Sharp stability of the Brunn-Minkowski inequality

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Minkowski sum

Minkowski sum

For $A, B \subset \mathbb{R}^n$ and $\lambda \in \mathbb{R}$;

$$A + B := \{a + b : a \in A, b \in B\},$$

$$\lambda A := \{\lambda a : a \in A\},$$

$$\mathrm{co}(A) := \text{ smallest convex set containing } A,$$

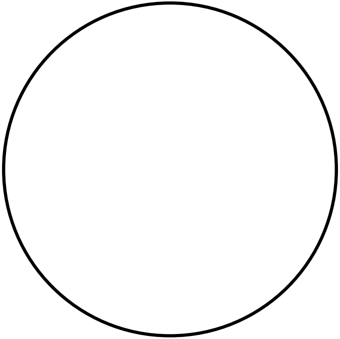
$$\text{e.g. } A \subset \frac{A+A}{2} \subset \mathrm{co}(A).$$

Brunn-Minkowski inequality for $A = B$ and $t = 1/2$

$$\left|\frac{A+A}{2}\right| \geq |A|,$$

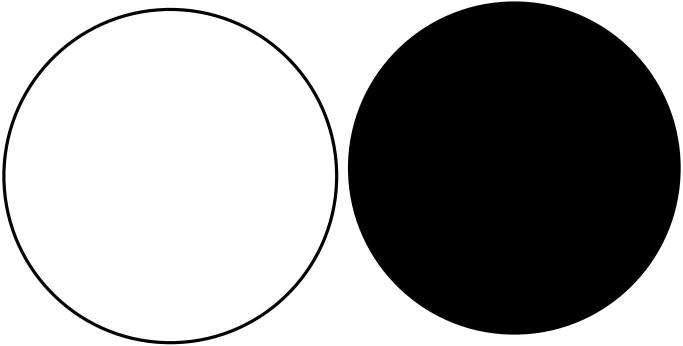
with equality iff A is convex.

Examples of semisums



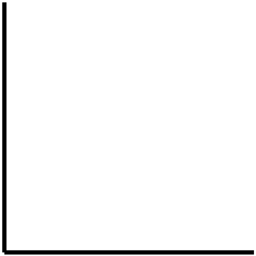
A

Examples of semisums



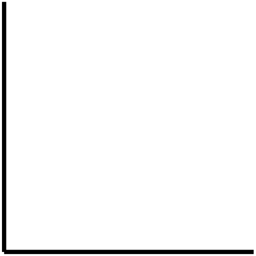
A $\qquad$ $\frac{A+A}{2}$

Examples



A

Examples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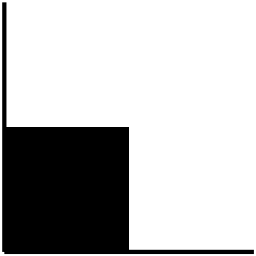
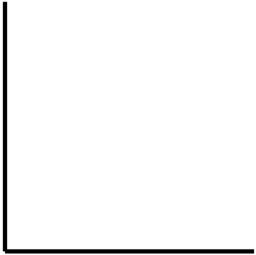

$\frac{A+A}{2}$

Examples



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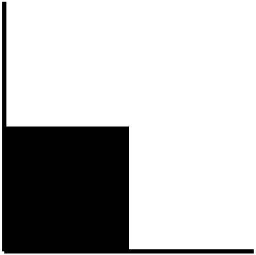
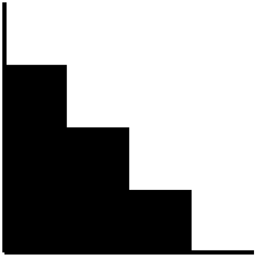
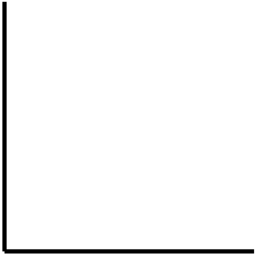


$\frac{A+A}{2}$



$\frac{\frac{A+A}{2}+\frac{A+A}{2}}{2} = \frac{A+A+A+A}{4}$

Examples



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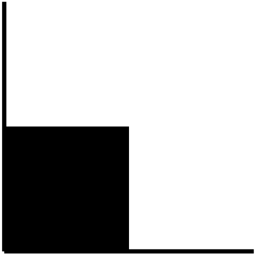
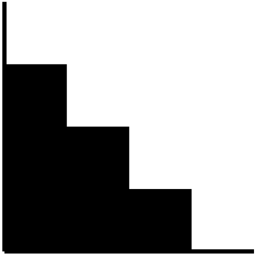


$\frac{A+A}{2}$

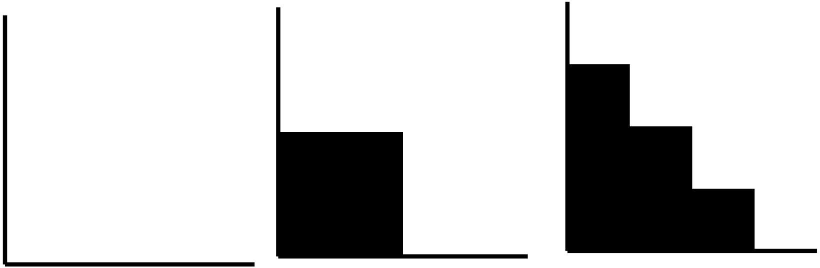


$\frac{\frac{A+A}{2}+\frac{A+A}{2}}{2} = \frac{A+A+A+A}{4}$

Observations

$$\frac{\overbrace{A + \ldots + A}^{k \text{ times}}}{k} \subset \mathrm{co}(A)$$

Examples



A $\qquad \frac{A+A}{2}$ $\qquad \frac{\frac{A+A}{2}+\frac{A+A}{2}}{2} = \frac{A+A+A+A}{4}$

Observations

Suspicion:

$$\left| \mathrm{co}(A) \setminus \frac{\overbrace{A + \ldots + A}^{k \text{ times}}}{k} \right| \to 0 \text{ as } k \to \infty$$

$A + A$

Theorem (Figalli and Jerison, 2015)

For all $A \subset \mathbb{R}^n$, we have

$$\frac{|\mathrm{co}(A) \setminus A|}{|A|} \to 0 \text{ as } \frac{|\frac{A+A}{2} \setminus A|}{|A|} \to 0.$$

Theorem (Figalli and Jerison, 2019)

For $n = 1, 2, 3$, $\exists C_n, d_n > 0$, such that the following holds. For all $A \subset \mathbb{R}^n$, with $|\frac{A+A}{2} \setminus A| \leq d_n |A|$, we have

$$|\mathrm{co}(A) \setminus A| \leq C_n \left| \frac{A+A}{2} \setminus A \right|.$$

Conjecture (Figalli and Jerison, 2019)

This holds for all $n \in \mathbb{N}$.

Linearity constants

Theorem (vH, Spink, T 2022)

$\exists C_n, d_n > 0$, such that the following holds. For all $A \subset \mathbb{R}^n$, with $|\frac{A+A}{2} \setminus A| \leq d_n|A|$, we have

$$|\text{co}(A) \setminus A| \leq C_n \left|\frac{A+A}{2} \setminus A\right|,$$

where $C_n = n^{O(n)}$.

Theorem (Figalli, vH, Spink, T 2023+)

$C_n = \exp(O(n))$ and for $n \leq 4$, we find C_n optimal.

Brunn-Minkowski inequality

Brunn-Minkowski inequality (Natural form)

For $A, B \subset \mathbb{R}^n$ measurable with $|A| = |B| > 0$, and $t \in (0, 1/2]$, we have

$$|tA + (1-t)B| \geq |A|$$

with equality iff $A = B$ is convex.

Brunn-Minkowski inequality (equivalent traditional formulation)

For $A, B \subset \mathbb{R}^n$ measurable with $|A|, |B| > 0$,

$$|A + B|^{\frac{1}{n}} \geq |A|^{\frac{1}{n}} + |B|^{\frac{1}{n}}$$

with equality iff A, B are homothetic convex bodies.

Stability parameters

Parameters

$$\delta := \frac{|tA + (1-t)B| - |A|}{|A|},$$

$$\omega := \min_{x \in \mathbb{R}^n} \frac{|\mathrm{co}(A \cup (x+B)) \setminus A|}{|A|},$$

$$\gamma := \frac{|\mathrm{co}(A) \setminus A| + |\mathrm{co}(B) \setminus B|}{|A|}.$$

$$\alpha := \min_{x \in \mathbb{R}^n} \frac{|A \triangle (x+B)|}{|A|}.$$

Initial Qualitative Result

Theorem (Christ, 2012)

For fixed t,

$$\omega, \gamma, \alpha \to 0, \text{ as } \delta \to 0.$$

Folklore conjectures

Conjectures

For $\delta \leq d_{n,t}$, we have

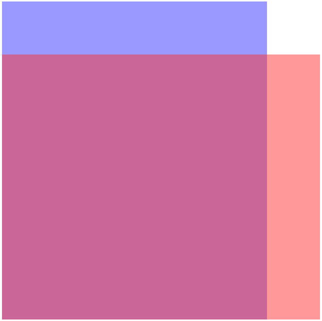
$$\min_{x \in \mathbb{R}^n} \frac{|\text{co}(A \cup (x+B)) \setminus A|}{|A|} = \omega \leq C_n \sqrt{\frac{\delta}{t}}$$

and

$$\frac{|\text{co}(A) \setminus A| + |\text{co}(B) \setminus B|}{|A|} = \gamma \leq C_n t^{-n+1} \delta$$

Optimality of the quadratic conjecture

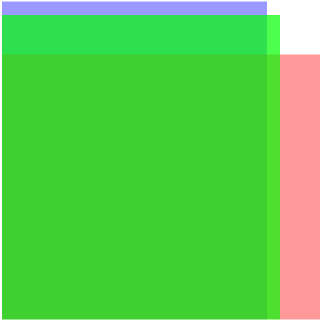
$A = [0,1] \times [0,1+\epsilon]$



$B = [0,1+\epsilon] \times [0,1]$

Optimality of the quadratic conjecture

$A = [0,1] \times [0, 1+\epsilon]$

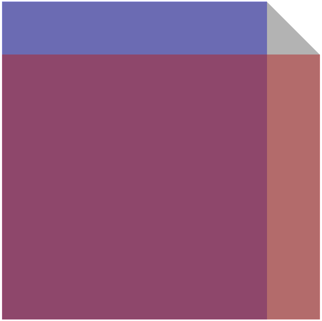


$$tA + (1-t)B \\ = [0, 1+(1-t)\epsilon] \times [0, 1+t\epsilon]$$

$B = [0, 1+\epsilon] \times [0,1]$

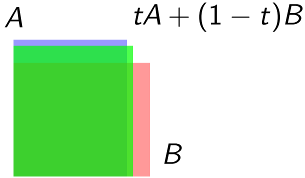
Optimality of the quadratic conjecture

$A = [0,1] \times [0, 1+\epsilon]$



$B = [0, 1+\epsilon] \times [0,1]$

Optimality of the quadratic conjecture



Calculation

$$\begin{aligned}\delta &\approx |tA + (1-t)B| - t|A| - (1-t)|B| \\ &= (1+t\epsilon)(1+(1-t)\epsilon) - (1+\epsilon) \\ &= (1-t)t\epsilon^2 \approx t\epsilon^2\end{aligned}$$

Hence,

$$\omega = \frac{|\mathrm{co}(A \cup B) \setminus A|}{|A|} = \frac{\epsilon + \frac{1}{2}\epsilon^2}{1+\epsilon} \approx \epsilon \approx \sqrt{\frac{\delta}{t}}$$

Folklore conjectures

Conjectures

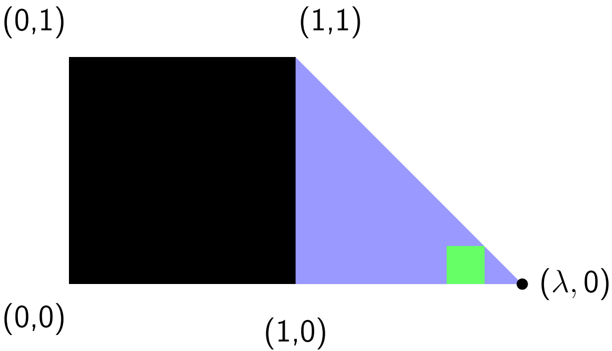
For $\delta \leq d_{n,t}$, we have

$$\omega \leq C_n \sqrt{\frac{\delta}{t}}$$

and

$$\gamma \leq C_n t^{-n+1} \delta$$

Bounded doubling (example)



$$A = [0,1]^2 \text{ and } B = [0,1]^2 \cup \{(\lambda, 0)\}, \text{ then}$$

$$|tA + (1-t)B| = 1 + t^n, \text{ so } \delta = t^n,$$

$$|\mathrm{co}(B)| \to \infty \text{ as } \lambda \to \infty$$

Bounded doubling

Bounded doubling

Need $\delta < t^n$ to say anything about ω.

Theorem (vH, Keevash 2023+)

If $\delta < t^n$, then (up to translation) $|\mathrm{co}(A \cup B)| \leq O_{n,t}(|A|)$.

Convention

Always: $\delta \leq d_{n,t}$

Previous Results for ω

Theorem (Figalli, Jerison, 2017)

$\exists C_n > 0$, so that for all $A, B \subset \mathbb{R}^n$, and $\delta \leq d_{n,t}$, we have

$$\omega \leq C_{n,t}\delta^{a_{n,t}},$$

where $a_{n,t} = \left(\frac{t}{8n\log(1/t)}\right)^{3^n}$.

Theorem (vH, Spink, T, 2023)

For $A, B \subset \mathbb{R}^2$ and $\delta \leq d_{2,t}$, we have

$$\omega \leq C\sqrt{\frac{\delta}{t}}.$$

Previous Results for α

Asymetry index

$$\alpha(A,B) = \min_{x\in\mathbb{R}^n} \frac{|(x+A)\Delta B|}{|A|}$$

Previous results

- (Figalli,Maggi, Pratelli, 2009) For A, B convex, $\alpha(A,B) \le C_n \delta^{1/2} t^{-1/2}$
- (Figalli, Maggi, Mooney, 2018) For A a ball and B arbitrary, $\alpha(A,B) \le C_n \delta^{1/2} t^{-1/2}$
- (Carlen, Maggi, 2018) For A convex and B arbitrary, $\alpha(A,B) \le C_n \delta^{n+\frac{3}{4}} t^{-1/2}$
- (Barchiesi, Julin, 2017) For A convex and B arbitrary, $\alpha(A,B) \le C_n \delta^{1/2} t^{-1/2}$

Main Results

Theorem (Figalli, vH, T 2023+)

$\exists C_n > 0$, so that for all $A, B \subset \mathbb{R}^n$, and $\delta \leq d_{n,t}$, we have

$$\omega \leq C_n \sqrt{\frac{\delta}{t}},$$

and

$$\gamma \leq C_n t^{-Cn^8} \delta.$$

Main Results

Theorem (Figalli, vH, T 2023+)

$\exists C_n > 0$, so that for all $A, B \subset \mathbb{R}^n$ of equal volume and $t \in (0, 1/2]$, if $|tA + (1-t)B| = (1+\delta)|A|$ and $\delta \leq d_{n,t}$, we have (up to translation)

$$|\mathrm{co}(A \cup B) \setminus A| \leq C_n \sqrt{\frac{\delta}{t}} |A|,$$

and

$$|\mathrm{co}(A) \setminus A| + |\mathrm{co}(B) \setminus B| \leq C_n t^{-Cn^8} \delta.$$

One Idea from the Proof: Cones!

Proposition

Consider $A, B \subset \mathbb{R}^d$ with $|A| = |B|$, then (up to translation) there exists a partition $\mathcal{C}$ of $\mathbb{R}^d$ into convex cones C with apex at the origin so that:

- $|C \cap A| = |C \cap B|$ for all $C \in \mathcal{C}$,
- C is narrow in all but at most one direction for almost all $C \in \mathcal{C}$, and
- C is essentially the convex hull of few lines for almost all $C \in \mathcal{C}$.

Note

$$t(A \cap C) + (1-t)(B \cap C) \subset C, \text{ so}$$

$$\sum_{C \in \mathcal{C}} |t(A \cap C) + (1-t)(B \cap C)| - |A \cap C| \leq |tA + (1-t)B| - |A|.$$

$$\sum_{C \in \mathcal{C}} |(A \cap C) \triangle (B \cap C)| = |A \triangle B|$$

Crucial Property: Alignment

Proposition

If $A, B \subset \mathbb{R}^d$ are near convex (i.e. $|\text{co}(A) \setminus A| + |\text{co}(B) \setminus B| \leq \alpha |A|$) and for some translate $x \in \mathbb{R}^d$:

$$|(A \cap C) \triangle (x + (B \cap C))| \leq \alpha |A|,$$

then in fact

$$|(A \cap C) \triangle (B \cap C)| \leq O_d(\alpha) |A|,$$

Cones construction

Process (3 dimensions)

Given a cone C so that $|A \cap C| = |B \cap C|$ and a line $\ell \ni o$, then there exists a hyperplane $H \supset \ell$, so that

$$|A \cap C \cap H^{\pm}| = |B \cap C \cap H^{\pm}|.$$

Thank you!

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