FACULTY
OF MATHEMATICS
AND PHYSICS
Charles University

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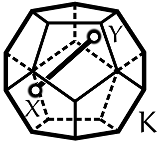
Mean distance in convex polyhedra, mean tetrahedron volume and related problems

Conference on Convex Geometry and Geometric Probability, Salzburg

September 2023



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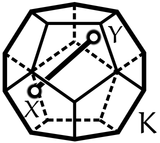
- $X, Y \in K$ uniformly and independently selected

$$\Lambda(K) := \frac{\mathbb{E}\,\|X - Y\|}{\sqrt[d]{\operatorname{vol}_d K}}$$



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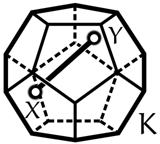
K	MC	$\Lambda(K)$
ball[1]	0.6381	$\frac{18}{35}\sqrt[3]{\frac{6}{\pi}}$
icosahedron	0.6413	?
dodecahedron	0.6425	?
octahedron	0.6585	?
cube[2]	0.6617	$\frac{4}{105} + \frac{17\sqrt{2}}{105} - \frac{2\sqrt{3}}{35} - \frac{\pi}{15} + \frac{1}{5}\operatorname{arccoth}\sqrt{2} + \frac{4}{5}\operatorname{arccoth}\sqrt{3}$
tetrahedron	0.7295	?

[1] trivial

[2] David Robbins and Theodore Bolis. "Average Distance between Two Points in a Box.". In: *Amer. Math. Monthly* 85 (1978), p. 278



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icosahedron	0.6413	?
dodecahedron	0.6425	?
octahedron	0.6585	$\sqrt[3]{\frac{3}{4}}\left(\frac{4}{105}+\frac{13\sqrt{2}}{105}-\frac{4\pi}{45}+\frac{109\ln 3}{630\sqrt{2}}+\frac{16\operatorname{arccot}\sqrt{2}}{315}+\frac{158\operatorname{arccoth}\sqrt{2}}{315}\sqrt{2}\right)$
cube[2]	0.6617	$\frac{4}{105}+\frac{17\sqrt{2}}{105}-\frac{2\sqrt{3}}{35}-\frac{\pi}{15}+\frac{1}{5}\operatorname{arccoth}\sqrt{2}+\frac{4}{5}\operatorname{arccoth}\sqrt{3}$
tetrahedron	0.7295	$\sqrt[3]{3}\left(\frac{\sqrt{2}}{7}-\frac{37\pi}{315}+\frac{4}{15}\arctan\sqrt{2}+\frac{113\ln 3}{210\sqrt{2}}\right)$

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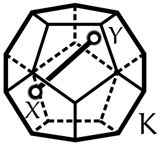

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Platonic solids – new results

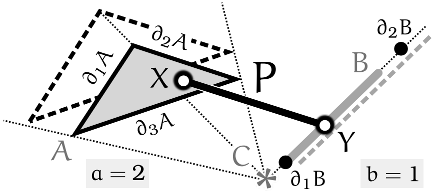
icosahedron	$\frac{1}{2}\sqrt[3]{\frac{9}{5}-\frac{3}{\sqrt{5}}}\left(\frac{197}{525}+\frac{239}{525\sqrt{5}}-\frac{44}{525}\sqrt{\frac{2\left(5+\sqrt{5}\right)}{5}}-\frac{\left(17226+6269\sqrt{5}\right)\pi}{157500}-\frac{\left(2186+1413\sqrt{5}\right)\operatorname{arccot}\phi}{15750}+\frac{\left(82-75\sqrt{5}\right)\operatorname{arccot}\left(\phi^{2}\right)}{5250}+\frac{4\left(2139+881\sqrt{5}\right)\operatorname{arccsch}\phi}{7875}+\frac{\left(15969+7151\sqrt{5}\right)\operatorname{arccoth}\phi}{12600}+\frac{\left(4449-1685\sqrt{5}\right)\ln 3}{42000}-\frac{\left(75783+37789\sqrt{5}\right)\ln 5}{252000}\right)$
dodecahedron	$\frac{1}{\sqrt[3]{30+14\sqrt{5}}}\left(\frac{1516}{1575}+\frac{2\sqrt{\frac{2}{5}}}{45}-\frac{124\sqrt{\frac{3}{5}}}{175}-\frac{71\sqrt{2}}{1575}-\frac{12\sqrt{3}}{35}+\frac{342}{175\sqrt{5}}+\frac{493\pi}{23625}+\frac{67\pi}{945\sqrt{5}}+\frac{\left(397-244\sqrt{5}\right)\operatorname{arccot}2}{18900}+\frac{\left(24023+11788\sqrt{5}\right)\left(\arccos\frac{2}{3}-\arccos\frac{1}{3}\right)}{94500}-\frac{\left(461+212\sqrt{5}\right)\left(\arccos\frac{23}{41}+\arccos\frac{39}{41}\right)}{1000}-\frac{\left(1031+521\sqrt{5}\right)\operatorname{arccosh}\frac{13}{3}}{75600}+\frac{\left(367+163\sqrt{5}\right)\operatorname{arccosh}9}{16800}+\frac{\left(22197+8149\sqrt{5}\right)\left(\operatorname{arccosh}\frac{121}{41}-\operatorname{arccosh}\frac{57}{41}\right)}{84000}+\frac{\left(15763+7063\sqrt{5}\right)\left(\operatorname{arccosh}\frac{7}{3}-\operatorname{arccosh}3\right)}{21000}+\frac{2\left(423+187\sqrt{5}\right)\left(\operatorname{arccosh}4-\operatorname{arccosh}2\right)}{875}+\frac{\left(288889+129739\sqrt{5}\right)\ln 3}{378000}+\frac{\left(109-3143\sqrt{5}\right)\ln 5}{151200}\right)$

Crofton Reduction Technique

Definition. $P_{AB} = \mathbb{E}\left[P(X,Y) \,|\, X \in A, Y \in B, \text{uniform and independent}\right]$, $L_{AB}^{(p)} = P_{AB}$ with $P = L^p = \|X - Y\|^p$, $\mathcal{A}(A)$ affine hull of A, $\mathcal{P}(\mathbb{R}^d)$ set of all polytopes in $\mathbb{R}^d$.

Lemma (CRT)[3]**.** Let $P : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be homogeneous of order p (scales like L^p) and $A, B \in \mathcal{P}(\mathbb{R}^d)$, $a = \dim A$, $b = \dim B$, then for any $C \in \mathcal{A}(A) \cap \mathcal{A}(B)$ (scaling point)

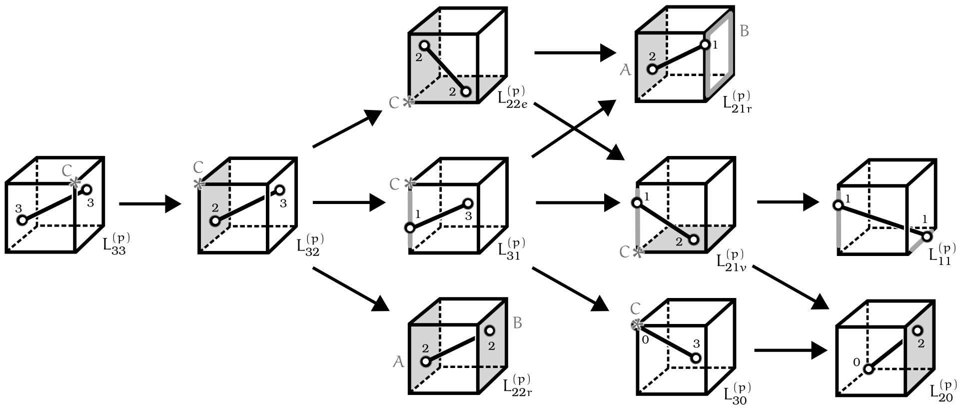
$$pP_{AB} = a(P_{\partial A\, B} - P_{AB}) + b(P_{A\, \partial B} - P_{AB}).$$



[3] H Ruben and WJ Reed. "A more general form of a theorem of Crofton". In: *Journal of Applied Probability* (1973), pp. 479–482.

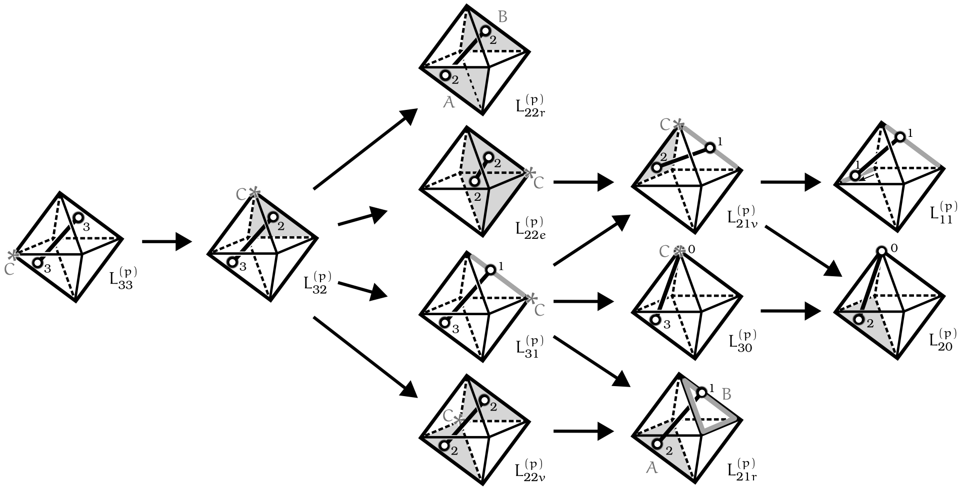
Cube line picking

$L_{ab}^{(p)}$ configurations

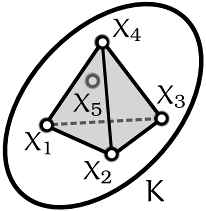


$$L_{33} = \frac{3L_{11}}{35} + \frac{3L_{20}}{35} + \frac{8L_{21r}}{35} + \frac{L_{22r}}{7}.$$

Octahedron line picking — $L_{ab}^{(p)}$ configurations



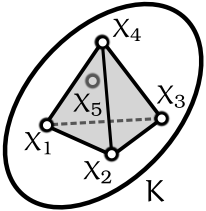
3 D. B. "Mean distance in polyhedra". In: *arXiv preprint arXiv:2309* (2023)



- $K \subset \mathbb{R}^d$ convex compact, $\dim K = d$
- $X_1, \ldots, X_n \in K$ uniform and independent
- $H_n = \text{hull}(X_1, \ldots, X_n)$.

$$V_n(K) := \frac{\mathbb{E}\,\text{vol}_d\, H_n}{\text{vol}_d\, K}$$

- $V(K) := V_{d+1}(K)$ mean simplex volume



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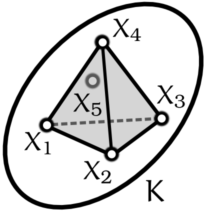
K	Monte Carlo	$V(K)$
ball[a]	0.01259	$\frac{9}{715}$
icosahedron	0.01277	?
dodecahedron	0.01282	?
octahedron	0.01363	?
cube[b]	0.01384	$\frac{3977}{216000} - \frac{\pi^2}{2160}$
tetrahedron[c]	0.01740	$\frac{13}{720} - \frac{\pi^2}{15015}$

[a] Bohuslav Hostinský. *Sur les probabilités géométriques*. Přírodovědecká fakulta Masarykovy university, 1925.

[b] Alessandro Zinani. "The expected volume of a tetrahedron whose vertices are chosen at random in the interior of a cube". In: *Monatshefte für Mathematik* 139.4 (2003), pp. 341–348.

[c] Christian Buchta and Matthias Reitzner. "What is the expected volume of a tetrahedron whose vertices are chosen at random from a given tetrahedron?" In: *Anz. Österreich. Akad. Wiss. Math.-Natur. Kl* 129 (1992), pp. 63–68.

Definition



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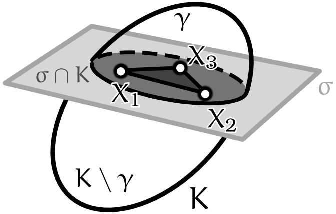
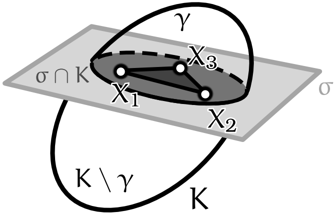


- $\sigma = \mathcal{A}(X_1, \ldots, X_d)$, $\Sigma = \{\sigma \,|\, \sigma \cap K \neq \emptyset\}$
- γ and $K \setminus \gamma$ two halves on which K is divided by σ
- $\Gamma = \mathrm{vol}_d\, \gamma / \mathrm{vol}_d\, K$

$$\begin{cases} \mathbb{E}\, f_0(H_n) = n(1 - V_{n-1}(K)) \\ \mathbb{E}\, f_d(H_n) = \binom{n}{d} \mathbb{E}\left[\Gamma^{n-d} + (1-\Gamma)^{n-d}\right] \end{cases}$$

4 Bradley Efron. "The convex hull of a random set of points". In: *Biometrika* 52.3-4 (1965), pp. 331–343



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Theorem. (Efron[4]) Efron's formula

If $\dim K = 3$, we can connect $f_0(H_n)$ with $f_2(H_n)$ via Euler's polyhedral formula, and thus

$$V_{n-1}(K) = 1 - \frac{2}{n} - \frac{1}{2n}\binom{n}{3}\mathbb{E}\left[\Gamma^{n-3} + (1-\Gamma)^{n-3}\right].$$

Corollary. $\dim K = 3,\ n = 5$: $\quad V(K) = \frac{3}{5} - \mathbb{E}\left[\Gamma^2 + (1-\Gamma)^2\right]$

[4] Bradley Efron. "The convex hull of a random set of points". In: *Biometrika* 52.3-4 (1965), pp. 331–343

Lemma. (Kingman[5], Zinani[6])

$$\mathbb{E}\left[\Gamma^{n-d}+(1-\Gamma)^{n-d}\right]=\int_{K^d}(\Gamma^{n-d}+(1-\Gamma)^{n-d})\frac{\mathrm{d}X_1\ldots\mathrm{d}X_d}{(\mathrm{vol}_d\,K)^d}$$

$$=\int_{\Sigma}(\Gamma^{n-d}+(1-\Gamma)^{n-d})\mathrm{vol}_{d-1}(\sigma\cap K)^{d+1}V(\sigma\cap K)m(\mathrm{d}\sigma),$$

where $m(\mathrm{d}\sigma)$ is affine invariant measure on $SA_{d-1}(\mathbb{R}^d)$.

- The integral is integrated over all plane sections
- For octahedron, there are 3 types of sections according to different vertices separation
- The integration itself is performed with the help of *Mathematica* CAS

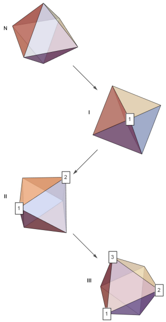


Figure 1: Octahedron sections genealogy

[5] John FC Kingman. "Random secants of a convex body". In: *Journal of Applied Probability* (1969), pp. 660–672

[6] Alessandro Zinani. "The expected volume of a tetrahedron whose vertices are chosen at random in the interior of a cube". In: *Monatshefte für Mathematik* 139.4 (2003), pp. 341–348

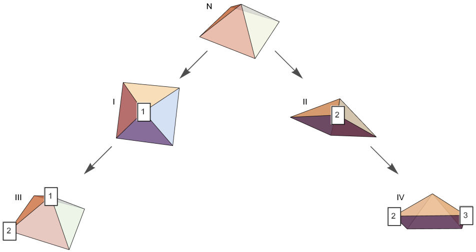


Figure 2: Square pyramid sections genealogy

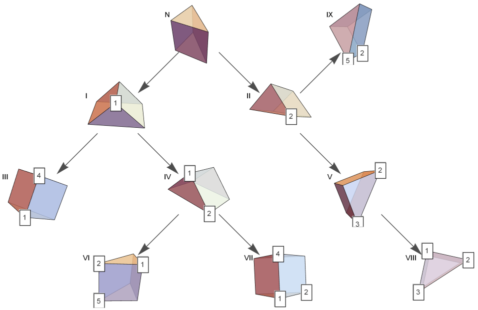


Figure 3: Triangular prism sections genealogy

K	Monte Carlo	$V(K)$
square pyramid	0.01573	$\frac{941\pi^2}{72072} - \frac{977}{8640}$
triangular prism	0.01536	$\frac{1859}{116640} - \frac{\pi^2}{17010}$

Yet another genealogies

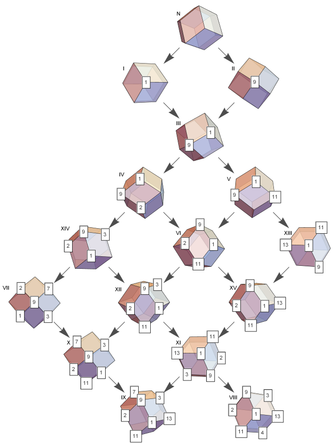


Figure 4: Rhombic dodecahedron sections genealogy

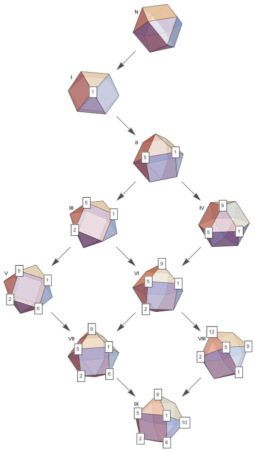


Figure 5: Cuboctahedron sections genealogy

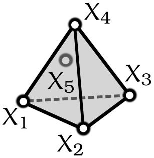
... and new results

K	MC	$V(K)$
rhombic dodecahedron	0.0129385	$\frac{2421179003623}{17933819904000}+\frac{37061863\pi^2}{29889699840}-\frac{9406373047\ln 2}{9340531200}-\frac{1757220593\ln^2 2}{2490808320}+\frac{282589831\ln 3}{283852800}-\frac{6078271\,\mathrm{Li}_2\left(\frac{1}{4}\right)}{8515584}$
cuboctahedron	0.0130025	$\frac{117410162173}{525525000000}+\frac{8752199\pi^2}{2402400000}-\frac{192940695481\ln 2}{105105000000}-\frac{318759601\ln^2 2}{250250000}+\frac{506316394917\ln 3}{280280000000}-\frac{648098487\,\mathrm{Li}_2\left(\frac{1}{4}\right)}{500500000}$
truncated tetrahedron	0.0148451	$\frac{35604506258521}{162358039443600}-\frac{13447020779\pi^2}{96641690145}+\frac{9972537226592\ln 2}{3382459155075}+\frac{3485442712\ln^2 2}{1400604205}-\frac{8953623027\ln 3}{7884520175}-\frac{53493528168\ln 2\ln 3}{32213896715}+\frac{53162662164\,\mathrm{Li}_2\left(\frac{1}{4}\right)}{32213896715}$

Observation. If $n = d + 2$, there are a.s. only two possible configurations:

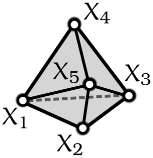
- T_d d-dimensional simplex
- T_d^+ two T_d sharing one face

T_3 :



$$\begin{cases} f_0 = 4 \\ f_2 = 4 \end{cases}$$

T_3^+ :



$$\begin{cases} f_0 = 5 \\ f_2 = 6 \end{cases}$$

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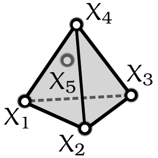
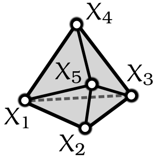


$\begin{cases} f_0 = 4 \\ f_2 = 4 \end{cases}$

T_3^+ :



$\begin{cases} f_0 = 5 \\ f_2 = 6 \end{cases}$

Invariant: $2f_0 - f_2 = 8 - 4 = 10 - 6 = 4$,

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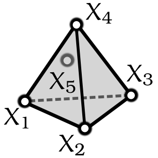

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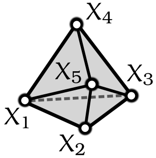

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Invariant: $2f_0 - f_2 = 8 - 4 = 10 - 6 = 4$,

$$\begin{cases} f_0(T_d) = d + 1 \\ f_{d-1}(T_d) = d + 2 \end{cases} \qquad \begin{cases} f_0(T_d^+) = d + 2 \\ f_{d-1}(T_d^+) = 2d \end{cases}$$

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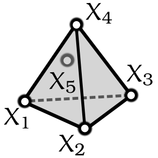
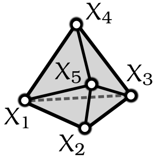


$$\begin{cases} f_0 = 4 \\ f_2 = 4 \end{cases}$$

T_3^+ :



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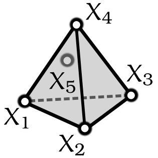
$$\begin{cases} f_0(T_d) = d + 1 \\ f_{d-1}(T_d) = d + 2 \end{cases} \qquad \begin{cases} f_0(T_d^+) = d + 2 \\ f_{d-1}(T_d^+) = 2d \end{cases}$$

$$(d-1)f_0 - f_{d-1} = (d-2)(d+1)$$

Observation. If $n = d + 2$, there are a.s. only two possible configurations:

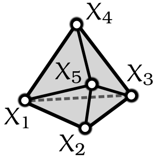
- T_d d-dimensional simplex
- T_d^+ two T_d sharing one face

T_3 :



$$\begin{cases} f_0 = 4 \\ f_2 = 4 \end{cases}$$

T_3^+ :



$$\begin{cases} f_0 = 5 \\ f_2 = 6 \end{cases}$$

Invariant: $2f_0 - f_2 = 8 - 4 = 10 - 6 = 4$,

$$\begin{cases} f_0(T_d) = d + 1 \\ f_{d-1}(T_d) = d + 2 \end{cases} \qquad \begin{cases} f_0(T_d^+) = d + 2 \\ f_{d-1}(T_d^+) = 2d \end{cases}$$

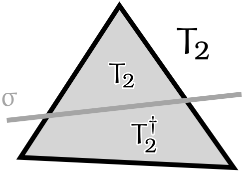
$$(d-1)f_0 - f_{d-1} = (d-2)(d+1)$$

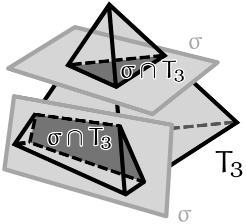
Theorem (12/2020 | B. D.[7]). Efron's formula in higher dimensions.

Let $\dim K = d$, then

$$V(K) = \frac{2d}{(d-1)(2+d)} - \frac{d+1}{2(d-1)} \mathbb{E}\left[\Gamma^2 + (1-\Gamma)^2\right],$$

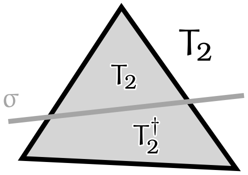
[7] D. B. "Efron's Mean Volume Formula in Higher Dimensions". In: *arXiv preprint arXiv:2308.02854* (2023)

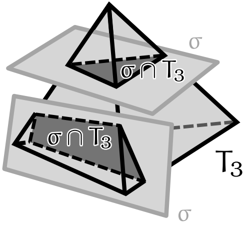




Observation. $\sigma \cap T_{d+1} \in \{T_d, T_d^\dagger\}$,

- $T_d \subset \mathbb{R}^d$ denotes d-simplex
- $\gamma(T_d) \in \{T_d, T_d^\dagger\}$, $T_d^\dagger$ truncated T_d





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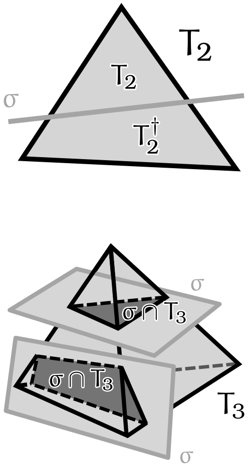
Therefore, $\sigma \cap T_4 \in \{T_3, T_3^\dagger\}$, so in order to calculate

$$V(T_4) = \frac{2}{9} - \frac{5}{6} \int_\Sigma \left(\Gamma^2 + (1-\Gamma)^2\right) \mathrm{vol}_3(\sigma \cap T_4)^5 V(\sigma \cap T_4) m(\mathrm{d}\sigma),$$

we only need to compute $V(T_3^\dagger)$ since $V(T_3) = \frac{13}{720} - \frac{\pi^2}{15015}$.



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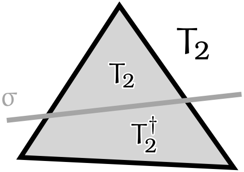
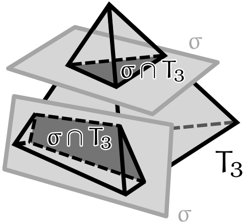
- **Monte Carlo.** $V(T_4) \in (0.00318034, 0.00318043)$ the 95% confidence interval.





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- **Monte Carlo.** $V(T_4) \in (0.00318034, 0.00318043)$ the 95% confidence interval.
- **Conjecture.** $V(T_4) = p_1 + p_2\zeta(2) + p_3\zeta(3) + p_4\zeta(4) + p_5\zeta(5) + p_6\zeta(2)\zeta(3)$, where p_i are rationals

- Consider a disk D of unit radius in which we pick three points randomly uniformly forming a triangle with side lengths a, b, c and perimeter $P = a + b + c$. We solved $\mathbb{E}\left[P^2\right]$ proposed by Finch[8] and many others means:

[8] Steven Finch. "Perimeter variance of uniform random triangles". In: *arXiv preprint arXiv:1007.0261* (2010).

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Function	Monte Carlo	Exact mean value
ab	0.8378	$\frac{3383}{432\pi^2} + \frac{35\zeta(3)}{96\pi^2}$
P^2	8.027	$3 + \frac{3383}{72\pi^2} + \frac{35\zeta(3)}{16\pi^2}$
abc	0.7531	$\frac{8(173+64\ln 2)}{735\pi}$
ab^2	0.9442	$\frac{4672}{1575\pi}$
P^3	23.351	$\frac{776848}{11025\pi} + \frac{1024\ln 2}{245\pi}$
a/b	1.5108	$\frac{289}{27\pi^2} + \frac{7\zeta(3)}{2\pi^2}$

[8] Steven Finch. "Perimeter variance of uniform random triangles". In: *arXiv preprint arXiv:1007.0261* (2010).

Triangle in a rectangle

Theorem. (Langford[9]) Probability of a triangle obtusity in a rectangle

Let K be a rectangle with aspect ratio $a \geqslant 1$. The probability $P(a)$ that a random triangle picked from K is obtuse is given by

$$P(a) = \begin{cases} \frac{1}{3} + \frac{47}{300}\left(a^2 + \frac{1}{a^2}\right) + \frac{1}{80}\pi\left(a^3 + \frac{1}{a^3}\right) - \frac{1}{5}\ln a\left(a^2 - \frac{1}{a^2}\right), & 1 \leqslant a \leqslant 2 \\ \frac{1}{3} + \frac{1}{a^2}\left(\frac{\pi}{80a} + \frac{47}{300} + \frac{\ln a}{5}\right) + \frac{47a^2}{300} - \frac{1}{5}a^2\ln a + \frac{a^3}{40}\arcsin\frac{2}{a} \\ \quad + \left(\frac{a^2}{10} - \frac{3}{5a^2}\right)\operatorname{arccosh}\frac{a^2-2}{2} + \frac{a\sqrt{a^2-4}}{150}\left(-31 + \frac{63}{a^2} + \frac{64}{a^4}\right), & a > 2 \end{cases}$$

[9] Eric Langford. "The probability that a random triangle is obtuse". In: *Biometrika* 56.3 (1969), pp. 689–690

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- **Corollary.** Probability a triangle is obtuse in a square is $P(1) = \frac{97}{150} + \frac{\pi}{40}$

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$$P(\alpha, \beta) = \begin{cases} P_{ac}(\alpha, \beta), & ABC \text{ acute} \\ P_{ob}(\alpha, \beta), & ABC \text{ obtuse at } C \end{cases}$$

(The functions P_{ac} and P_{ob} are on the next slide)

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Triangle in a triangle

$$
\begin{aligned}
&\frac{1}{1920}\Big(-128\,(\mathrm{Csc}[\gamma]^4\,\mathrm{Log}[\mathrm{Sin}[\alpha]\,\mathrm{Sin}[\beta]]+\mathrm{Csc}[\beta]^4\,\mathrm{Log}[\mathrm{Sin}[\alpha]\,\mathrm{Sin}[\gamma]]+\mathrm{Csc}[\alpha]^4\,\mathrm{Log}[\mathrm{Sin}[\beta]\,\mathrm{Sin}[\gamma]])\,\mathrm{Sin}[\alpha]^2\\
&\quad \mathrm{Sin}[\beta]^2\,\mathrm{Sin}[\gamma]^2+\mathrm{Csc}[\alpha]^2\,\mathrm{Csc}[\beta]^2\,\mathrm{Csc}[\gamma]^2\\
&\quad\big(-2\,(-36+20\,\mathrm{Cos}[2\alpha]+11\,\mathrm{Cos}[4\alpha]-2\,\mathrm{Cos}[2\,(\alpha-2\beta)]-2\,\mathrm{Cos}[4\alpha-2\beta]-\\
&\qquad 16\,\mathrm{Cos}[2\,(\alpha-\beta)]+\mathrm{Cos}[4\,(\alpha-\beta)]+20\,\mathrm{Cos}[2\beta]+11\,\mathrm{Cos}[4\beta]-16\,\mathrm{Cos}[2\,(2\alpha+\beta)]+\\
&\qquad \mathrm{Cos}[4\,(2\alpha+\beta)]-2\,\mathrm{Cos}[2\,(3\alpha+\beta)]-16\,\mathrm{Cos}[2\,(\alpha+2\beta)]+\mathrm{Cos}[4\,(\alpha+2\beta)]-\\
&\qquad 2\,\mathrm{Cos}[2\,(\alpha+3\beta)]-2\,\mathrm{Cos}[6\alpha+4\beta]-2\,\mathrm{Cos}[4\alpha+6\beta]+20\,\mathrm{Cos}[2\gamma]+11\,\mathrm{Cos}[4\gamma])+\\
&\quad \alpha\,(40\,\mathrm{Sin}[2\alpha]-5\,\mathrm{Sin}[4\alpha]+4\,\mathrm{Sin}[2\,(\alpha-2\beta)]+4\,\mathrm{Sin}[4\alpha-2\beta]-20\,\mathrm{Sin}[2\,(\alpha-\beta)]-\\
&\qquad \mathrm{Sin}[4\,(\alpha-\beta)]-20\,\mathrm{Sin}[2\,(2\alpha+\beta)]+8\,\mathrm{Cos}[\beta]\,\mathrm{Sin}[3\,(2\alpha+\beta)]-\mathrm{Sin}[4\,(2\alpha+\beta)])-\\
&\quad \beta\,(4\,\mathrm{Sin}[2\,(\alpha-2\beta)]+4\,\mathrm{Sin}[4\alpha-2\beta]-20\,\mathrm{Sin}[2\,(\alpha-\beta)]-\mathrm{Sin}[4\,(\alpha-\beta)]-40\,\mathrm{Sin}[2\beta]+\\
&\qquad 5\,\mathrm{Sin}[4\beta]+20\,\mathrm{Sin}[2\,(\alpha+2\beta)]-8\,\mathrm{Cos}[\alpha]\,\mathrm{Sin}[3\,(\alpha+2\beta)]+\mathrm{Sin}[4\,(\alpha+2\beta)])+\\
&\quad (\pi-\alpha-\beta)\,(20\,\mathrm{Sin}[2\,(2\alpha+\beta)]+\mathrm{Sin}[4\,(2\alpha+\beta)]-4\,\mathrm{Sin}[2\,(3\alpha+\beta)]+20\,\mathrm{Sin}[2\,(\alpha+2\beta)]+\mathrm{Sin}[\\
&\qquad 4\,(\alpha+2\beta)]-4\,\mathrm{Sin}[2\,(\alpha+3\beta)]-4\,\mathrm{Sin}[6\alpha+4\beta]-4\,\mathrm{Sin}[4\alpha+6\beta]+40\,\mathrm{Sin}[2\gamma]-5\,\mathrm{Sin}[4\gamma])\big)+\\
&\quad 2\,\mathrm{Csc}[\gamma]^2\,(64\,(6\,\mathrm{Log}[\mathrm{Sin}[\alpha]]\,\mathrm{Sec}[\alpha]^2\,\mathrm{Sin}[\beta]^2\,\mathrm{Tan}[\alpha]^2+\mathrm{Sin}[\alpha]^2\,(-1+\mathrm{Tan}[\beta]^2))+\\
&\qquad \mathrm{Sec}[\beta]^2\,((33+\mathrm{Cos}[4\alpha]\,(-17+\mathrm{Cos}[4\beta])+(-17+8\,\mathrm{Cos}[2\alpha])\,\mathrm{Cos}[4\beta])\,\mathrm{Sec}[\alpha]^2+\\
&\qquad 8\,(-1+\mathrm{Tan}[\alpha]^2+48\,\mathrm{Log}[\mathrm{Sin}[\beta]]\,\mathrm{Sin}[\alpha]^2\,\mathrm{Tan}[\beta]^2)))+\\
&\quad 2\,\mathrm{Csc}[\beta]^2\,(\mathrm{Sec}[\alpha]^2\,((33-8\,\mathrm{Cos}[2\gamma]-17\,\mathrm{Cos}[4\gamma]+\mathrm{Cos}[4\alpha]\,(-17+8\,\mathrm{Cos}[2\gamma]+\mathrm{Cos}[4\gamma]))\,\mathrm{Sec}[\gamma]^2-\\
&\qquad 64\,\mathrm{Sin}[\gamma]^2\,(\mathrm{Cos}[2\alpha]-6\,\mathrm{Log}[\mathrm{Sin}[\alpha]]\,\mathrm{Tan}[\alpha]^2))+384\,\mathrm{Log}[\mathrm{Sin}[\gamma]]\,\mathrm{Sec}[\gamma]^2\,\mathrm{Sin}[\alpha]^2\,\mathrm{Tan}[\gamma]^2)+\\
&\quad 2\,\mathrm{Csc}[\alpha]^2\,(\mathrm{Sec}[\beta]^2\,((33+\mathrm{Cos}[4\beta]\,(-17+\mathrm{Cos}[4\gamma])-17\,\mathrm{Cos}[4\gamma])\,\mathrm{Sec}[\gamma]^2-64\,\mathrm{Sin}[\gamma]^2\\
&\qquad (\mathrm{Cos}[2\beta]-6\,\mathrm{Log}[\mathrm{Sin}[\beta]]\,\mathrm{Tan}[\beta]^2))-64\,\mathrm{Sec}[\gamma]^2\,\mathrm{Sin}[\beta]^2\,(\mathrm{Cos}[2\gamma]-6\,\mathrm{Log}[\mathrm{Sin}[\gamma]]\,\mathrm{Tan}[\gamma]^2))+\\
&\quad 64\,(18+4\,\mathrm{Sec}[\beta]\,\mathrm{Sec}[\gamma]\,\mathrm{Sin}[\alpha]\,\mathrm{Tan}[\alpha]+(9-5\,\mathrm{Cos}[2\alpha]+\mathrm{Cos}[2\beta]-5\,\mathrm{Cos}[2\gamma])\\
&\qquad \mathrm{Csc}[\alpha]\,\mathrm{Csc}[\gamma]\,\mathrm{Log}[\mathrm{Sin}[\beta]]\,\mathrm{Sec}[\beta]\,\mathrm{Tan}[\beta]^2+4\,\mathrm{Tan}[\beta]\,\mathrm{Tan}[\gamma]+\\
&\qquad \mathrm{Csc}[\beta]\,((9+\mathrm{Cos}[2\alpha]-5\,\mathrm{Cos}[2\beta]-5\,\mathrm{Cos}[2\gamma])\,\mathrm{Csc}[\gamma]\,\mathrm{Log}[\mathrm{Sin}[\alpha]]\,\mathrm{Sec}[\alpha]\,\mathrm{Tan}[\alpha]^2+\\
&\qquad (9-5\,\mathrm{Cos}[2\alpha]-5\,\mathrm{Cos}[2\beta]+\mathrm{Cos}[2\gamma])\,\mathrm{Csc}[\alpha]\,\mathrm{Log}[\mathrm{Sin}[\gamma]]\,\mathrm{Sec}[\gamma]\,\mathrm{Tan}[\gamma]^2)))
\end{aligned}
$$

Figure 6: The function P_{ac}

$$
\begin{aligned}
&\frac{1}{480}\Big(8\,\mathrm{Cos}[2\alpha]+8\,(19+7\,\mathrm{Cos}[2\beta])\,\mathrm{Sec}[\alpha]^2\,\mathrm{Sec}[\beta]^2-8\,(-44+\mathrm{Cos}[2\beta]+10\,\mathrm{Sec}[\beta]^2)+\\
&\quad 2\,(52-57\,\mathrm{Cos}[2\beta]-12\,\mathrm{Cos}[4\beta]+\mathrm{Cos}[6\beta])\,\mathrm{Csc}[\gamma]\,\mathrm{Sec}[\beta]^3\,\mathrm{Sin}[\alpha]+\\
&\quad 16\,\beta\,(-2\,\mathrm{Cos}[\alpha+2\beta]+\mathrm{Cos}[\alpha]\,\mathrm{Cos}[2\alpha+4\beta])\,\mathrm{Csc}[\beta]^2\,\mathrm{Csc}[\gamma]^2\,\mathrm{Sin}[\alpha]^3+\\
&\quad 8\,\mathrm{Cot}[\beta]\,\mathrm{Sin}[2\alpha]+16\,\alpha\,(-2\,\mathrm{Cos}[2\alpha+\beta]+\mathrm{Cos}[\beta]\,\mathrm{Cos}[4\alpha+2\beta])\,\mathrm{Csc}[\alpha]^2\,\mathrm{Csc}[\gamma]^2\,\mathrm{Sin}[\beta]^3+\\
&\quad \mathrm{Cot}[\alpha]\,\mathrm{Sec}[\beta]^3\,(74\,\mathrm{Sin}[\beta]-21\,\mathrm{Sin}[3\beta]+\mathrm{Sin}[5\beta])+96\,\mathrm{Cot}[\beta]\,\mathrm{Sec}[\alpha]^2\,\mathrm{Tan}[\alpha]-\\
&\quad 4\,(23-12\,\mathrm{Cos}[2\beta]+13\,\mathrm{Cos}[4\beta])\,\mathrm{Csc}[\beta]\,\mathrm{Sec}[\beta]^3\,\mathrm{Tan}[\alpha]+4\,\mathrm{Log}[\mathrm{Sin}[\alpha]]\,\mathrm{Tan}[\alpha]^2\\
&\qquad (4\,(7+\mathrm{Cos}[2\alpha]-5\,\mathrm{Cos}[2\beta]-3\,\mathrm{Cos}[2\gamma])\,\mathrm{Csc}[\beta]\,\mathrm{Csc}[\gamma]\,\mathrm{Sec}[\alpha]+4\,(15+4\,\mathrm{Cos}[2\alpha]+\mathrm{Cos}[4\alpha])\\
&\qquad \mathrm{Csc}[\gamma]^2\,\mathrm{Sec}[\alpha]^2\,\mathrm{Sin}[\beta]^2-(-21+4\,\mathrm{Cos}[2\alpha]+\mathrm{Cos}[4\alpha])\,(1+\mathrm{Cot}[\beta]\,\mathrm{Tan}[\alpha])^2)+\\
&\quad 32\,\mathrm{Log}[\mathrm{Cos}[\beta]]\,(-5\,\mathrm{Csc}[\gamma]^2\,\mathrm{Sin}[\alpha]^2\,\mathrm{Sin}[\beta]^2+\mathrm{Tan}[\alpha]^2\,(1+\mathrm{Cot}[\beta]\,\mathrm{Tan}[\alpha])\,(5+3\,\mathrm{Cot}[\beta]\,\mathrm{Tan}[\alpha]))+\\
&\quad 4\,(19-28\,\mathrm{Cos}[2\beta]+\mathrm{Cos}[4\beta])\,\mathrm{Csc}[\gamma]^2\,\mathrm{Tan}[\beta]^2+4\,\mathrm{Log}[\mathrm{Sin}[\beta]]\,\mathrm{Tan}[\beta]^2\\
&\qquad (-4\,(-7+5\,\mathrm{Cos}[2\alpha]-\mathrm{Cos}[2\beta]+3\,\mathrm{Cos}[2\gamma])\,\mathrm{Csc}[\alpha]\,\mathrm{Csc}[\gamma]\,\mathrm{Sec}[\beta]+4\,(15+4\,\mathrm{Cos}[2\beta]+\mathrm{Cos}[4\beta])\\
&\qquad \mathrm{Csc}[\gamma]^2\,\mathrm{Sec}[\beta]^2\,\mathrm{Sin}[\alpha]^2-(-21+4\,\mathrm{Cos}[2\beta]+\mathrm{Cos}[4\beta])\,(1+\mathrm{Cot}[\alpha]\,\mathrm{Tan}[\beta])^2)+\\
&\quad 32\,\mathrm{Log}[\mathrm{Cos}[\alpha]]\,(-5\,\mathrm{Csc}[\gamma]^2\,\mathrm{Sin}[\alpha]^2\,\mathrm{Sin}[\beta]^2+\mathrm{Tan}[\beta]^2\,(1+\mathrm{Cot}[\alpha]\,\mathrm{Tan}[\beta])\,(5+3\,\mathrm{Cot}[\alpha]\,\mathrm{Tan}[\beta]))+\\
&\quad 32\,\mathrm{Log}[\mathrm{Sin}[\gamma]]\,((\mathrm{Csc}[\alpha]^4+\mathrm{Csc}[\beta]^4)\,\mathrm{Sin}[\alpha]^2\,\mathrm{Sin}[\beta]^2\,\mathrm{Sin}[\gamma]^2-\\
&\qquad (3\,\mathrm{Cot}[\alpha]^2-4\,\mathrm{Cot}[\alpha]\,\mathrm{Cot}[\beta]+3\,\mathrm{Cot}[\beta]^2)\,(\mathrm{Tan}[\alpha]+\mathrm{Tan}[\beta])^4-2\,\mathrm{Tan}[\gamma]^2)\Big)
\end{aligned}
$$

Figure 7: The function P_{ob}

List of new results of mean solid/characteristic picking

- Mean tetrahedron volume in
  - octahedron
  - square pyramid
  - triangular prism
  - rhombic dodecahedron
  - cuboctahedron
  - truncated tetrahedron
- Mean line lenght in
  - tetrahedron
  - octahedron
  - icosahedron
  - dodecahedron

- Means of triangle characteristics in a unit disk
  - perimeter squared
  - perimeter cubed
  - product of two sides
  - product of three sides
  - product of a side and a square of another
  - quotient of two sides
- Probability of a triangle being obtuse in a given triangle

$$\frac{117410162173}{525525000000}+\frac{8752199\pi^2}{2402400000}-\frac{192940695481\ln 2}{105105000000}-\frac{318759601\ln^2 2}{250250000}+\frac{506316394917\ln 3}{280280000000}-\frac{648098487\,\mathrm{Li}_2\left(\frac{1}{4}\right)}{500500000}$$

Thank you for your attention!

img source: Foundation TV Series

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