Inequalities and Counterexamples for Functional Intrinsic Volumes

Fabian Mussnig

joint work with Jacopo Ulivelli

TECHNISCHE UNIVERSITÄT WIEN

Intrinsic and Mixed Volumes

$\mathcal{K}^n$ ... convex bodies (non-empty, compact, convex subsets of $\mathbb{R}^n$)

Intrinsic and Mixed Volumes

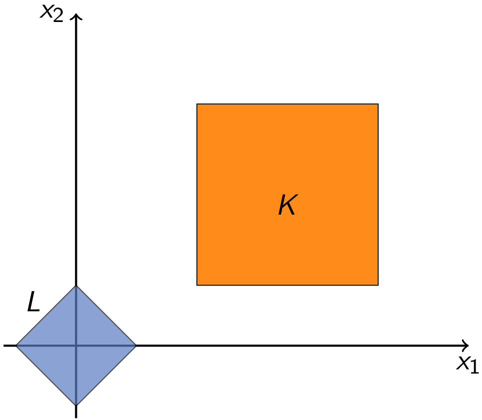
$\mathcal{K}^n$ ... convex bodies (non-empty, compact, convex subsets of $\mathbb{R}^n$)

For $K, L \in \mathcal{K}^n$ let $K + L = \{x + y : x \in K, y \in L\}$.

Intrinsic and Mixed Volumes

$\mathcal{K}^n$ ... convex bodies (non-empty, compact, convex subsets of $\mathbb{R}^n$)

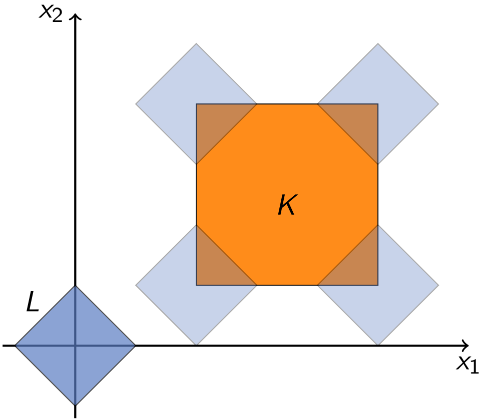
For $K, L \in \mathcal{K}^n$ let $K + L = \{x + y : x \in K, y \in L\}$.



Intrinsic and Mixed Volumes

$\mathcal{K}^n$ ... convex bodies (non-empty, compact, convex subsets of $\mathbb{R}^n$)

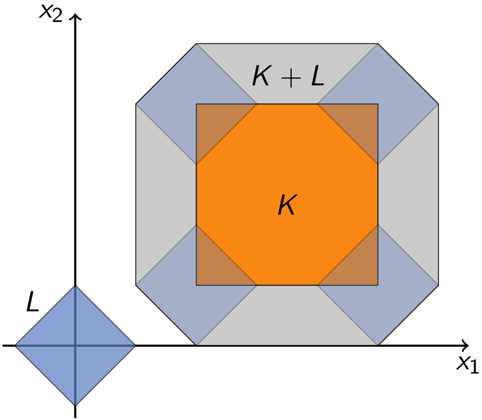
For $K, L \in \mathcal{K}^n$ let $K + L = \{x + y : x \in K, y \in L\}$.



Intrinsic and Mixed Volumes

$\mathcal{K}^n$ ... convex bodies (non-empty, compact, convex subsets of $\mathbb{R}^n$)

For $K, L \in \mathcal{K}^n$ let $K + L = \{x + y : x \in K, y \in L\}$.



Intrinsic and Mixed Volumes

$\mathcal{K}^n$ ... convex bodies (non-empty, compact, convex subsets of $\mathbb{R}^n$)
For $K, L \in \mathcal{K}^n$ let $K + L = \{x + y : x \in K, y \in L\}$.

Recall: Steiner Formula

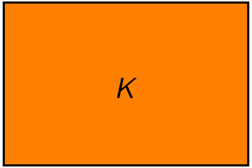
$$\mathrm{vol}_n(K + rB^n) \qquad\qquad r > 0, K \in \mathcal{K}^n$$

Intrinsic and Mixed Volumes

$\mathcal{K}^n$ ... convex bodies (non-empty, compact, convex subsets of $\mathbb{R}^n$)
For $K, L \in \mathcal{K}^n$ let $K + L = \{x + y : x \in K, y \in L\}$.

Recall: Steiner Formula

$$\mathrm{vol}_n(K + rB^n) \qquad r > 0, K \in \mathcal{K}^n$$

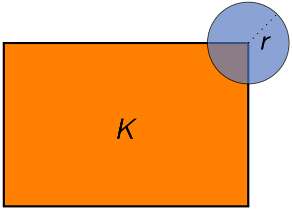


Intrinsic and Mixed Volumes

$\mathcal{K}^n$ ... convex bodies (non-empty, compact, convex subsets of $\mathbb{R}^n$)
For $K, L \in \mathcal{K}^n$ let $K + L = \{x + y : x \in K, y \in L\}$.

Recall: Steiner Formula

$$\mathrm{vol}_n(K + rB^n) \qquad r > 0, K \in \mathcal{K}^n$$

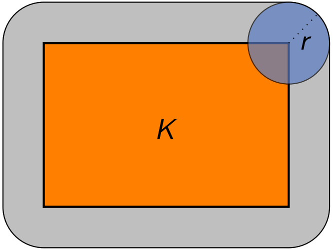


Intrinsic and Mixed Volumes

$\mathcal{K}^n$ ... convex bodies (non-empty, compact, convex subsets of $\mathbb{R}^n$)
For $K, L \in \mathcal{K}^n$ let $K + L = \{x + y : x \in K, y \in L\}$.

Recall: Steiner Formula

$$\mathrm{vol}_n(K + rB^n) \qquad r > 0, K \in \mathcal{K}^n$$

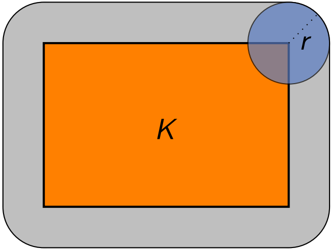


Intrinsic and Mixed Volumes

$\mathcal{K}^n$ ... convex bodies (non-empty, compact, convex subsets of $\mathbb{R}^n$)
For $K, L \in \mathcal{K}^n$ let $K + L = \{x + y : x \in K, y \in L\}$.

Recall: Steiner Formula

$$\mathrm{vol}_n(K + rB^n) = \sum_{j=0}^{n} r^{n-j}\,\mathrm{vol}_{n-j}(B^{n-j})V_j(K), \quad r > 0, K \in \mathcal{K}^n$$



Intrinsic and Mixed Volumes

$\mathcal{K}^n$ ... convex bodies (non-empty, compact, convex subsets of $\mathbb{R}^n$)
For $K, L \in \mathcal{K}^n$ let $K + L = \{x + y : x \in K, y \in L\}$.

Recall: Steiner Formula

$$\mathrm{vol}_n(K + rB^n) = \sum_{j=0}^{n} r^{n-j} \,\mathrm{vol}_{n-j}(B^{n-j}) V_j(K), \quad r > 0, K \in \mathcal{K}^n$$

Recall: Mixed Volumes

For $m \in \mathbb{N}$, $K_1, \dots, K_m \in \mathcal{K}^n$, $\lambda_1, \dots, \lambda_m \geq 0$,

$$\mathrm{vol}_n(\lambda_1 K_1 + \cdots + \lambda_m K_m) = \sum_{i_1,\dots,i_n=1}^{m} \lambda_{i_1} \cdots \lambda_{i_n} V(K_{i_1}, \dots, K_{i_n}).$$

Inequalities

Inequalities

Isoperimetric Inequality

For $K \in \mathcal{K}^n$,

$$\left(\frac{V_{n-1}(K)}{V_{n-1}(B^n)}\right)^{\frac{1}{n-1}} \geq \left(\frac{V_n(K)}{V_n(B^n)}\right)^{\frac{1}{n}}.$$

Inequalities

Isoperimetric Inequality

For $K \in \mathcal{K}^n$,

$$\left(\frac{V_1(K)}{V_1(B^n)}\right)^{\frac{1}{1}} \geq \cdots \geq \left(\frac{V_{n-1}(K)}{V_{n-1}(B^n)}\right)^{\frac{1}{n-1}} \geq \left(\frac{V_n(K)}{V_n(B^n)}\right)^{\frac{1}{n}}.$$

Inequalities

Isoperimetric Inequality

For $K \in \mathcal{K}^n$,

$$\left(\frac{V_1(K)}{V_1(B^n)}\right)^{\frac{1}{1}} \geq \cdots \geq \left(\frac{V_{n-1}(K)}{V_{n-1}(B^n)}\right)^{\frac{1}{n-1}} \geq \left(\frac{V_n(K)}{V_n(B^n)}\right)^{\frac{1}{n}}.$$

Brunn–Minkowski Inequality

For $K, L \in \mathcal{K}^n$,

$$V_n(K+L)^{\frac{1}{n}} \geq V_n(K)^{\frac{1}{n}} + V_n(L)^{\frac{1}{n}}.$$

Inequalities

Isoperimetric Inequality

For $K \in \mathcal{K}^n$,

$$\left(\frac{V_1(K)}{V_1(B^n)}\right)^{\frac{1}{1}} \geq \cdots \geq \left(\frac{V_{n-1}(K)}{V_{n-1}(B^n)}\right)^{\frac{1}{n-1}} \geq \left(\frac{V_n(K)}{V_n(B^n)}\right)^{\frac{1}{n}}.$$

Brunn–Minkowski Inequality

For $K, L \in \mathcal{K}^n$, $1 \leq j \leq n$,

$$V_j(K+L)^{\frac{1}{j}} \geq V_j(K)^{\frac{1}{j}} + V_j(L)^{\frac{1}{j}}.$$

Inequalities

Isoperimetric Inequality

For $K \in \mathcal{K}^n$,

$$\left(\frac{V_1(K)}{V_1(B^n)}\right)^{\frac{1}{1}} \geq \cdots \geq \left(\frac{V_{n-1}(K)}{V_{n-1}(B^n)}\right)^{\frac{1}{n-1}} \geq \left(\frac{V_n(K)}{V_n(B^n)}\right)^{\frac{1}{n}}.$$

Brunn–Minkowski Inequality

For $K, L \in \mathcal{K}^n$, $1 \leq j \leq n$,

$$V_j(K+L)^{\frac{1}{j}} \geq V_j(K)^{\frac{1}{j}} + V_j(L)^{\frac{1}{j}}.$$

Alexandrov–Fenchel Inequality

For $K_1, \ldots, K_n \in \mathcal{K}^n$,

$$V(K_1, K_2, \ldots, K_n)^2 \geq V(K_1, K_1, K_3, \ldots, K_n)V(K_2, K_2, K_3, \ldots, K_n).$$

Super-coercive Convex Functions

$$\mathrm{Conv_{sc}}(\mathbb{R}^n) := \Big\{ u\colon \mathbb{R}^n \to (-\infty, \infty] \,:\, \text{convex, l.s.c., } u \not\equiv +\infty,$$
$$\text{super-coercive: } \lim_{|x|\to+\infty} \frac{u(x)}{|x|} = +\infty \Big\}$$

Super-coercive Convex Functions

$$\mathrm{Conv_{sc}}(\mathbb{R}^n) := \Big\{ u\colon \mathbb{R}^n \to (-\infty, \infty] \ :\ \text{convex, l.s.c., } u \not\equiv +\infty,$$
$$\text{super-coercive: } \lim_{|x|\to+\infty} \frac{u(x)}{|x|} = +\infty \Big\}$$

Super-coercive Convex Functions

$$\mathrm{Conv_{sc}}(\mathbb{R}^n) := \Big\{ u\colon \mathbb{R}^n \to (-\infty, \infty] \ : \ \text{convex, l.s.c., } u \not\equiv +\infty, \\ \text{super-coercive: } \lim_{|x|\to+\infty} \frac{u(x)}{|x|} = +\infty \Big\}$$

Super-coercive Convex Functions

$$\mathrm{Conv_{sc}}(\mathbb{R}^n) := \Big\{ u\colon \mathbb{R}^n \to (-\infty, \infty] \ : \ \text{convex, l.s.c., } u \not\equiv +\infty,$$
$$\text{super-coercive: } \lim_{|x|\to+\infty} \frac{u(x)}{|x|} = +\infty \Big\}$$

Super-coercive Convex Functions

$$\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) := \Big\{ u\colon \mathbb{R}^n \to (-\infty, \infty] \; : \; \text{convex, l.s.c., } u \not\equiv +\infty, \\ \text{super-coercive: } \lim_{|x|\to+\infty} \frac{u(x)}{|x|} = +\infty \Big\}$$

Super-coercive Convex Functions

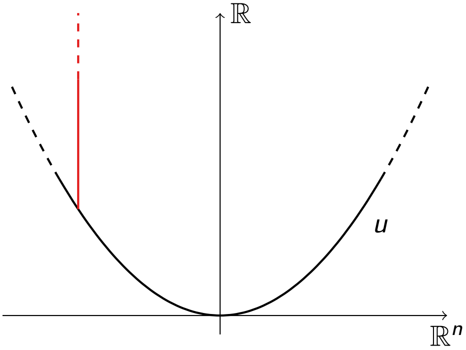
$$\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) := \Big\{ u \colon \mathbb{R}^n \to (-\infty, \infty] \, : \, \text{convex, l.s.c.,} \; u \not\equiv +\infty,$$
$$\text{super-coercive:} \;\; \lim_{|x| \to +\infty} \frac{u(x)}{|x|} = +\infty \Big\}$$

Super-coercive Convex Functions

$$\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) := \Big\{ u \colon \mathbb{R}^n \to (-\infty, \infty] \ : \ \text{convex, l.s.c., } u \not\equiv +\infty,$$
$$\text{super-coercive: } \lim_{|x|\to+\infty} \frac{u(x)}{|x|} = +\infty \Big\}$$

Super-coercive Convex Functions

$$\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) := \Big\{ u\colon \mathbb{R}^n \to (-\infty,\infty] \,:\, \text{convex, l.s.c., } u \not\equiv +\infty,$$
$$\text{super-coercive: } \lim_{|x|\to+\infty} \frac{u(x)}{|x|} = +\infty \Big\}$$



Super-coercive Convex Functions

$$\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) := \Big\{ u\colon \mathbb{R}^n \to (-\infty, \infty] \ : \ \text{convex, l.s.c., } u \not\equiv +\infty, \\ \text{super-coercive: } \lim_{|x|\to+\infty} \frac{u(x)}{|x|} = +\infty \Big\}$$

Every convex body $K \in \mathcal{K}^n$ is represented by its indicator function $\mathrm{I}_K^\infty \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$,

$$\mathrm{I}_K^\infty(x) = \begin{cases} 0 & \text{if } x \in K, \\ +\infty & \text{if } x \notin K. \end{cases}$$

Super-coercive Convex Functions

$$\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) := \Big\{ u\colon \mathbb{R}^n \to (-\infty, \infty] \ : \ \text{convex, l.s.c., } u \not\equiv +\infty, \\ \text{super-coercive: } \lim_{|x|\to+\infty} \frac{u(x)}{|x|} = +\infty \Big\}$$

Every convex body $K \in \mathcal{K}^n$ is represented by its indicator function $\mathrm{I}_K^\infty \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$,

$$\mathrm{I}_K^\infty(x) = \begin{cases} 0 & \text{if } x \in K, \\ +\infty & \text{if } x \notin K. \end{cases}$$

$\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ is dual to $\{v\colon \mathbb{R}^n \to \mathbb{R} : v \text{ is convex}\}$ via the Legendre–Fenchel transform.

Functional Intrinsic and Mixed Volumes

For $\alpha \in C_c([0,\infty))$ set $\overline{\mathrm{V}}_{n,\alpha}(u) := \int_{\mathrm{dom}\,(u)} \alpha(|\nabla u(x)|)\,\mathrm{d}x,\ u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$.

Functional Intrinsic and Mixed Volumes

For $\alpha \in C_c([0,\infty))$ set $\overline{\mathrm{V}}_{n,\alpha}(u) := \int_{\mathrm{dom}\,(u)} \alpha(|\nabla u(x)|)\,\mathrm{d}x,\ u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$.

$$\overline{\mathrm{V}}_{n,\alpha}(\mathrm{I}_K^\infty) = \int_K \alpha(0)\,\mathrm{d}x = \alpha(0)\,V_n(K), \quad K \in \mathcal{K}^n.$$

Functional Intrinsic and Mixed Volumes

For $\alpha \in C_c([0,\infty))$ set $\overline{\mathrm{V}}_{n,\alpha}(u) := \int_{\mathrm{dom}\,(u)} \alpha(|\nabla u(x)|)\,\mathrm{d}x,\ u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$.

Functional Steiner Formula (Colesanti, Ludwig, M., Calc. Var. PDE 2022)

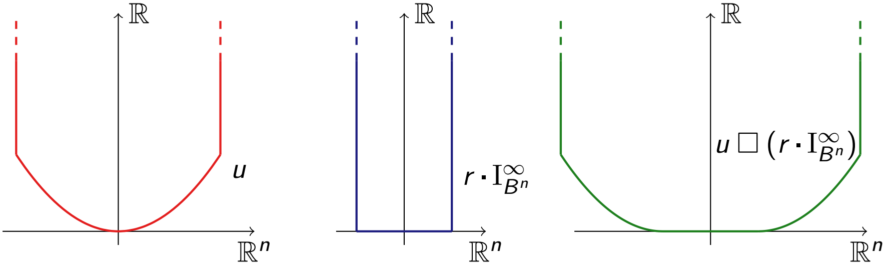
$$\overline{\mathrm{V}}_{n,\alpha}(u \,\Box\, (r \cdot \mathrm{I}^{\infty}_{B^n})) \qquad r > 0,\ u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$$

Functional Intrinsic and Mixed Volumes

For $\alpha \in C_c([0,\infty))$ set $\overline{\mathrm{V}}_{n,\alpha}(u) := \int_{\mathrm{dom}\,(u)} \alpha(|\nabla u(x)|)\,\mathrm{d}x,\ u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$.

Functional Steiner Formula (Colesanti, Ludwig, M., Calc. Var. PDE 2022)

$\overline{\mathrm{V}}_{n,\alpha}(u \,\square\, (r \cdot \mathrm{I}_{B^n}^{\infty}))$ $\qquad r > 0,\ u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$



Functional Intrinsic and Mixed Volumes

For $\alpha \in C_c([0,\infty))$ set $\overline{\mathrm{V}}_{n,\alpha}(u) := \int_{\mathrm{dom}\,(u)} \alpha(|\nabla u(x)|)\,\mathrm{d}x$, $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$.

Functional Steiner Formula (Colesanti, Ludwig, M., Calc. Var. PDE 2022)

$$\overline{\mathrm{V}}_{n,\alpha}(u \,\square\, (r \cdot \mathrm{I}^{\infty}_{B^n})) = \sum_{j=0}^{n} r^{n-j}\,\mathrm{vol}_{n-j}(B^{n-j})\overline{\mathrm{V}}_{j,\alpha}(u), \quad r > 0, u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$$

Functional Intrinsic and Mixed Volumes

For $\alpha \in C_c([0,\infty))$ set $\overline{\mathrm{V}}_{n,\alpha}(u) := \int_{\mathrm{dom}\,(u)} \alpha(|\nabla u(x)|)\,\mathrm{d}x,\ u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$.

Functional Steiner Formula (Colesanti, Ludwig, M., Calc. Var. PDE 2022)

$$\overline{\mathrm{V}}_{n,\alpha}(u \,\square\, (r \cdot \mathrm{I}^{\infty}_{B^n})) = \sum_{j=0}^{n} r^{n-j} \,\mathrm{vol}_{n-j}(B^{n-j})\overline{\mathrm{V}}_{j,\alpha}(u), \quad r > 0, u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$$

$\overline{\mathrm{V}}_{j,\alpha}(u)$

Functional Intrinsic and Mixed Volumes

For $\alpha \in C_c([0,\infty))$ set $\overline{\mathrm{V}}_{n,\alpha}(u) := \int_{\mathrm{dom}\,(u)} \alpha(|\nabla u(x)|)\,\mathrm{d}x,\ u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$.

Functional Steiner Formula (Colesanti, Ludwig, M., Calc. Var. PDE 2022)

$$\overline{\mathrm{V}}_{n,\alpha}(u \,\square\, (r \cdot \mathrm{I}^{\infty}_{B^n})) = \sum_{j=0}^{n} r^{n-j}\,\mathrm{vol}_{n-j}(B^{n-j})\overline{\mathrm{V}}_{j,\alpha}(u), \quad r > 0, u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$$

$$\overline{\mathrm{V}}_{j,\alpha}(u) \quad = \int_{\mathbb{R}^n} \alpha(|y|) \qquad \mathrm{dMA}^*_j(u;y)$$

Functional Intrinsic and Mixed Volumes

For $\alpha \in C_c([0,\infty))$ set $\overline{\mathrm{V}}_{n,\alpha}(u) := \int_{\mathrm{dom}\,(u)} \alpha(|\nabla u(x)|)\,\mathrm{d}x,\ u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$.

Functional Steiner Formula (Colesanti, Ludwig, M., Calc. Var. PDE 2022)

$$\overline{\mathrm{V}}_{n,\alpha}(u \,\square\, (r \cdot \mathrm{I}^\infty_{B^n})) = \sum_{j=0}^{n} r^{n-j}\,\mathrm{vol}_{n-j}(B^{n-j})\overline{\mathrm{V}}_{j,\alpha}(u), \quad r > 0, u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$$

$$\overline{\mathrm{V}}_{j,\alpha}(u) \quad = \int_{\mathbb{R}^n} \alpha(|y|) \quad \underbrace{\mathrm{dMA}^*_j(u;y)}_{\cong\ \det(\mathrm{D}^2 u^*(y)[j],\mathrm{D}^2|y|[n-j])\,\mathrm{d}y}$$

Functional Intrinsic and Mixed Volumes

For $\alpha \in C_c([0,\infty))$ set $\overline{\mathrm{V}}_{n,\alpha}(u) := \int_{\mathrm{dom}\,(u)} \alpha(|\nabla u(x)|)\,\mathrm{d}x$, $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$.

Functional Steiner Formula (Colesanti, Ludwig, M., Calc. Var. PDE 2022)

$$\overline{\mathrm{V}}_{n,\alpha}(u \,\square\, (r \cdot \mathrm{I}^\infty_{B^n})) = \sum_{j=0}^{n} r^{n-j}\,\mathrm{vol}_{n-j}(B^{n-j})\overline{\mathrm{V}}_{j,\alpha}(u), \quad r > 0, u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$$

$$\overline{\mathrm{V}}_{j,\alpha}(u) = \int_{\mathbb{R}^n} \alpha(|y|) \underbrace{\mathrm{dMA}_j^*(u;y)}_{\cong\ \det(\mathrm{D}^2 u^*(y)[j],\mathrm{D}^2|y|[n-j])\,\mathrm{d}y}$$

$$\overset{u\in C^2_+(\mathbb{R}^n)}{=} \int_{\mathbb{R}^n} \zeta(|\nabla u(x)|)[\mathrm{D}^2 u(x)]_{n-j}\,\mathrm{d}x$$

Functional Intrinsic and Mixed Volumes

For $\alpha \in C_c([0,\infty))$ set $\overline{\mathrm{V}}_{n,\alpha}(u) := \int_{\mathrm{dom}\,(u)} \alpha(|\nabla u(x)|)\,\mathrm{d}x,\ u \in \mathrm{Conv_{sc}}(\mathbb{R}^n)$.

Functional Steiner Formula (Colesanti, Ludwig, M., Calc. Var. PDE 2022)

$$\overline{\mathrm{V}}_{n,\alpha}(u \,\square\, (r \cdot \mathrm{I}^\infty_{B^n})) = \sum_{j=0}^{n} r^{n-j}\,\mathrm{vol}_{n-j}(B^{n-j})\overline{\mathrm{V}}_{j,\alpha}(u), \quad r > 0, u \in \mathrm{Conv_{sc}}(\mathbb{R}^n)$$

Functional Mixed Volumes (Colesanti, Ludwig, M., JFA 2020; Knoerr JFA 2021)

For $m \in \mathbb{N}$, $u_1, \ldots, u_m \in \mathrm{Conv_{sc}}(\mathbb{R}^n)$, $\lambda_1, \ldots, \lambda_m \geq 0$,

$$\overline{\mathrm{V}}_{n,\alpha}((\lambda_1 \cdot u_1) \,\square \cdots \square\, (\lambda_m \cdot u_m)) = \sum_{i_1,\ldots,i_n=1}^{m} \lambda_{i_1} \cdots \lambda_{i_n} \overline{\mathrm{V}}_{\alpha}(u_{i_1}, \ldots, u_{i_n}).$$

Properties and Results for Functional Intrinsic Volumes

Properties and Results (Colesanti, Ludwig, M.; Knoerr; Knoerr, Ulivelli)

Properties and Results for Functional Intrinsic Volumes

Properties and Results (Colesanti, Ludwig, M.; Knoerr; Knoerr, Ulivelli)

- $\overline{\mathrm{V}}_{j,\alpha}(\mathrm{I}_K^\infty) = \alpha(0)V_j(K)$ for $K \in \mathcal{K}^n$.

Properties and Results for Functional Intrinsic Volumes

Properties and Results (Colesanti, Ludwig, M.; Knoerr; Knoerr, Ulivelli)

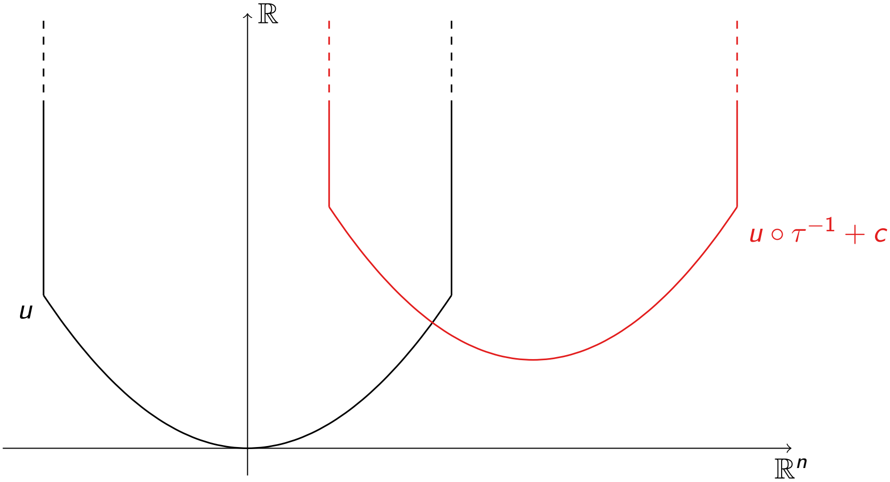
- $\overline{\mathrm{V}}_{j,\alpha}(\mathrm{I}_K^\infty) = \alpha(0) V_j(K)$ for $K \in \mathcal{K}^n$.
- Continuous w.r.t. epi-convergence.

Properties and Results for Functional Intrinsic Volumes

Properties and Results (Colesanti, Ludwig, M.; Knoerr; Knoerr, Ulivelli)

- $\overline{\mathrm{V}}_{j,\alpha}(\mathrm{I}_K^\infty) = \alpha(0) V_j(K)$ for $K \in \mathcal{K}^n$.
- Continuous w.r.t. epi-convergence.
- Epi-translation invariant.

Properties and Results for Functional Intrinsic Volumes



Properties and Results for Functional Intrinsic Volumes

Properties and Results (Colesanti, Ludwig, M.; Knoerr; Knoerr, Ulivelli)

- $\overline{\mathrm{V}}_{j,\alpha}(\mathrm{I}_K^\infty) = \alpha(0)V_j(K)$ for $K \in \mathcal{K}^n$.
- Continuous w.r.t. epi-convergence.
- Epi-translation invariant.
- Rotation invariant: $\overline{\mathrm{V}}_{j,\alpha}(u \circ \vartheta^{-1}) = \overline{\mathrm{V}}_{j,\alpha}(u)$ for $\vartheta \in \mathrm{SO}(n)$, $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$.

Properties and Results for Functional Intrinsic Volumes

Properties and Results (Colesanti, Ludwig, M.; Knoerr; Knoerr, Ulivelli)

- $\overline{\mathrm{V}}_{j,\alpha}(\mathrm{I}_K^\infty) = \alpha(0) V_j(K)$ for $K \in \mathcal{K}^n$.
- Continuous w.r.t. epi-convergence.
- Epi-translation invariant.
- Rotation invariant: $\overline{\mathrm{V}}_{j,\alpha}(u \circ \vartheta^{-1}) = \overline{\mathrm{V}}_{j,\alpha}(u)$ for $\vartheta \in \mathrm{SO}(n)$, $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$.
- Valuation: $\overline{\mathrm{V}}_{j,\alpha}(u \vee v) + \overline{\mathrm{V}}_{j,\alpha}(u \wedge v) = \overline{\mathrm{V}}_{j,\alpha}(u) + \overline{\mathrm{V}}_{j,\alpha}(v)$

Properties and Results for Functional Intrinsic Volumes

Properties and Results (Colesanti, Ludwig, M.; Knoerr; Knoerr, Ulivelli)

- $\overline{\mathrm{V}}_{j,\alpha}(\mathrm{I}_K^\infty) = \alpha(0) V_j(K)$ for $K \in \mathcal{K}^n$.
- Continuous w.r.t. epi-convergence.
- Epi-translation invariant.
- Rotation invariant: $\overline{\mathrm{V}}_{j,\alpha}(u \circ \vartheta^{-1}) = \overline{\mathrm{V}}_{j,\alpha}(u)$ for $\vartheta \in \mathrm{SO}(n)$, $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$.
- Valuation: $\overline{\mathrm{V}}_{j,\alpha}(u \vee v) + \overline{\mathrm{V}}_{j,\alpha}(u \wedge v) = \overline{\mathrm{V}}_{j,\alpha}(u) + \overline{\mathrm{V}}_{j,\alpha}(v)$

Properties and Results for Functional Intrinsic Volumes

Properties and Results (Colesanti, Ludwig, M.; Knoerr; Knoerr, Ulivelli)

- $\overline{\mathrm{V}}_{j,\alpha}(\mathrm{I}_K^\infty) = \alpha(0) V_j(K)$ for $K \in \mathcal{K}^n$.
- Continuous w.r.t. epi-convergence.
- Epi-translation invariant.
- Rotation invariant: $\overline{\mathrm{V}}_{j,\alpha}(u \circ \vartheta^{-1}) = \overline{\mathrm{V}}_{j,\alpha}(u)$ for $\vartheta \in \mathrm{SO}(n)$, $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$.
- Valuation: $\overline{\mathrm{V}}_{j,\alpha}(u \vee v) + \overline{\mathrm{V}}_{j,\alpha}(u \wedge v) = \overline{\mathrm{V}}_{j,\alpha}(u) + \overline{\mathrm{V}}_{j,\alpha}(v)$
- Characterized by a Hadwiger-type theorem.

Properties and Results for Functional Intrinsic Volumes

Properties and Results (Colesanti, Ludwig, M.; Knoerr; Knoerr, Ulivelli)

- $\overline{\mathrm{V}}_{j,\alpha}(\mathrm{I}_K^\infty) = \alpha(0) V_j(K)$ for $K \in \mathcal{K}^n$.
- Continuous w.r.t. epi-convergence.
- Epi-translation invariant.
- Rotation invariant: $\overline{\mathrm{V}}_{j,\alpha}(u \circ \vartheta^{-1}) = \overline{\mathrm{V}}_{j,\alpha}(u)$ for $\vartheta \in \mathrm{SO}(n)$, $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$.
- Valuation: $\overline{\mathrm{V}}_{j,\alpha}(u \vee v) + \overline{\mathrm{V}}_{j,\alpha}(u \wedge v) = \overline{\mathrm{V}}_{j,\alpha}(u) + \overline{\mathrm{V}}_{j,\alpha}(v)$
- Characterized by a Hadwiger-type theorem.
- Cauchy–Kubota formulas.

Properties and Results for Functional Intrinsic Volumes

Properties and Results (Colesanti, Ludwig, M.; Knoerr; Knoerr, Ulivelli)

- $\overline{\mathrm{V}}_{j,\alpha}(\mathrm{I}_K^\infty) = \alpha(0) V_j(K)$ for $K \in \mathcal{K}^n$.
- Continuous w.r.t. epi-convergence.
- Epi-translation invariant.
- Rotation invariant: $\overline{\mathrm{V}}_{j,\alpha}(u \circ \vartheta^{-1}) = \overline{\mathrm{V}}_{j,\alpha}(u)$ for $\vartheta \in \mathrm{SO}(n)$, $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$.
- Valuation: $\overline{\mathrm{V}}_{j,\alpha}(u \vee v) + \overline{\mathrm{V}}_{j,\alpha}(u \wedge v) = \overline{\mathrm{V}}_{j,\alpha}(u) + \overline{\mathrm{V}}_{j,\alpha}(v)$
- Characterized by a Hadwiger-type theorem.
- Cauchy–Kubota formulas.
- Theory of continuous epi-translation invariant valuations on $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$.

Properties and Results for Functional Intrinsic Volumes

Properties and Results (Colesanti, Ludwig, M.; Knoerr; Knoerr, Ulivelli)

- $\overline{\mathrm{V}}_{j,\alpha}(\mathrm{I}_K^\infty) = \alpha(0)V_j(K)$ for $K \in \mathcal{K}^n$.
- Continuous w.r.t. epi-convergence.
- Epi-translation invariant.
- Rotation invariant: $\overline{\mathrm{V}}_{j,\alpha}(u \circ \vartheta^{-1}) = \overline{\mathrm{V}}_{j,\alpha}(u)$ for $\vartheta \in \mathrm{SO}(n)$, $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$.
- Valuation: $\overline{\mathrm{V}}_{j,\alpha}(u \vee v) + \overline{\mathrm{V}}_{j,\alpha}(u \wedge v) = \overline{\mathrm{V}}_{j,\alpha}(u) + \overline{\mathrm{V}}_{j,\alpha}(v)$
- Characterized by a Hadwiger-type theorem.
- Cauchy–Kubota formulas.
- Theory of continuous epi-translation invariant valuations on $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$.
- Theory of smooth valuations.

Open Problem

Question

Do functional intrinsic and mixed volumes satisfy inequalities?

Open Problem

Question

Do functional intrinsic and mixed volumes satisfy inequalities? In particular,

- isoperimetric inequalities,

Open Problem

Question

Do functional intrinsic and mixed volumes satisfy inequalities? In particular,

- isoperimetric inequalities,
- Brunn–Minkowski inequalities,

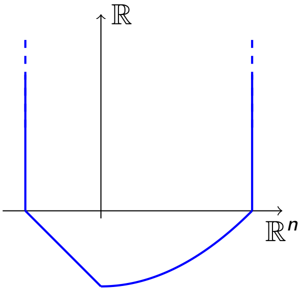
Open Problem

Question

Do functional intrinsic and mixed volumes satisfy inequalities? In particular,

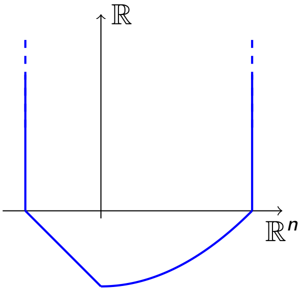
- isoperimetric inequalities,
- Brunn–Minkowski inequalities,
- Alexandrov–Fenchel inequalities.

What is Known?



$$\mathrm{Conv}_{\mathrm{sc},0}(\mathbb{R}^n) := \left\{ u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) : \partial(\mathrm{dom}\,(u)) = \{u = 0\} \right\}$$

What is Known?

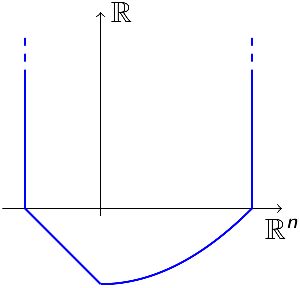


$$\mathrm{Conv}_{\mathrm{sc},0}(\mathbb{R}^n) := \left\{ u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) : \partial(\mathrm{dom}\,(u)) = \{u = 0\} \right\}$$

Theorem (Klartag, GAFA Lecture Notes 2007)

For $u_0, u_1, \dots, u_n \in \mathrm{Conv}_{\mathrm{sc},0}(\mathbb{R}^n)$,

What is Known?



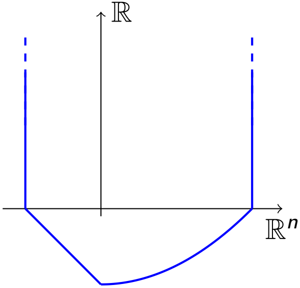
$$\mathrm{Conv}_{\mathrm{sc},0}(\mathbb{R}^n) := \big\{ u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) : \partial(\mathrm{dom}\,(u)) = \{u = 0\} \big\}$$

Theorem (Klartag, GAFA Lecture Notes 2007)

For $u_0, u_1, \dots, u_n \in \mathrm{Conv}_{\mathrm{sc},0}(\mathbb{R}^n)$,

$$\begin{aligned} \overline{\mathrm{V}}_{u_0}(u_1, \dots, u_n)^2 \\ \geq \overline{\mathrm{V}}_{u_0}(u_1, u_1, u_3 \dots, u_n)\overline{\mathrm{V}}_{u_0}(u_2, u_2, u_3, \dots, u_n). \end{aligned}$$

What is Known?



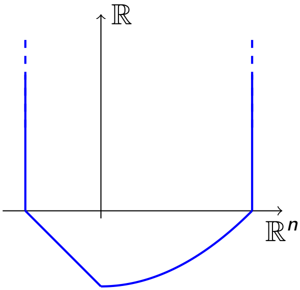
$$\mathrm{Conv}_{\mathrm{sc},0}(\mathbb{R}^n) := \left\{u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) : \partial(\mathrm{dom}\,(u)) = \{u = 0\}\right\}$$

Theorem (Klartag, GAFA Lecture Notes 2007)

For $u_0, u_1, \ldots, u_n \in \mathrm{Conv}_{\mathrm{sc},0}(\mathbb{R}^n)$,

$$\overline{\mathrm{V}}_{u_0}(u_1, \ldots, u_n)^2 \geq \overline{\mathrm{V}}_{u_0}(u_1, u_1, u_3 \ldots, u_n)\overline{\mathrm{V}}_{u_0}(u_2, u_2, u_3, \ldots, u_n).$$

What is Known?



$$\mathrm{Conv}_{\mathrm{sc},0}(\mathbb{R}^n) := \left\{u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) : \partial(\mathrm{dom}\,(u)) = \{u = 0\}\right\}$$

Theorem (Klartag, GAFA Lecture Notes 2007)

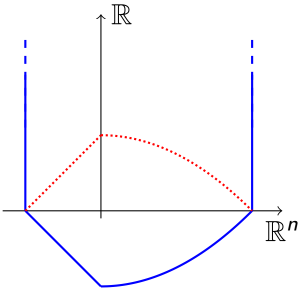
For $u_0, u_1, \dots, u_n \in \mathrm{Conv}_{\mathrm{sc},0}(\mathbb{R}^n)$,

$$\overline{\mathrm{V}}_{u_0}(u_1, \dots, u_n)^2 \geq \overline{\mathrm{V}}_{u_0}(u_1, u_1, u_3 \dots, u_n)\overline{\mathrm{V}}_{u_0}(u_2, u_2, u_3, \dots, u_n).$$

Observation

$$\overline{\mathrm{V}}_{u_0}(u_1, \dots, u_n) = V(K_{u_0}, \dots, K_{u_n}) \text{ for some } K_{u_0}, \dots, K_{u_n} \in \mathcal{K}^{n+1}.$$

What is Known?



$$\mathrm{Conv}_{\mathrm{sc},0}(\mathbb{R}^n) := \left\{u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) : \partial(\mathrm{dom}\,(u)) = \{u = 0\}\right\}$$

Theorem (Klartag, GAFA Lecture Notes 2007)

For $u_0, u_1, \dots, u_n \in \mathrm{Conv}_{\mathrm{sc},0}(\mathbb{R}^n)$,

$$\overline{\mathrm{V}}_{u_0}(u_1, \dots, u_n)^2 \geq \overline{\mathrm{V}}_{u_0}(u_1, u_1, u_3 \dots, u_n)\overline{\mathrm{V}}_{u_0}(u_2, u_2, u_3, \dots, u_n).$$

Observation

$$\overline{\mathrm{V}}_{u_0}(u_1, \dots, u_n) = V(K_{u_0}, \dots, K_{u_n}) \text{ for some } K_{u_0}, \dots, K_{u_n} \in \mathcal{K}^{n+1}.$$

Functional Brunn–Minkowski Inequality

Functional Brunn–Minkowski Inequality

Wishful Thinking

For non-negative $\alpha \in C_c([0,\infty))$, $u, v \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$, $1 \leq j \leq n$,

$$\overline{\mathrm{V}}_{j,\alpha}(u \,\Box\, v)^{\frac{1}{j}} \geq \overline{\mathrm{V}}_{j,\alpha}(u)^{\frac{1}{j}} + \overline{\mathrm{V}}_{j,\alpha}(v)^{\frac{1}{j}}.$$

Functional Brunn–Minkowski Inequality

Wishful Thinking

For non-negative $\alpha \in C_c([0,\infty))$, $u, v \in \mathrm{Conv}_{sc}(\mathbb{R}^n)$, $1 \le j \le n$,

$$\overline{V}_{j,\alpha}(u \,\square\, v)^{\frac{1}{j}} \ge \overline{V}_{j,\alpha}(u)^{\frac{1}{j}} + \overline{V}_{j,\alpha}(v)^{\frac{1}{j}}.$$

True for Special Cases

Functional Brunn–Minkowski Inequality

Wishful Thinking

For non-negative $\alpha \in C_c([0,\infty))$, $u, v \in \mathrm{Conv_{sc}}(\mathbb{R}^n)$, $1 \leq j \leq n$,

$$\overline{\mathrm{V}}_{j,\alpha}(u \,\square\, v)^{\frac{1}{j}} \geq \overline{\mathrm{V}}_{j,\alpha}(u)^{\frac{1}{j}} + \overline{\mathrm{V}}_{j,\alpha}(v)^{\frac{1}{j}}.$$

True for Special Cases

- $u = \mathrm{I}_K^\infty$, $v = \mathrm{I}_L^\infty$ with $K, L \in \mathcal{K}^n$.

Functional Brunn–Minkowski Inequality

Wishful Thinking

For non-negative $\alpha \in C_c([0,\infty))$, $u, v \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$, $1 \leq j \leq n$,

$$\overline{\mathrm{V}}_{j,\alpha}(u \mathbin{\square} v)^{\frac{1}{j}} \geq \overline{\mathrm{V}}_{j,\alpha}(u)^{\frac{1}{j}} + \overline{\mathrm{V}}_{j,\alpha}(v)^{\frac{1}{j}}.$$

True for Special Cases

- $u = \mathrm{I}_K^\infty$, $v = \mathrm{I}_L^\infty$ with $K, L \in \mathcal{K}^n$.
- $j = 1$.

Functional Brunn–Minkowski Inequality

Wishful Thinking

For non-negative $\alpha \in C_c([0,\infty))$, $u, v \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$, $1 \le j \le n$,

$$\overline{\mathrm{V}}_{j,\alpha}(u \,\square\, v)^{\frac{1}{j}} \ge \overline{\mathrm{V}}_{j,\alpha}(u)^{\frac{1}{j}} + \overline{\mathrm{V}}_{j,\alpha}(v)^{\frac{1}{j}}.$$

True for Special Cases

- $u = \mathrm{I}_K^\infty$, $v = \mathrm{I}_L^\infty$ with $K, L \in \mathcal{K}^n$.
- $j = 1$.

Proposition (M., Ulivelli, 2023+)

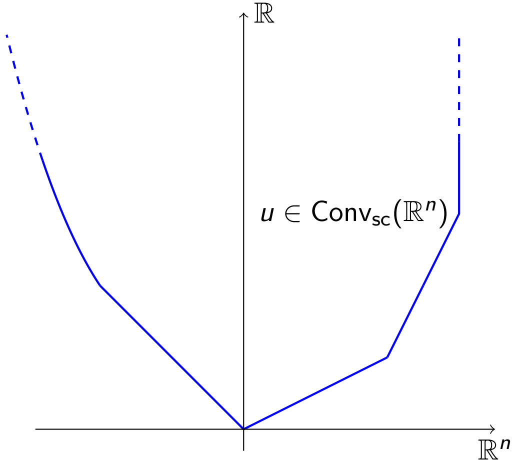
For every $2 \le j \le n$ and non-negative $\alpha \in C_c([0,\infty))$, $\alpha \not\equiv 0$, there exist functions $u, v \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ such that

$$\overline{\mathrm{V}}_{j,\alpha}(u \,\square\, v)^{\frac{1}{j}} < \overline{\mathrm{V}}_{j,\alpha}(u)^{\frac{1}{j}} + \overline{\mathrm{V}}_{j,\alpha}(v)^{\frac{1}{j}}.$$

From Functions to Bodies

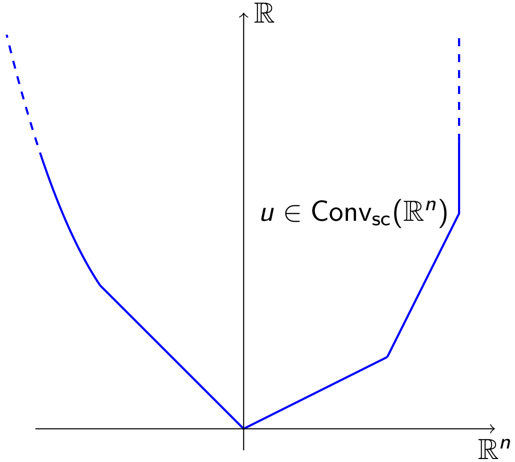
Fix $\alpha \in C_c([0, \infty))$.

From Functions to Bodies



Fix $\alpha \in C_c([0,\infty))$.

From Functions to Bodies

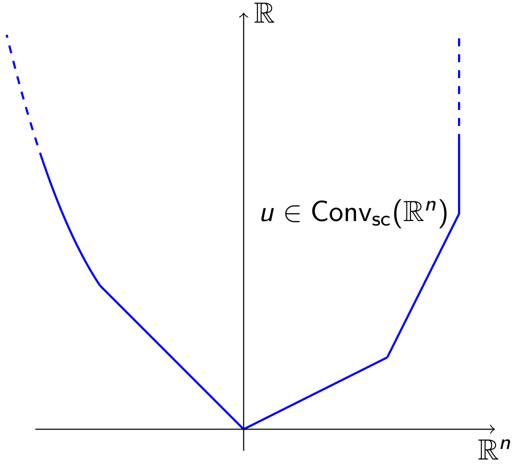


Fix $\alpha \in C_c([0,\infty))$.

Relating the Measures

$\overline{\mathrm{V}}_{n,\alpha}(u)$

From Functions to Bodies

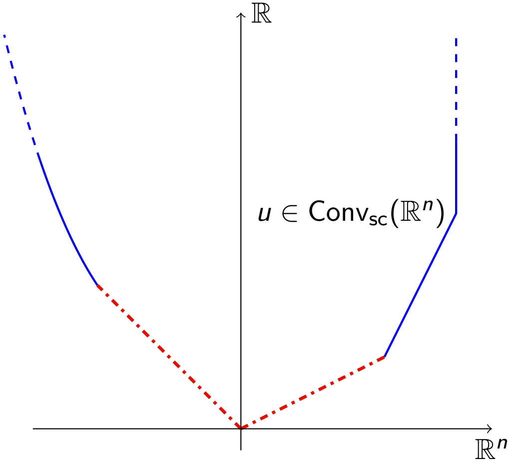


Fix $\alpha \in C_c([0,\infty))$.

Relating the Measures

$$\overline{\mathrm{V}}_{n,\alpha}(u) = \int_{\mathrm{dom}\,(u)} \alpha(|\nabla u(x)|)\,\mathrm{d}x$$

From Functions to Bodies

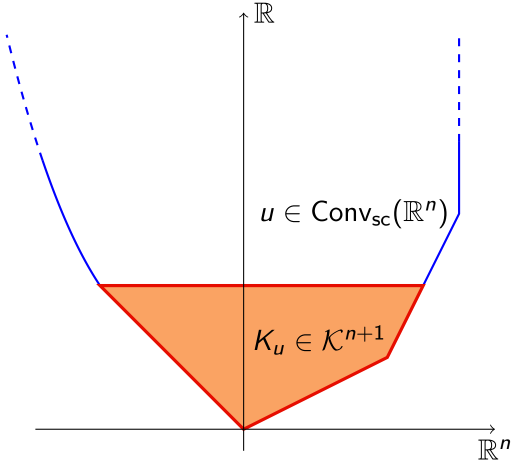


Fix $\alpha \in C_c([0,\infty))$.

Relating the Measures

$$\overline{\mathrm{V}}_{n,\alpha}(u) = \int_{\mathrm{dom}\,(u)} \alpha(|\nabla u(x)|)\,\mathrm{d}x$$

From Functions to Bodies

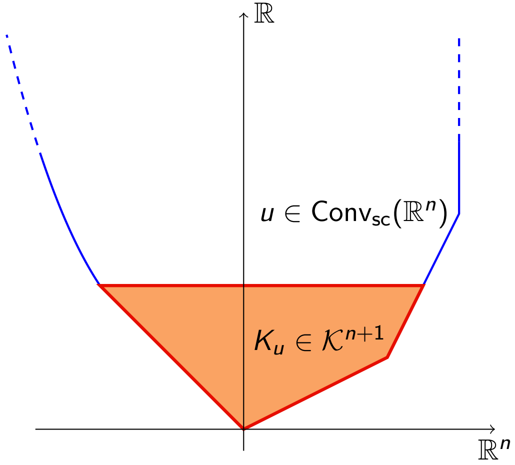


Fix $\alpha \in C_c([0,\infty))$.

Relating the Measures

$$\overline{\mathrm{V}}_{n,\alpha}(u) = \int_{\mathrm{dom}\,(u)} \alpha(|\nabla u(x)|)\,\mathrm{d}x$$

From Functions to Bodies



Fix $\alpha \in C_c([0,\infty))$.

Relating the Measures

$$\overline{\mathrm{V}}_{n,\alpha}(u) = \int_{\mathrm{dom}\,(u)} \alpha(|\nabla u(x)|)\,\mathrm{d}x$$
$$= \int_{\mathbb{S}^n_-} \tilde{\alpha}(|\langle z, e_{n+1}\rangle|)\,\mathrm{d}S_n(K_u, z)$$

where

$$\tilde{\alpha}(|\langle z, e_{n+1}\rangle|) = \frac{\alpha(|\operatorname{gno}(z)|)}{\sqrt{1+|\operatorname{gno}(z)|^2}}$$

for $z \in \mathbb{S}^n_-$.

An Explanation

Representation of $\overline{\mathrm{V}}_{n,\alpha}(u)$

$$\overline{\mathrm{V}}_{n,\alpha}(u) = \int_{\mathbb{S}^n_-} \tilde{\alpha}(|\langle z, e_{n+1}\rangle|)\,\mathrm{d}S_n(K_u, z)$$

An Explanation

Representation of $\overline{\mathrm{V}}_{n,\alpha}(u)$

$$\overline{\mathrm{V}}_{n,\alpha}(u) = \int_{\mathbb{S}^n_-} \tilde{\alpha}(|\langle z, e_{n+1}\rangle|)\,\mathrm{d}S_n(K_u, z)$$

Theorem (Colesanti, Hug, Saorín Gómez, J. Geom. Anal. 2014)

For $K \in \mathcal{K}^{n+1}$ let $\mathrm{F}(K) = \int_{\mathbb{S}^n} f(z)\,\mathrm{d}S_n(K, z)$ with some $f \in C(\mathbb{S}^n)$.

An Explanation

Representation of $\overline{\mathrm{V}}_{n,\alpha}(u)$

$$\overline{\mathrm{V}}_{n,\alpha}(u) = \int_{\mathbb{S}^n_-} \tilde{\alpha}(|\langle z, e_{n+1}\rangle|)\,\mathrm{d}S_n(K_u, z)$$

Theorem (Colesanti, Hug, Saorín Gómez, J. Geom. Anal. 2014)

For $K \in \mathcal{K}^{n+1}$ let $\mathrm{F}(K) = \int_{\mathbb{S}^n} f(z)\,\mathrm{d}S_n(K, z)$ with some $f \in C(\mathbb{S}^n)$. The inequality

$$F(K+L)^{\frac{1}{n}} \geq F(K)^{\frac{1}{n}} + F(L)^{\frac{1}{n}}$$

holds for all $K, L \in \mathcal{K}^{n+1}$, if and only if f is a support function.

An Explanation

Representation of $\overline{\mathrm{V}}_{n,\alpha}(u)$

$$\overline{\mathrm{V}}_{n,\alpha}(u) = \int_{\mathbb{S}^n_-} \tilde{\alpha}(|\langle z, e_{n+1}\rangle|)\,\mathrm{d}S_n(K_u, z)$$

Theorem (Colesanti, Hug, Saorín Gómez, J. Geom. Anal. 2014)

For $K \in \mathcal{K}^{n+1}$ let $\mathrm{F}(K) = \int_{\mathbb{S}^n} f(z)\,\mathrm{d}S_n(K, z)$ with some $f \in C(\mathbb{S}^n)$. The inequality

$$F(K+L)^{\frac{1}{n}} \geq F(K)^{\frac{1}{n}} + F(L)^{\frac{1}{n}}$$

holds for all $K, L \in \mathcal{K}^{n+1}$, if and only if f is a support function.

Remark

Since α has compact support, $z \mapsto \tilde{\alpha}(|\langle z, e_{n+1}\rangle|)$ vanishes in a neighborhood of the equator.

Isoperimetric Inequality

Isoperimetric Inequality

Proposition (M., Ulivelli, 2023+)

For every non-negative $\alpha \in C_c([0,\infty))$, $\alpha \not\equiv 0$, the operator $\overline{\mathrm{V}}_{n-1,\alpha}$ does not attain a minimum on

$$A(n,\alpha) = \{u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) : \overline{\mathrm{V}}_{n,\alpha}(u) = 1\}.$$

Isoperimetric Inequality

Proposition (M., Ulivelli, 2023+)

For every non-negative $\alpha \in C_c([0,\infty))$, $\alpha \not\equiv 0$, the operator $\overline{\mathrm{V}}_{n-1,\alpha}$ does not attain a minimum on

$$A(n,\alpha) = \{u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) : \overline{\mathrm{V}}_{n,\alpha}(u) = 1\}.$$

In fact, $\overline{\mathrm{V}}_{n-1,\alpha}(u) > 0$ for every $u \in A(n,\alpha)$, while

$$\inf_{u \in A(n,\alpha)} \overline{\mathrm{V}}_{n-1,\alpha}(u) = 0.$$

Isoperimetric Inequality

Proposition (M., Ulivelli, 2023+)

For every $1 \leq j < k \leq n$ and non-negative $\alpha \in C_c([0,\infty))$, $\alpha \not\equiv 0$, the operator $\overline{\mathrm{V}}_{j,\alpha}$ does not attain a minimum on

$$A(k,\alpha) = \{u \in \mathrm{Conv_{sc}}(\mathbb{R}^n) : \overline{\mathrm{V}}_{k,\alpha}(u) = 1\}.$$

In fact, $\overline{\mathrm{V}}_{j,\alpha}(u) > 0$ for every $u \in A(k,\alpha)$, while

$$\inf_{u \in A(k,\alpha)} \overline{\mathrm{V}}_{j,\alpha}(u) = 0.$$

Short Explanation

The measures $\mathrm{MA}_j^*(u;\cdot)$ correspond to mixed area measures $S(K_u[j], B_H^n[n-j], \cdot)$, where

$$B_H^n = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + \cdots + x_n^2 \le 1, x_{n+1} = 0\}.$$

Short Explanation

The measures $\mathrm{MA}_j^*(u;\cdot)$ correspond to mixed area measures $S(K_u[j], B_H^n[n-j],\cdot)$, where

$$B_H^n = \{(x_1,\dots,x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2+\cdots+x_n^2 \leq 1, x_{n+1}=0\}.$$

Mixed Area Measure

$$S_{n-1}(\lambda_1 K_1+\cdots+\lambda_m K_m,\cdot) = \sum_{i_1,\dots,i_{n-1}=1}^{m} \lambda_{i_1}\cdots\lambda_{i_{n-1}} S(K_{i_1},\dots,K_{i_{n-1}},\cdot)$$

for every $K_1,\dots,K_m \in \mathcal{K}^n$, $\lambda_1\dots,\lambda_m \geq 0$ and $m \in \mathbb{N}$.

Short Explanation

The measures $\mathrm{MA}_j^*(u;\cdot)$ correspond to mixed area measures $S(K_u[j], B_H^n[n-j], \cdot)$, where

$$B_H^n = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + \cdots + x_n^2 \leq 1, x_{n+1} = 0\}.$$

Theorem (Schneider, Geom. Dedicata 1988)

For every $0 \leq j \leq n$ *and* $K \in \mathcal{K}^{n+1}$,

$$\operatorname{supp} S(K[j], B_H^n[n-j], \cdot) \subseteq \operatorname{cl}\operatorname{ext}(K[j], B_H^n[n-j]).$$

Short Explanation

The measures $\mathrm{MA}_j^*(u;\cdot)$ correspond to mixed area measures $S(K_u[j], B_H^n[n-j],\cdot)$, where

$$B_H^n = \{(x_1,\dots,x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2+\cdots+x_n^2 \le 1, x_{n+1}=0\}.$$

Theorem (Hug, M., Ulivelli, 2023+)

For every $0 \le j \le n$ *and* $K \in \mathcal{K}^{n+1}$,

$$\operatorname{supp} S(K[j], B_H^n[n-j],\cdot) = \operatorname{cl}\operatorname{ext}(K[j], B_H^n[n-j]).$$

Short Explanation

The measures $\mathrm{MA}_j^*(u;\cdot)$ correspond to mixed area measures $S(K_u[j], B_H^n[n-j],\cdot)$, where

$$B_H^n = \{(x_1,\dots,x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2+\cdots+x_n^2 \le 1, x_{n+1}=0\}.$$

Theorem (Hug, M., Ulivelli, 2023+)

For every $0 \le j \le n$ *and* $K \in \mathcal{K}^{n+1}$,

$$\operatorname{supp} S(K[j], B_H^n[n-j],\cdot) = \operatorname{cl}\operatorname{ext}(K[j], B_H^n[n-j]).$$

Corollary

$$\operatorname{supp} S(K[n], B_H^n[0],\cdot) \subseteq \cdots \subseteq \operatorname{supp} S(K[1], B_H^n[n-1],\cdot)$$

Short Explanation

The measures $\mathrm{MA}_j^*(u;\cdot)$ correspond to mixed area measures $S(K_u[j], B_H^n[n-j], \cdot)$, where

$$B_H^n = \{(x_1,\dots,x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + \cdots + x_n^2 \leq 1, x_{n+1} = 0\}.$$

Theorem (Hug, M., Ulivelli, 2023+)

For every $0 \leq j \leq n$ and $K \in \mathcal{K}^{n+1}$,

$$\operatorname{supp} S(K[j], B_H^n[n-j], \cdot) = \operatorname{cl} \operatorname{ext}(K[j], B_H^n[n-j]).$$

Corollary

$$\operatorname{supp} S(K[n], B_H^n[0], \cdot) \subseteq \cdots \subseteq \operatorname{supp} S(K[1], B_H^n[n-1], \cdot)$$

Corollary

$$\operatorname{supp} \mathrm{MA}_n^*(u;\cdot) \subseteq \cdots \subseteq \operatorname{supp} \mathrm{MA}_1^*(u;\cdot)$$

Wulff-type Inequalities

For $u \in \mathrm{Conv}_{\mathrm{sc},0}(\mathbb{R}^n)$ set $\overline{\mathrm{V}}_{n+1}(u) = \int_{\mathrm{dom}\,(u)} |u(x)|\,\mathrm{d}x$.

Wulff-type Inequalities

For $u \in \mathrm{Conv}_{\mathrm{sc},0}(\mathbb{R}^n)$ set $\overline{\mathrm{V}}_{n+1}(u) = \int_{\mathrm{dom}\,(u)} |u(x)|\,\mathrm{d}x$.

Theorem (M., Ulivelli, 2023+)

If $\beta \in C(\mathbb{R}^n)$ *is such that* $0 < c \leq \frac{\beta(x)}{\sqrt{1+|x|^2}} \leq C$ *for every* $x \in \mathbb{R}^n$ *with* $c, C > 0$

Wulff-type Inequalities

For $u \in \mathrm{Conv}_{\mathrm{sc},0}(\mathbb{R}^n)$ set $\overline{\mathrm{V}}_{n+1}(u) = \int_{\mathrm{dom}\,(u)} |u(x)|\,\mathrm{d}x$.

Theorem (M., Ulivelli, 2023+)

If $\beta \in C(\mathbb{R}^n)$ is such that $0 < c \leq \frac{\beta(x)}{\sqrt{1+|x|^2}} \leq C$ for every $x \in \mathbb{R}^n$ with $c, C > 0$, then

$$\overline{\mathrm{V}}_{n,\beta}(u)^{\frac{1}{n}} \geq d(n,\beta)\overline{\mathrm{V}}_{n+1}(u)^{\frac{1}{n+1}}$$

for every $u \in \mathrm{Conv}_{\mathrm{sc},0}(\mathbb{R}^n)$.

Wulff-type Inequalities

For $u \in \mathrm{Conv}_{\mathrm{sc},0}(\mathbb{R}^n)$ set $\overline{\mathrm{V}}_{n+1}(u) = \int_{\mathrm{dom}\,(u)} |u(x)|\,\mathrm{d}x$.

Theorem (M., Ulivelli, 2023+)

If $\beta \in C(\mathbb{R}^n)$ is such that $0 < c \leq \frac{\beta(x)}{\sqrt{1+|x|^2}} \leq C$ for every $x \in \mathbb{R}^n$ with $c, C > 0$, then

$$\overline{\mathrm{V}}_{n,\beta}(u)^{\frac{1}{n}} \geq d(n,\beta)\overline{\mathrm{V}}_{n+1}(u)^{\frac{1}{n+1}}$$

for every $u \in \mathrm{Conv}_{\mathrm{sc},0}(\mathbb{R}^n)$.

Remark

More general setting possible but it involves boundary terms.

Thank you very much,

Thank you very much,

Thank you very much,