Strengthened inequalities for the mean width and the ℓ-norm

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Disclaimer

This is talk is based on joint works with K.J. Böröczky (Budapest) and D. Hug (Karlsruhe).

The mean width

- Let K be a convex body K in $\mathbb{R}^n$ (compact convex set with nonempty interior).
- Its support function $h_K : \mathbb{R}^n \to \mathbb{R}$ is

$$h_K(x) = \max_{y \in K} \langle x, y \rangle \text{ for } x \in \mathbb{R}^n.$$

- The mean width of K is

$$W(K) = \frac{1}{n\kappa_n} \int_{S^{n-1}} h_K(u) - h_K(-u)\, du,$$

where integration is wrt the $(n-1)$-dimensional Hausdorff measure (that coincides with the spherical Lebesgue measure in this case).

The ℓ-norm

- For a convex body $K \subset \mathbb{R}^n$ with $o \in \operatorname{int} K$, the gauge function is

$$\|x\|_K = \min\{t \geq 0 : \ x \in tK\}, \ \text{for } x \in \mathbb{R}^n.$$

- γ_n is the standard Gaussian measure in $\mathbb{R}^n$ with density function $x \mapsto \frac{1}{\sqrt{2\pi}^n} e^{-\|x\|^2/2}$, $x \in \mathbb{R}^n$, wrt the Lebesgue measure.
- The ℓ-norm of K is defined as

$$\ell(K) = \int_{\mathbb{R}^n} \|x\|_K \, \gamma_n(dx) = \mathbb{E}\|X\|_K,$$

where X is a Gaussian random vector with distribution γ_n.

The ℓ-norm

- The polar body of K is

$$K^\circ = \{x \in \mathbb{R}^n : \langle x, y \rangle \leq 1 \; \forall y \in K\}.$$

- B_p^n is the unit ball of the l_p norm in $\mathbb{R}^n$ for $p \in [1, \infty]$.
- In particular, B_2^n is a Euclidean unit ball, B_1^n is a regular crosspolytope inscribed into B_2^n, and $(B_1^n)^\circ = B_\infty^n$ is a cube circumscribed around B_2^n.
- Integration in polar coordinates yields that

$$\ell(K) = \tfrac{\ell(B_2^n)}{2} W(K^\circ).$$

with

$$\lim_{n\to\infty} \frac{\ell(B_2^n)}{\sqrt{n}} = 1.$$

- The ℓ-norm of K can be expressed in the form (Barthe 1998)

$$\ell(K) = \int_{\mathbb{R}^n} \mathbb{P}(\|X\|_K > t)\, dt = \int_0^\infty (1 - \gamma_n(tK))\, dt.$$

John and Löwner ellipsoids

Theorem (John, Ball)

Let K be an n-dimensional convex body. Then there exists a unique maximal volume ellipsoid contained in K. Moreover, this maximal volume ellipsoid is the unit ball if and only if there exists vectors $u_1, \ldots, u_m \in \mathrm{bd}K \cap S^{n-1}$ and (positive) $c_1, \ldots, c_m$ such that

$$\sum_{i=1}^{m} c_i u_i \otimes u_i = \mathrm{Id}_n, \text{ and}$$

$$\sum_{i=1}^{m} c_i u_i = 0.$$

- The ellipsoid in the Theorem is called the John ellipsoid of K.
- By polar duality, there also exists a unique ellipsoid of minimal volume containing K, called the Löwner ellipsoid of K with a similar characterization.

Volume ratios

- Ball 1991 proved that simplices (cubes if K is origin symmetric) maximize the volume of K given the volume of the John ellipsoid of K, and thus simplices (cubes) determine the extremal inner volume ratio.
- For the dual problem, Barthe 1998 proved that simplices (regular crosspolytopes if K is origin symmetric) minimize the volume of K given the volume of the Löwner ellipsoid of K, hence simplices (regular crosspolytopes) determine the extremal outer volume ratio.
- Equality cases were characterized by Barthe 1998.

Let Δ_n be a regular simplex inscribed into B_2^n, and Δ_n° a regular simplex circumscribed around B_2^n.

Theorem (Barthe '98, Schmuckenschläger '99)

Let K be a convex body in $\mathbb{R}^n$.

(i) *If $B_2^n \supset K$ is the Löwner ellipsoid of K, then $\ell(K) \leq \ell(\Delta_n)$, and if $B_2^n \subset K$ is the John ellipsoid of K, then $W(K) \geq W(\Delta_n^\circ)$. Equality holds in either case if and only if K is a regular simplex.*

(ii) *If $B_2^n \subset K$ is the John ellipsoid of K, then $W(K) \leq W(\Delta_n^\circ)$, and if $B_2^n \supset K$ is the Löwner ellipsoid of K, then $W(K) \geq W(\Delta_n)$. Equality holds in either case if and only if K is a regular.*

- Hausdorff distance between compact subsets X and Y of $\mathbb{R}^n$:

$$\delta_H(X, Y) = \max\{\max_{y \in Y} d(y, X), \max_{x \in X} d(x, Y)\},$$

  where $d(x, Y) = \min_{y \in Y} \|x - y\|$.
- Symmetric difference distance for convex bodies K and C:

$$\delta_{\mathrm{vol}}(K, C) = V(K \backslash C) + V(C \backslash K).$$

- Let $O(n)$ denote the orthogonal group (rotation group) of $\mathbb{R}^n$.

Stability in the nonsymmetric case

Theorem (K.J. Böröczky, FF, D. Hug, JLMS 2021*)

Let B_2^n be the Löwner ellipsoid of a convex body $K \subset B_2^n$ in $\mathbb{R}^n$, let $c = n^{26n}$ and let $\varepsilon \in (0,1)$. If $\ell(K) \geq (1-\varepsilon)\ell(\Delta_n)$, then there exists a $T \in O(n)$ such that

(i) $\delta_{\mathrm{vol}}(K, T\Delta_n) \leq c\,\sqrt[4]{\varepsilon}$,

(ii) $\delta_H(K, T\Delta_n) \leq c\,\sqrt[4]{\varepsilon}$.

Theorem (K.J. Böröczky, FF, D. Hug, JLMS 2021*)

Let B_2^n be the John ellipsoid of a convex body $K \supset B_2^n$ in $\mathbb{R}^n$ and let $\varepsilon > 0$. If $\ell(K) \leq (1+\varepsilon)\ell(\Delta_n^\circ)$, then there exists a $T \in O(n)$ such that

(i) $\delta_{\mathrm{vol}}(K, T\Delta_n^\circ) \leq c\,\sqrt[4]{\varepsilon}$ *for* $c = n^{27n}$,

(ii) $\delta_H(K, T\Delta_n^\circ) \leq c\,\sqrt[4n]{\varepsilon}$ *for* $c = n^{27}$.

* K.J. Böröczky, FF, D. Hug, *Strengthened inequalities for the mean width and the ℓ-norm*, J. London Math Soc. **104** (2021), 233–268.

- For the first Theorem, add an $(n+2)$nd vertex $v_{n+2} \in S^{n-1}$ to the $n+1$ vertices $v_1, \ldots, v_{n+1}$ of Δ_n such that v_1 lies on the geodesic arc on S^{n-1} connecting v_2 and v_{n+1}, and such that $\angle(v_{n+1}, v_1) = c_1\varepsilon$ for a suitable $c_1 > 0$ depending on n. The polytope $K = \mathrm{conv}\{v_1, \ldots, v_{n+2}\}$ satisfies $\ell(K) \geq (1-\varepsilon)\ell(\Delta_n)$ on the one hand, and $\delta_{\mathrm{vol}}(K, T\Delta_n) \geq c_2\varepsilon$ and $\delta_H(K, T\Delta_n) \geq c_2\varepsilon$ for a suitable $c_2 > 0$, depending on n, and for any $T \in O(n)$, on the other hand.
- Using the polar of K for (i) of the second Theorem, we have $\ell(K^\circ) \leq (1+\varepsilon)\ell(\Delta_n^\circ)$ while $\delta_{\mathrm{vol}}(K^\circ, T\Delta_n^\circ) \geq c_3\varepsilon$ for a suitable $c_3 > 0$ depending on n and for any $T \in O(n)$.
- Cut off $n+1$ regular simplices of edge length $c_4\sqrt[n]{\varepsilon}$ at the vertices of Δ_n°, for a suitable $c_4 > 0$ depending on n. For the resulting polytope $\widetilde{K}$, $\ell(\widetilde{K}) \leq (1+\varepsilon)\ell(\Delta_n^\circ)$ and $\delta_H(\widetilde{K}, T\Delta_n^\circ) \geq c_5\sqrt[n]{\varepsilon}$ for any $T \in O(n)$ for some suitable $c_5 > 0$ depending on n.

Let $u_1, \ldots, u_k \in S^{n-1}$ and $c_1, \ldots, c_k > 0$ form a John decomposition of the identity, and let and $f_1, \ldots, f_k$ be non-negative measurable functions on $\mathbb{R}$.

The rank one geometric Brascamp-Lieb inequality:

$$\int_{\mathbb{R}^n} \prod_{i=1}^{k} f_i(\langle x, u_i \rangle)^{c_i} \, dx \le \prod_{i=1}^{k} \left(\int_{\mathbb{R}} f_i \right)^{c_i}.$$

The reverse rank one geometric Brascamp-Lieb inequality:

$$\int_{\mathbb{R}^n} \sup_{x = \sum_{i=1}^{k} c_i \theta_i u_i} \prod_{i=1}^{k} f_i(\theta_i)^{c_i} \, dx \ge \prod_{i=1}^{k} \left(\int_{\mathbb{R}} f_i \right)^{c_i}.$$

In the reverse Brascamp–Lieb inequality, we always assume that $\theta_1, \ldots, \theta_k \in \mathbb{R}$.

The origin symmetric case

Theorem (Barthe '98, Schechtmann, Schmuckenschläger '95)

Let K be an origin symmetric convex body in $\mathbb{R}^n$.

(i) *If $B_2^n \supset K$ is the Löwner ellipsoid of K, then $\ell(K) \leq \ell(B_1^n)$ and $W(K) \geq W(B_1^n)$. Equality holds in either case if and only if K is a regular crosspolytope.*

(ii) *If $B^n \subset K$ is the John ellipsoid of K, then $W(K) \leq W(B_\infty^n)$ and $\ell(K) \geq \ell(B_\infty^n)$. Equality holds in either case if and only if K is a cube.*

Stability in the origin symmetric case

Theorem (K.J. Böröczky, FF, D. Hug 2023+)

If K is an origin symmetric convex body in $\mathbb{R}^n$ such that $B_2^n \supset K$ is the Löwner ellipsoid of K, and $\ell(K) \geq (1-\varepsilon)\ell(B_1^n)$ for some $\varepsilon \in (0, \varepsilon_0)$, then there exists an orthogonal transformation $\Phi \in O(n)$ such that $\delta_H(K, \Phi B_1^n) \leq \mathrm{constant} \cdot \varepsilon$.

- The order of ε is optimal in the above Theorem.

Sketch of the proof

- Let $u_1, \dots, u_k \in S^{n-1}$ and $c_1, \dots, c_k > 0$, $k > n$, satisfy $\sum_{i=1}^k c_i u_i \otimes u_i = \mathrm{I}_n$.
- Let $0 < \eta \le 1/(3n)$. If for any $j \in \{n+1, \dots, k\}$ there exists some $i \in \{1, \dots, n\}$ satisfying $|\langle u_i, u_j \rangle| \ge \cos\eta$, then there exist an orthonormal basis $w_1, \dots, w_n$ and $\xi_j \in \{-1, 1\}$ for $j = 1, \dots, k$ such that

$$\delta_H\left(\{w_1, \dots, w_n\}, \{\xi_1 u_1, \dots, \xi_k u_k\}\right) < 4\sqrt{n}\,\eta.$$

- If $R > 0$ and $K \subset C \subset RB^n$ are convex bodies, then

$$\ell(K) - \ell(C) \ge \frac{1}{2}\left(\frac{n}{2\pi e}\right)^{\frac{n}{2}} R^{-(n+1)} V(C \setminus K).$$

Sketch of the proof

- We define $M = \mathrm{conv}\{\pm u_1, \ldots, \pm u_k\} \subset K$, and the core claim of our argument is that there exists $\Phi \in O(n)$ and $\gamma > 0$ depending on n such that

$$(1 - \gamma\varepsilon)\Phi B_1^n \subset M.$$

- Let $\tilde{\eta} \in [0, \frac{\pi}{2})$ be minimal s.t. for any $j = n+1, \ldots, k$ there exists $i \in \{1, \ldots, n\}$ with $|\langle u_i, u_j \rangle| \geq \cos\tilde{\eta}$, for η sufficiently small.
- In particular, we may assume that

$$|\langle u_i, u_k \rangle| \leq \cos\tilde{\eta} \quad \text{for } i = 1, \ldots, n.$$

- Using a suitably chosen cone, we estimate the ℓ-norm of K and deduce that $\eta \leq \gamma_1\varepsilon$.
- Thus, there exists an orthonormal basis $w_1, \ldots, w_n$ such that

$$\delta_H\left(\{\pm w_1, \ldots, \pm w_n\}, \{\pm u_1, \ldots, \pm u_k\}\right) < 4\sqrt{n}\,\eta < \gamma_2\varepsilon.$$

Sketch of the proof

- If $z \in K$ satisfies $\|z\|_{B_1^n} = (1+t)(1-\gamma\varepsilon)$ for $t \in (0,1)$, then consider
$$P_z = \mathrm{conv}\left\{\{\pm z\} \cup (1-\gamma\varepsilon)B_1^n\right\}$$
- Estimating the $\ell(P_z)$ puts an upper bound $t \leq \frac{2\gamma+1}{\gamma_4} \cdot \varepsilon = \gamma_5\varepsilon$, so
$$K \subset (1+\gamma_5\varepsilon)B_1^n.$$

Thank you for your attention.