2.2 (Lassak, 1989). In E^3, every centrally symmetric convex body C can be covered by eight translates of $int(C)$.

Theorem 2.3 (Rogers and Zong, 1997; …). In E^n, we have

$$\gamma(K) \le 4^{(1+o(1))n}$$

and, in particular,

$$\gamma(C) \le 2^{(1+o(1))n}.$$

Remark 2.1. Hadwiger's conjecture has several equivalent forms. For example, Boltyanski's illumination conjecture and Bezdek's separation conjecture.

Remark 2.2. There are many partial results about Borsuk's problem and Hadwiger's conjecture. If you want references, please write to me at: cmzong@math.pku.edu.cn.

3、The Boltyanski-Gohberg Conjecture

As a generalization of Borsuk's problem in normed linear spaces, V. Boltyanski and I. Gohberg proposed the following conjecture in 1965： In any given n-dimensional normed linear space $\{R^n, \|.\|\}$， every bounded set X can be divided into 2^n subsets of smaller diameters.

In $\{R^n, \|.\|\}$, let $\beta(X)$ denote the smallest number such that X can be divided into $\beta(X)$ subsets of smaller diameters. He conjectured that $\beta(X) \leq 2^n$.

Lemma 3.1. Let $\overline{X}$ denote the closure of the convex hull of X， then

$$\beta(X) \leq \gamma(\overline{X}).$$

Theorem 3.1 (Grünbaum， 1957). For every bounded set X in a two-dimensional normed linear space $\{R^2, \|.\|\}$, we have

$$\beta(X) \leq 4.$$

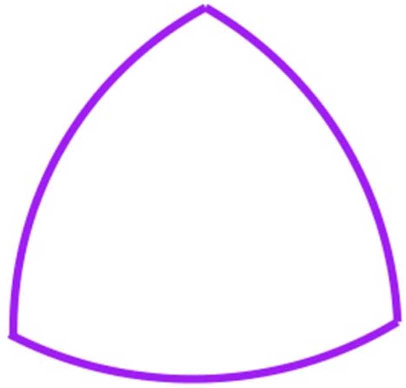
Corollary of Theorem 2.3. In $\{R^n, \|.\|\}$, for every bounded set X we have

$$\beta(X) \leq 4^{(1+o(1))n}.$$

4、 A Computer Program

First, we recall some well-known results in convexity:

Lemma 4.1 (Pál, 1920; Lebesgue, 1921). For every bounded set X in E^n there is a convex body C of constant width satisfying both $X \subseteq C$ and $d(X) = d(C)$.



Remark 4.1. Clearly, we have $\alpha(X) \leq \alpha(C)$. Therefore, to resolve Borsuk's problem in E^n, it is sufficient to deal with all the convex bodies of constant width 1.

Lemma 4.2 (Eggleston, 1958). Assume that C is an n-dimensional convex body of unit constant width. First, its insphere and circumsphere are concentric. Let r and R be their radii respectively, then we have

$$1-\sqrt{\frac{n}{2n+2}} \leq r \leq R \leq \sqrt{\frac{n}{2n+2}}.$$

Remark 4.2. Therefore, to resolve Borsuk's problem in E^n, it is sufficient to deal with all the convex bodies K satisfying $t\mathrm{B} \subseteq \mathrm{K} \subseteq \mathrm{B}$, where B is the unit ball centered at the origin and $t=\sqrt{(2n+2)/n}-1$.

Let G_n denote the set of all n-dimensional convex bodies K satisfying $B \subseteq K \subseteq tB$ associated with the Hausdorff metric, where $t = \left(\sqrt{(2n+2)/n} - 1\right)^{-1}$.

Remark 4.3. The set G_n is compact, with $\delta^{H}(K, L) = \inf\{r: K \subset L + rB,\ L \subset K + rB\}$.

For $K \in G_n$, we define $f(K)$ to be the smallest number such that K can be divided into n+1 subsets K_1, K_2, …, K_{n+1} satisfying

$$d(K_i) \le f(K)\, d(K), \qquad i = 1, 2, \dots, n+1.$$

Clearly, Borsuk's problem equivalent to: Does $f(\mathrm{K})<1$ hold for all $\mathrm{K} \in \mathrm{G}_n$?

Lemma 4.3 (Zong, 2022). The functional $f(\mathrm{K})$ is continuous on G_n. In particular, if $\|\mathrm{K}_1, \mathrm{K}_2\| \leq \varepsilon$, we have

$$|f(\mathrm{K}_1) - f(\mathrm{K}_2)| \leq 2\varepsilon.$$

By Remark 4.3 and Lemma 4.3, one can obtain:

Theorem 4.1 (Zong, 2022)**.** If Borsuk's problem has a positive answer in E^n, one can prove it by checking finite number of polytopes.

Borsuk's Problem in E^4, as an Example:

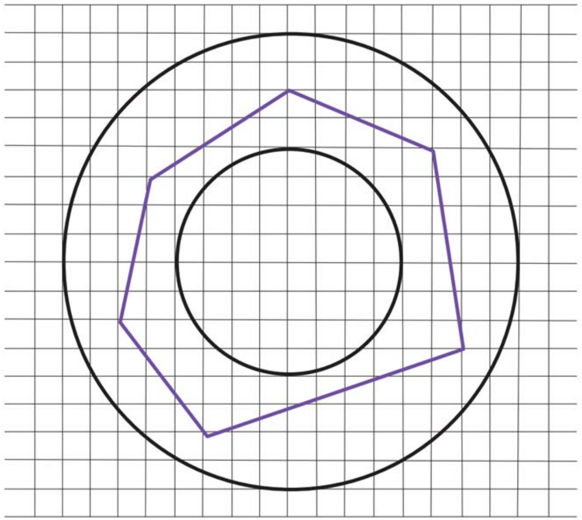
If every bonded set X in E^4 can be divided into five subsets $X_1, X_2, \ldots, X_5$ satisfying

$$d(X_i) \leq 0.995\, d(X), \qquad (4.1)$$

to resolve Borsuk's problem in E^4 it is sufficient to checking (4.1) for all $\mathbb{Z}^4$ lattice polytopes P satisfying

$$1598\, B^4 \subset P \subset 2756\, B^4,$$

Where B^4 is the 4-dimensional unit ball centered at the origin.



Question 4.1. Can this method verify the plane case?

Answer: Not yet. Although the considered lattice polygons are apparently finite, to enumerate them by a computer is very hard!

Question 4.2. Can one make similar approaches to Hadwiger's conjecture and the Boltyanski-Gohberg conjecture?

Answer: Yes, we did.

5、The Boltyanski-Gohberg Conjecture

If $\{\mathrm{R}^n, \|.\|\}$ is an n-dimensional normed linear space, then

$$\mathrm{C} = \{\mathbf{x} \in \mathrm{R}^n \colon \|\mathbf{o}, \mathbf{x}\| \leq 1\}$$

is a centrally symmetric convex body centered at the origin. On the other hand, if C is a centrally symmetric convex body centered at the origin of R^n, then

$$\|\mathbf{x}, \mathbf{y}\| = \min\{r > \mathbf{0} \colon \mathbf{x} - \mathbf{y} \in r\mathrm{C}\}$$

defines an norm on R^n.

Let F_n denote the family of all convex bodies K in $\{\mathrm{R}^n, \|.\|\}$, define $\mathrm{D(K)} = \{\mathbf{x} - \mathbf{y} \colon \mathbf{x}, \mathbf{y} \in \mathrm{K}\}$, and let $\beta_C(\mathrm{X})$ denote the smallest number m such that X can be divided into m subsets of smaller diameters.

Lemma 5.1 (Wang, Xue and Zong, 2023). In $\{R^n, \|.\|\}$, we have

$$\max_{K \in F_n} \beta_C(K) \leq \max_{K \in F_n} \beta_{D(K)}(K).$$

Remark 5.1. By this lemma, to prove the Boltyanski-Gohberg conjecture or to obtain general upper bounds for $\beta_C(X)$ it is sufficient to deal with $\beta_{D(K)}(K)$.

Assume that K is a convex body in R^n. We embed it into

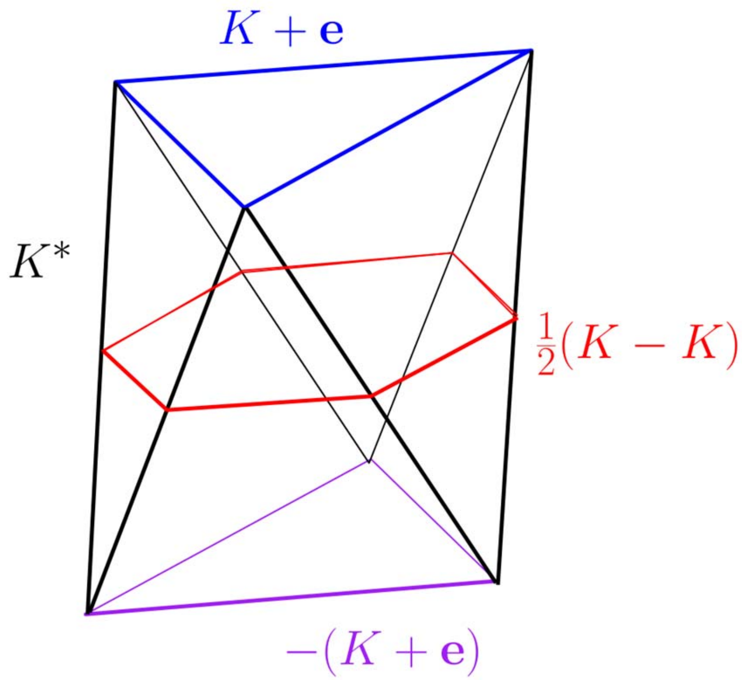
$$R^{n+1} = R^n \oplus R.$$

In R^{n+1} we take $\mathbf{e}$=(0, 0, …, 0, 1) and define K^* to be the convex hull of

$$(K + \mathbf{e}) \cup (-K - \mathbf{e})$$

Clearly, K^* is a centrally symmetric convex body in R^{n+1} and, therefore,

$$D(K^*)=2K^*.$$



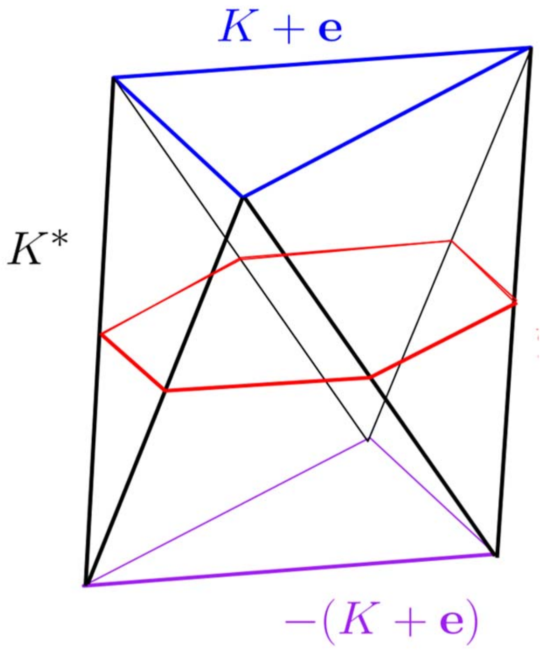
Lemma 5.2 (Wang, Xue and Zong, 2023).

$$\beta_{D(\mathrm{K}^*)}(\mathrm{K}^*) = 2\beta_{D(\mathrm{K})}(\mathrm{K}).$$

Remark 5.2. The key observation for the proof is

$$\|\mathbf{x},\mathbf{y}\| = 1$$

whenever $\mathbf{x} \in \mathrm{K} + \mathbf{e}$ and $\mathbf{y} \in -(\mathrm{K} + \mathbf{e})$, where $\|,\|$ is the norm of $D(\mathrm{K}^*)$.



Theorem 5.1 (Wang, Xue and Zong, 2023).
In $\{\mathrm{R}^n, \|.\|\}$, for every bounded set X we have

$$\beta(X) \le 2^{(1+o(1))n}.$$

The BG Conjecture:

$$\beta(X) \le 2^n.$$

Proof Idea:

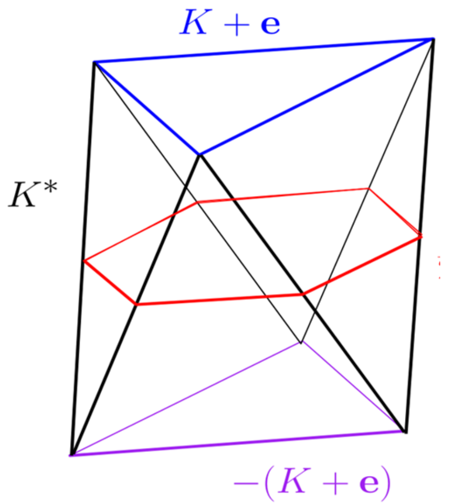
$$\mathrm{K} = \mathrm{conv}(\mathrm{X}),$$

$$\mathrm{K}^* = \mathrm{conv}\big((\mathrm{K} + \mathbf{e}) \cup (-\mathrm{K} - \mathbf{e})\big),$$

$$2\beta_{D(\mathrm{K})}(\mathrm{K}) = \beta_{D(\mathrm{K}^*)}(\mathrm{K}^*),$$

$$\beta_{D(\mathrm{K}^*)}(\mathrm{K}^*) \le \gamma(\mathrm{K}^*),$$

$$\gamma(\mathrm{K}^*) \le 2^{(1+o(1))(n+1)}.$$



References：

1. K. Bezdek and M. A. Khan, The geometry of homothetic covering and illumination. Discrete Geometry and Symmetry, Springer Proc. Math. Stat., 234, Springer, Cham, 2018, 1-30.
2. V. G. Boltyanski and I. T. Gohberg, Results and Problems in Combinatorial Geometry, Cambridge University Press, 1985; Nauka, Moscow 1965.
3. J. Kahn and G. Kalai, A counterexample to Borsuk's conjecture, Bull. Amer. Math. Soc. (N.S.) 29 (1993), 60-62.
4. C. Zong, Borsuk's partition conjecture. Japan. J. Math. 16 (2021), 185-201.
5. J. Wang, F. Xue and C. Zong, Borsuk's problem in normed spaces, Bull. LMS, in press.

Thank You Very Much!