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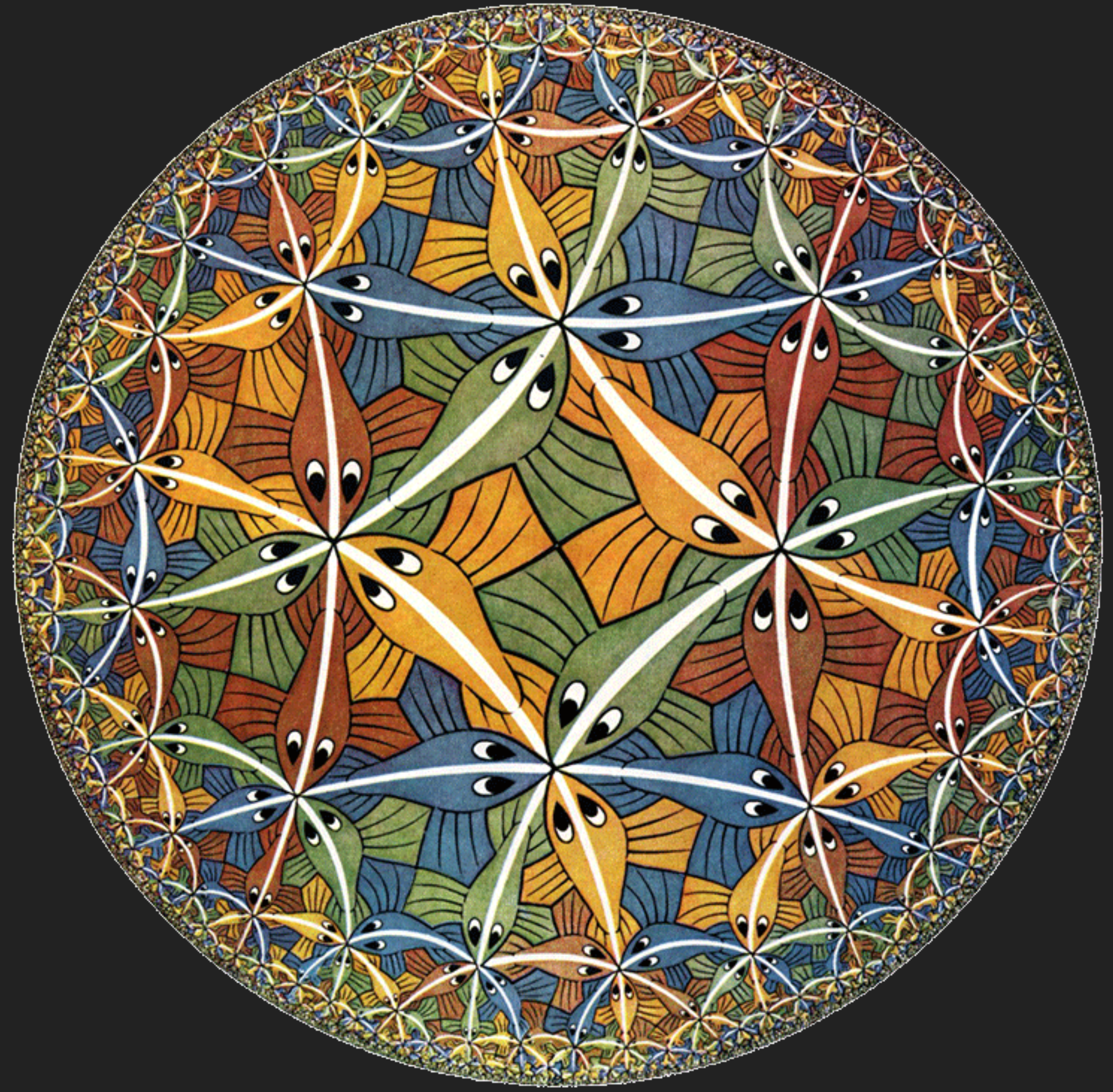


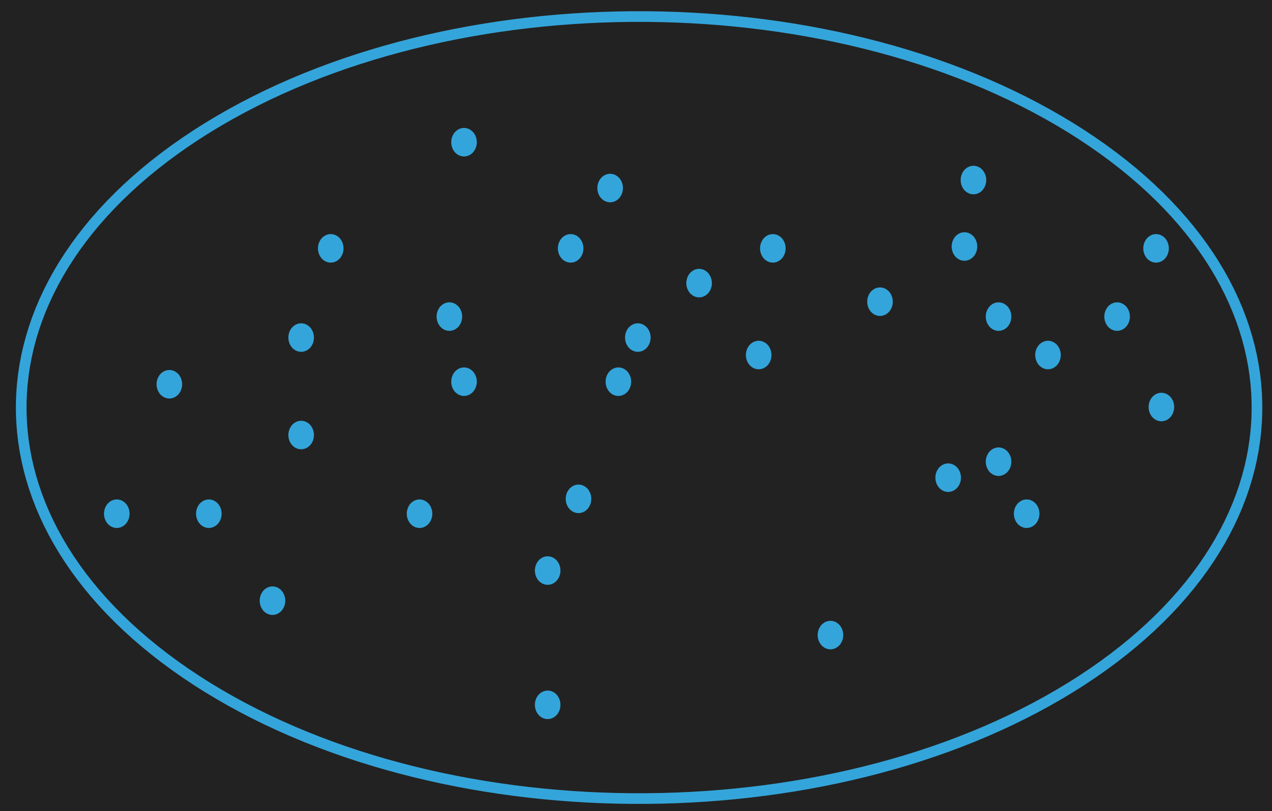
CHRISTOPH THÄLE

HOW COXETER AND ESCHER MEET POISSON

THE POISSON PROCESS

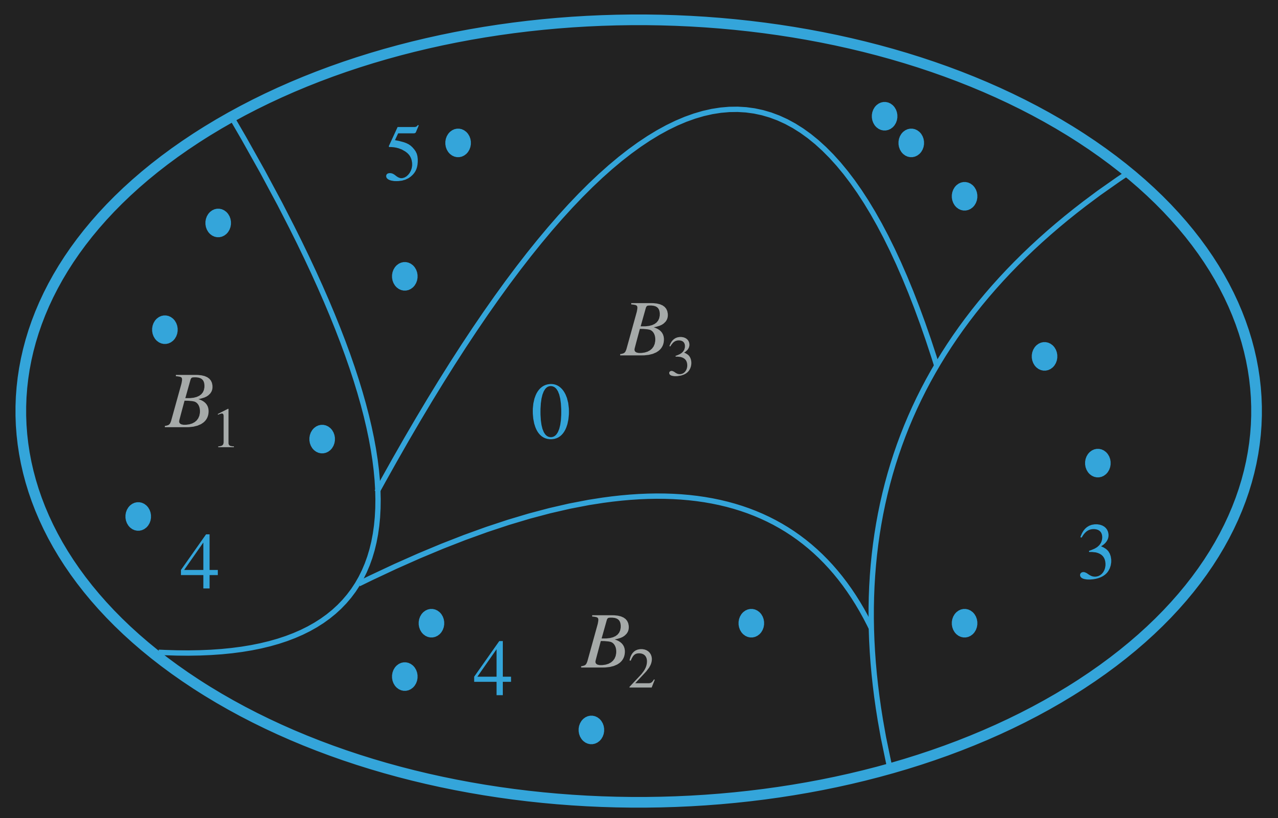
- $(X, \mathbb{X})$ measurable space
- μ probability/finite measure

- Random sample of n independent random points with distribution μ



- $(X, \mathbb{X})$ measurable space
- μ σ-finite measure

- Poisson process with intensity measure μ



Within each B_i :

- $\mu|_{B_i}$ is a finite measure
- $N_i \sim \text{Poisson}(\mu(B_i))$, that is, $\mathbb{P}[N_i = k] = \dfrac{\mu(B_i)^k}{k!} e^{-\mu(B_i)}$
- random sample of N_i points with distribution $\mu|_{B_i} / \mu(B_i)$

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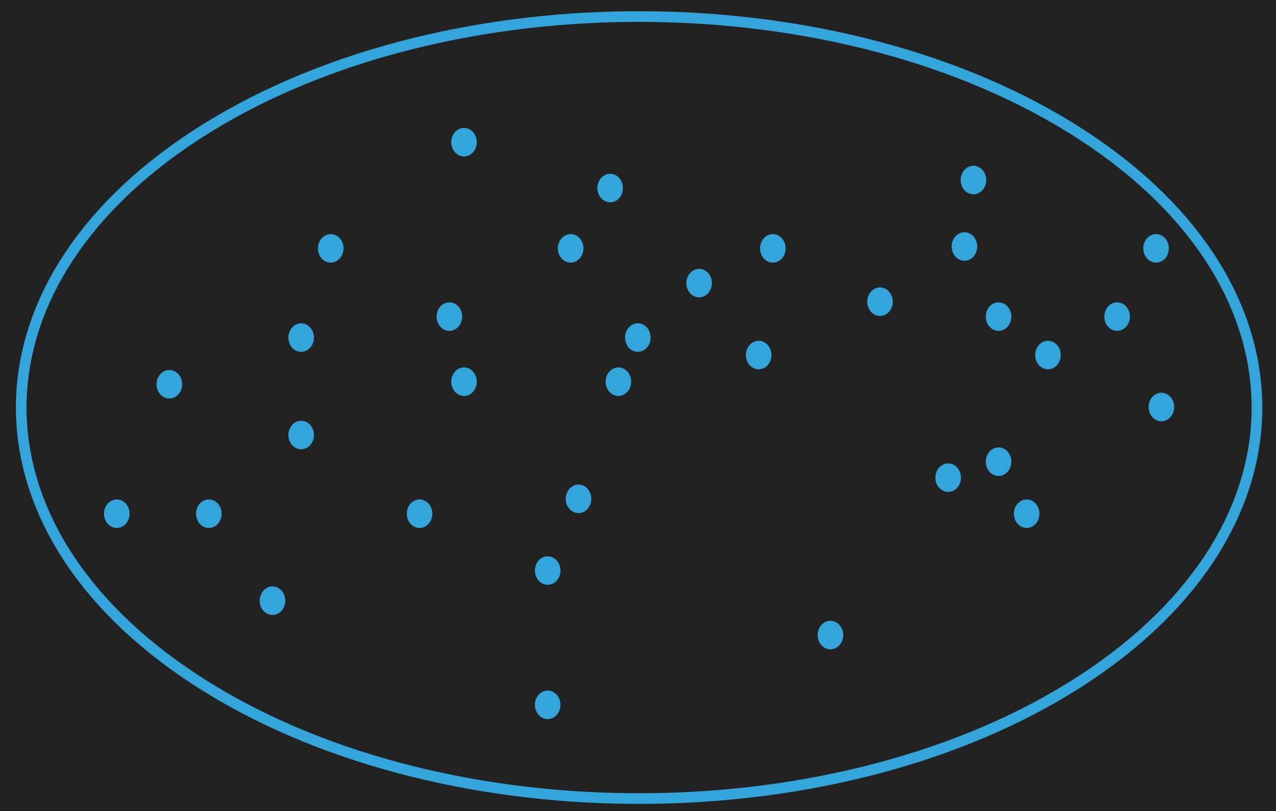

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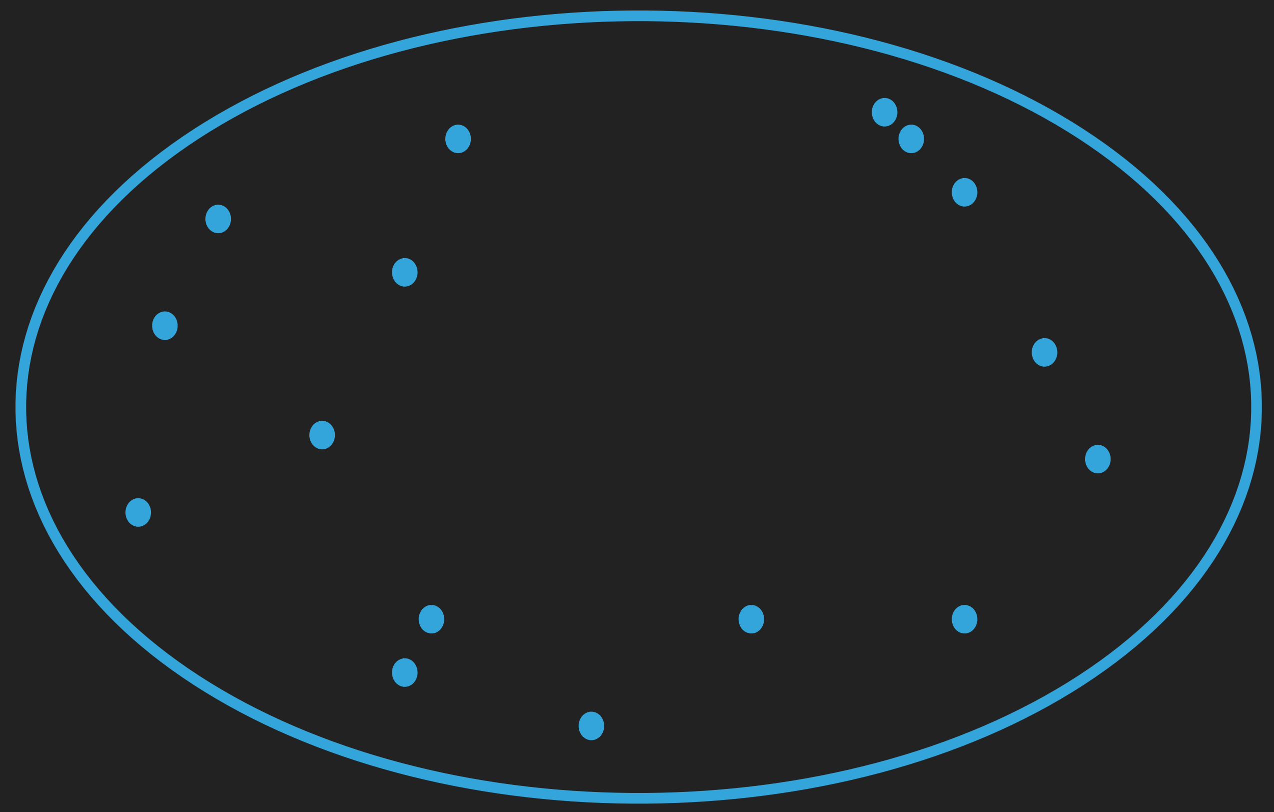


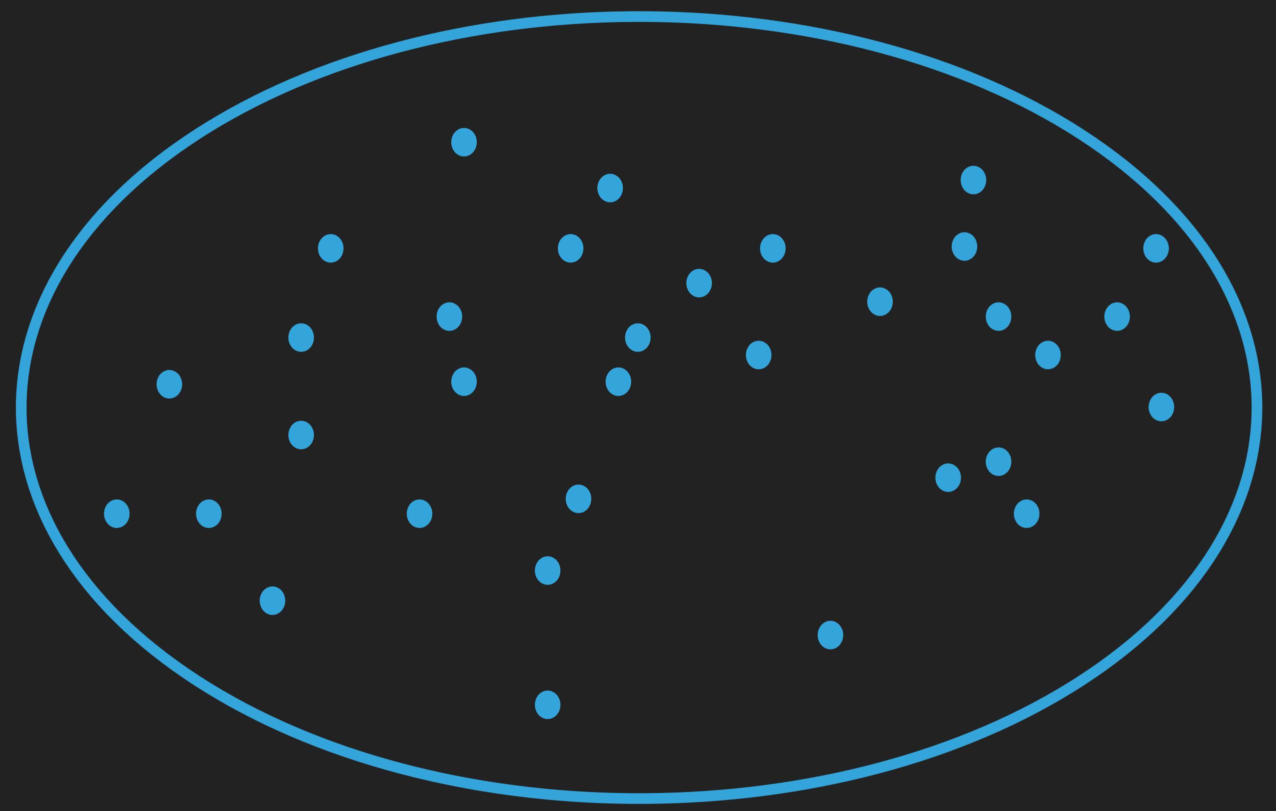
Now forget about the sets B_i and the numbers N_i

- Poisson process η with intensity measure μ

THE POISSON PROCESS

- $(X, \mathbb{X})$ measurable space
- μ probability/finite measure

- Random sample of n independent random points with distribution μ



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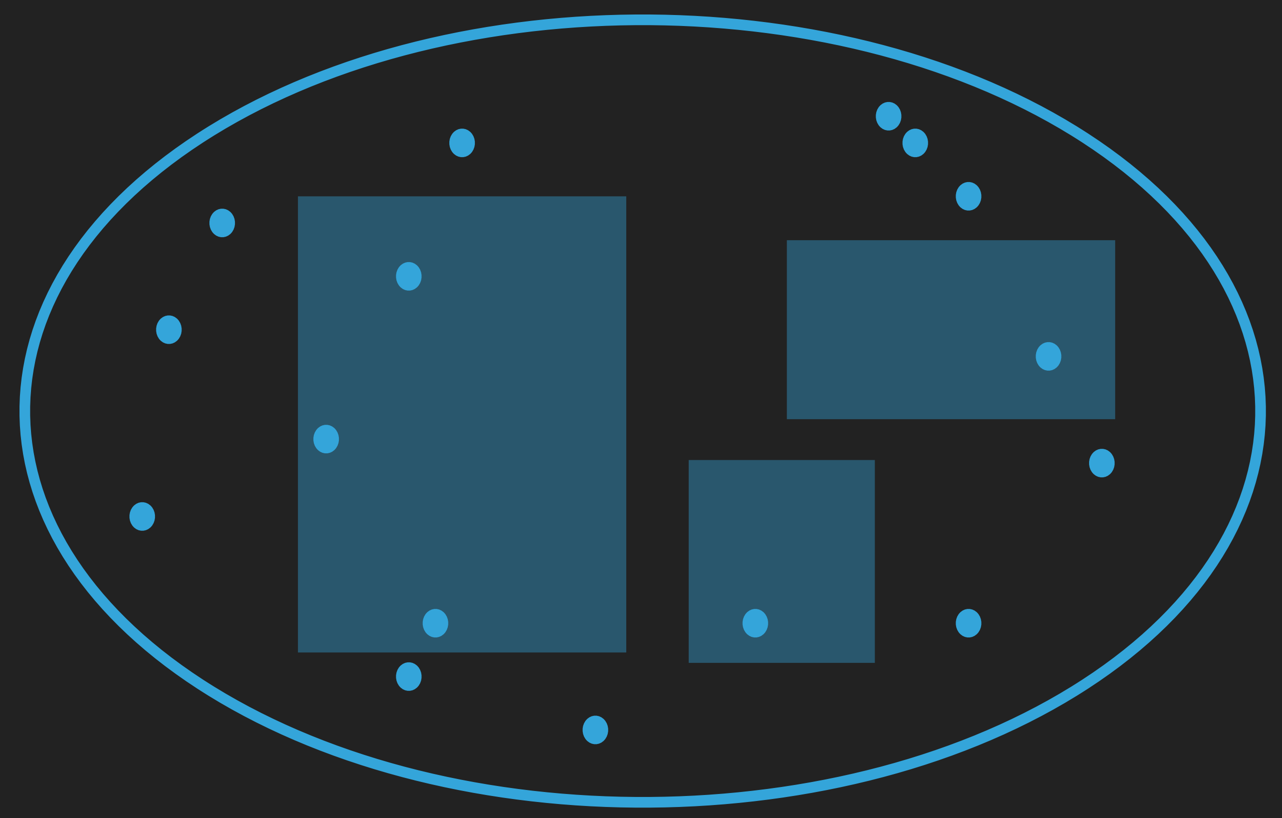


Key properties of a Poisson process η with intensity measure μ:

- $\eta(B) \sim \text{Poisson}(\mu(B))$
- $B_1, \ldots, B_n$ disjoint $\Longrightarrow \eta(B_1), \ldots, \eta(B_n)$ independent

THE POISSON HYPERPLANE PROCESS

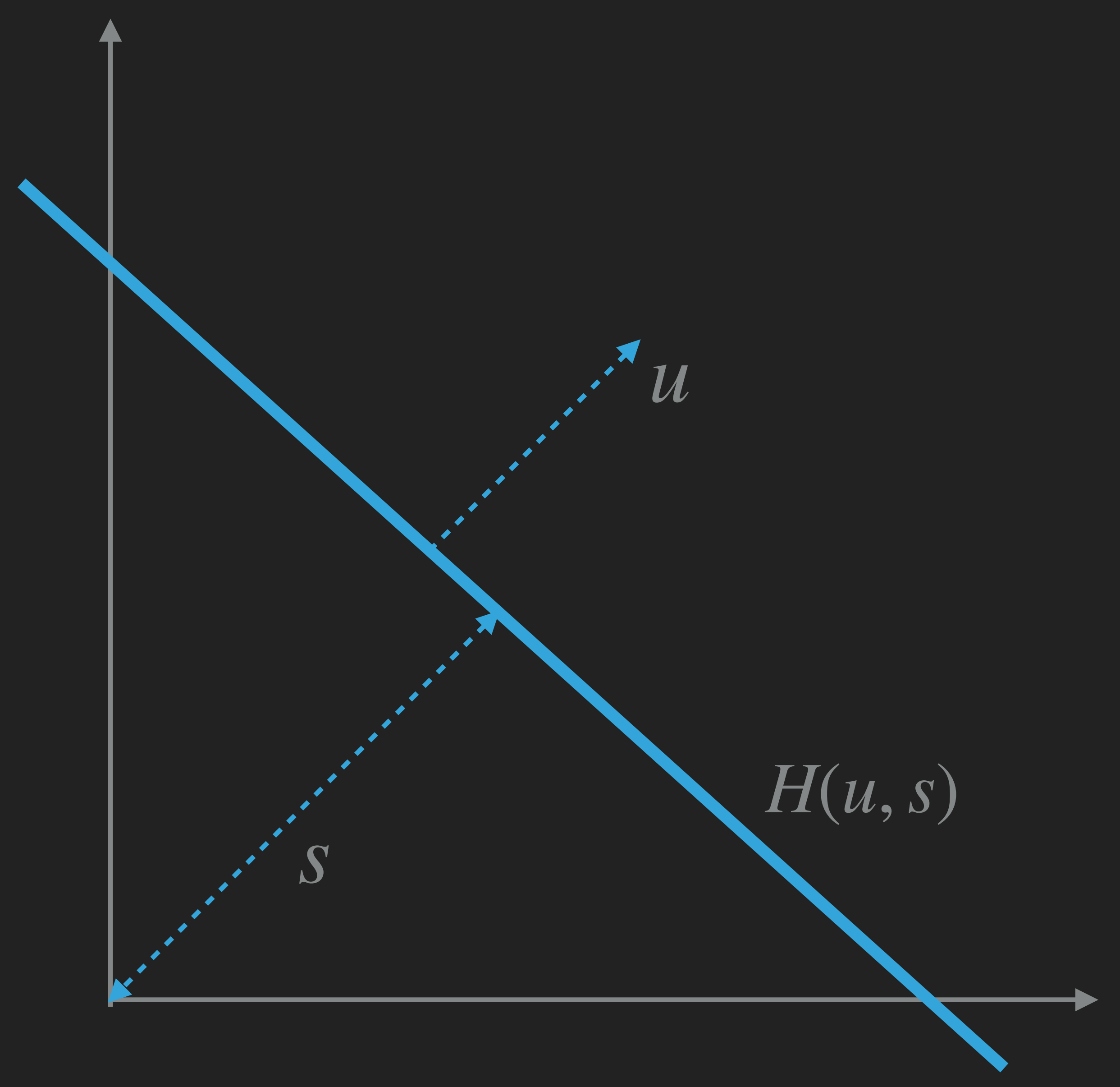
- $\mathbb{A}$ space of affine hyperplanes in $\mathbb{R}^d$
- parametrization: $H(u,s) := \{z \in \mathbb{R}^d : \langle z, u\rangle = s\}$
- invariant measure $\mathrm{d}\Lambda = \mathrm{d}s\mathrm{d}u$, that is

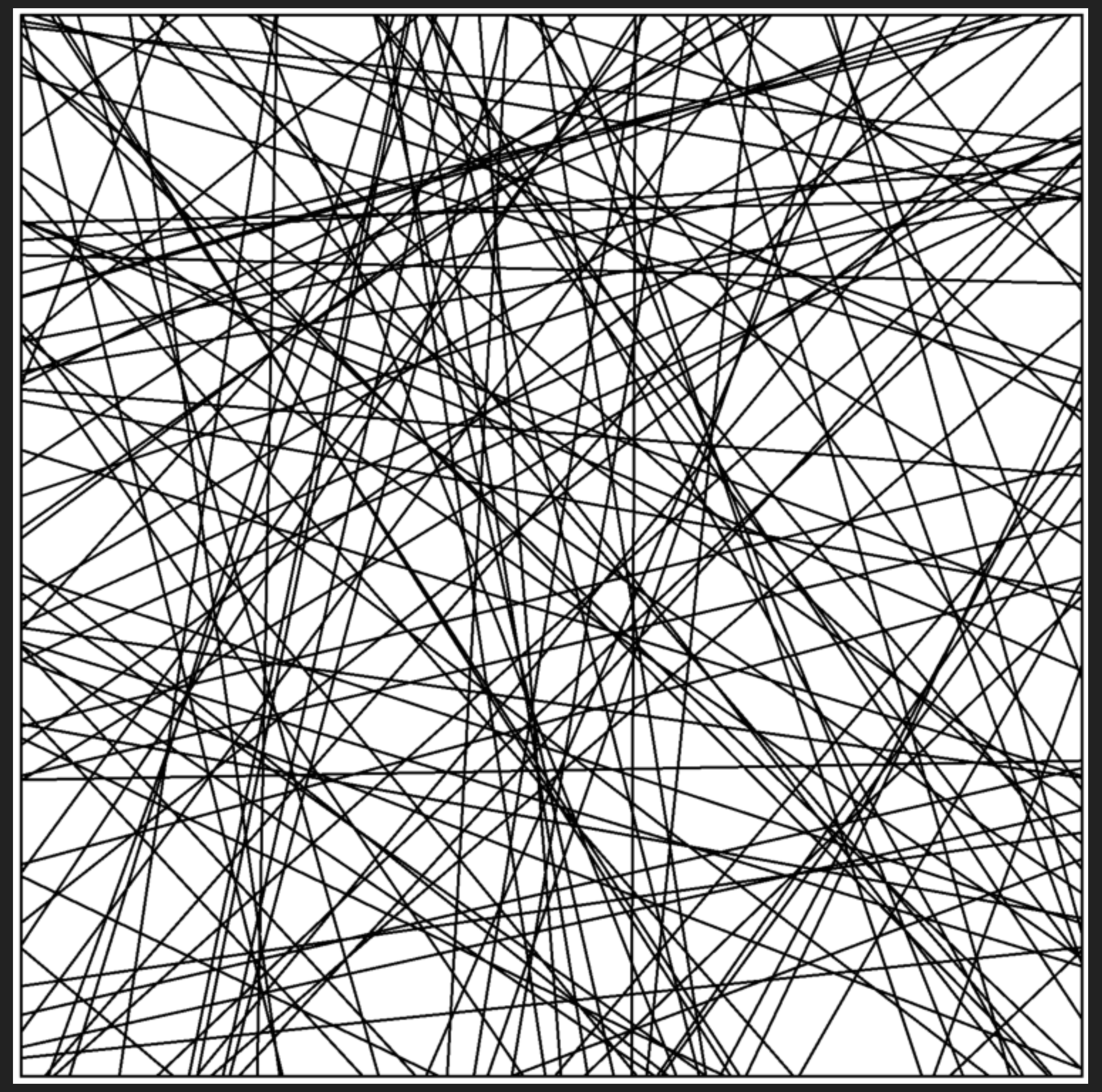
$$\int_{\mathbb{A}} f(H)\,\Lambda(\mathrm{d}H) = \int_{\mathbb{R}}\int_{\mathbb{S}^{d-1}} f(H(u,s))\,\mathrm{d}s\mathrm{d}u$$

- Poisson hyperplane process η

  = Poisson process on $\mathbb{A}$ with intensity measure Λ

- Classical model in stochastic geometry
- Many contributions by Calka, Hug, Kabluchko, Mecke, Miles, Reitzner, Santaló, Schneider …





FLUCTUATIONS OF THE SURFACE FUNCTIONAL

- invariant measure $\mathrm{d}\Lambda = \mathrm{d}s\mathrm{d}u$
- Poisson hyperplane process η
- Surface functional $S_R := \mathscr{H}^{d-1}\left(\bigcup_{H\in\eta} H \cap B_R\right)$
- Question: distributional behavior as $R \to \infty$?
- Answer: $\mathbb{E}S_R$, $\mathrm{Var}\,S_R$ and $\frac{S_R - \mathbb{E}S_R}{\sqrt{\mathrm{Var}\,S_R}} \xrightarrow{D} \mathscr{N}(0,1)$ as $R \to \infty$
- Paroux 1998 for $d = 2$

  Heinrich, Schmidt, Schmidt 2006

  Heinrich 2009

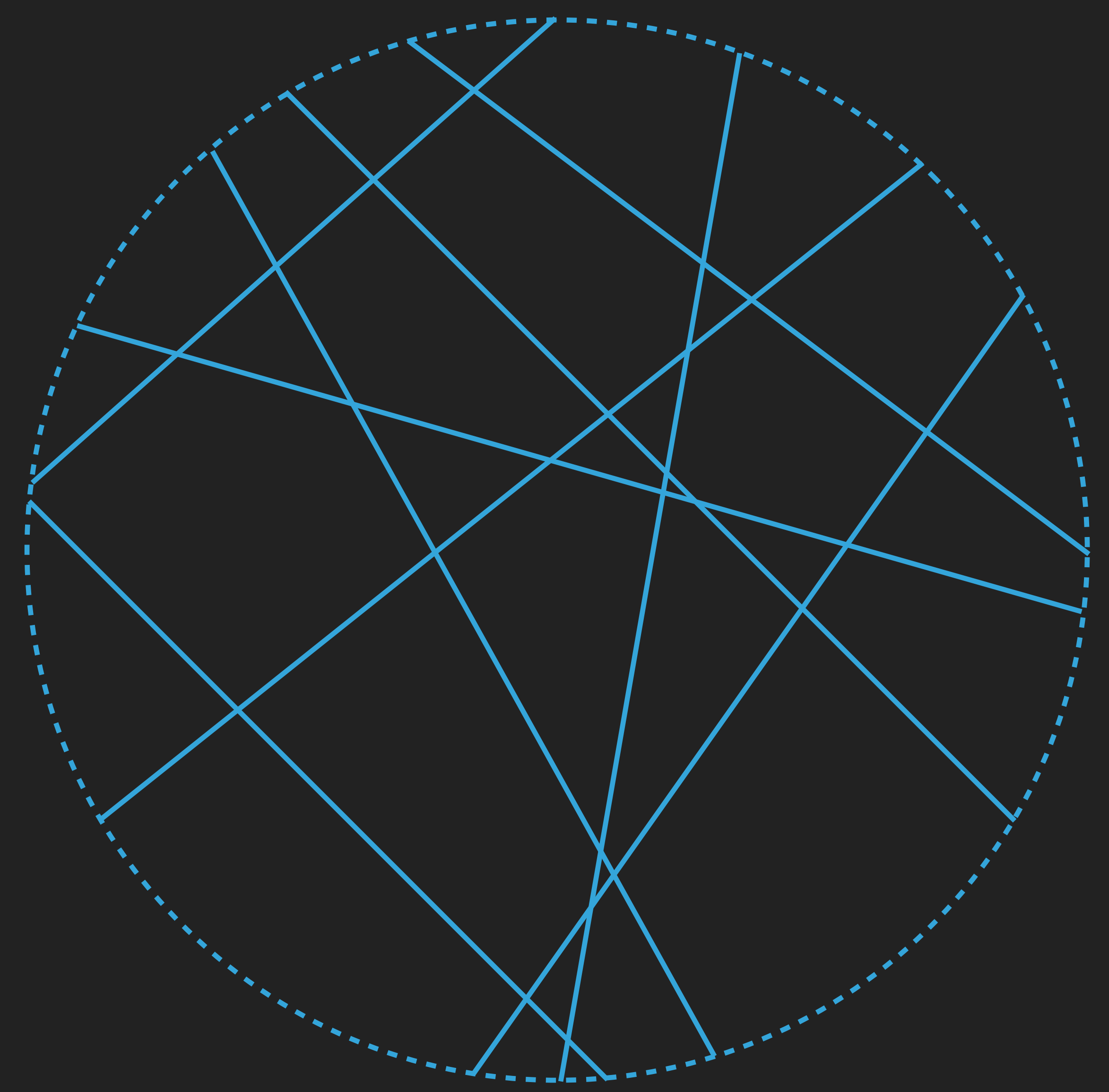
  Reitzner, Schulte 2013

  Last, Penrose, Schulte, T. 2014

  Eichelsbacher, Thäle 2014

  Schulte 2016

  + many others



HYPERBOLIC SPACE

- $\mathbb{H}^d$ = d-dimensional standard space of constant curvature -1

- Conformal ball model: $B^d = \{z \in \mathbb{R}^d : \|z\| < 1\}$

  with Riemannian metric $g_{\mathbb{H}^d} = \dfrac{4}{(1-\|z\|^2)^2} g_{\mathbb{R}^d}$

- Geodesics = Euclidean lines through the centre

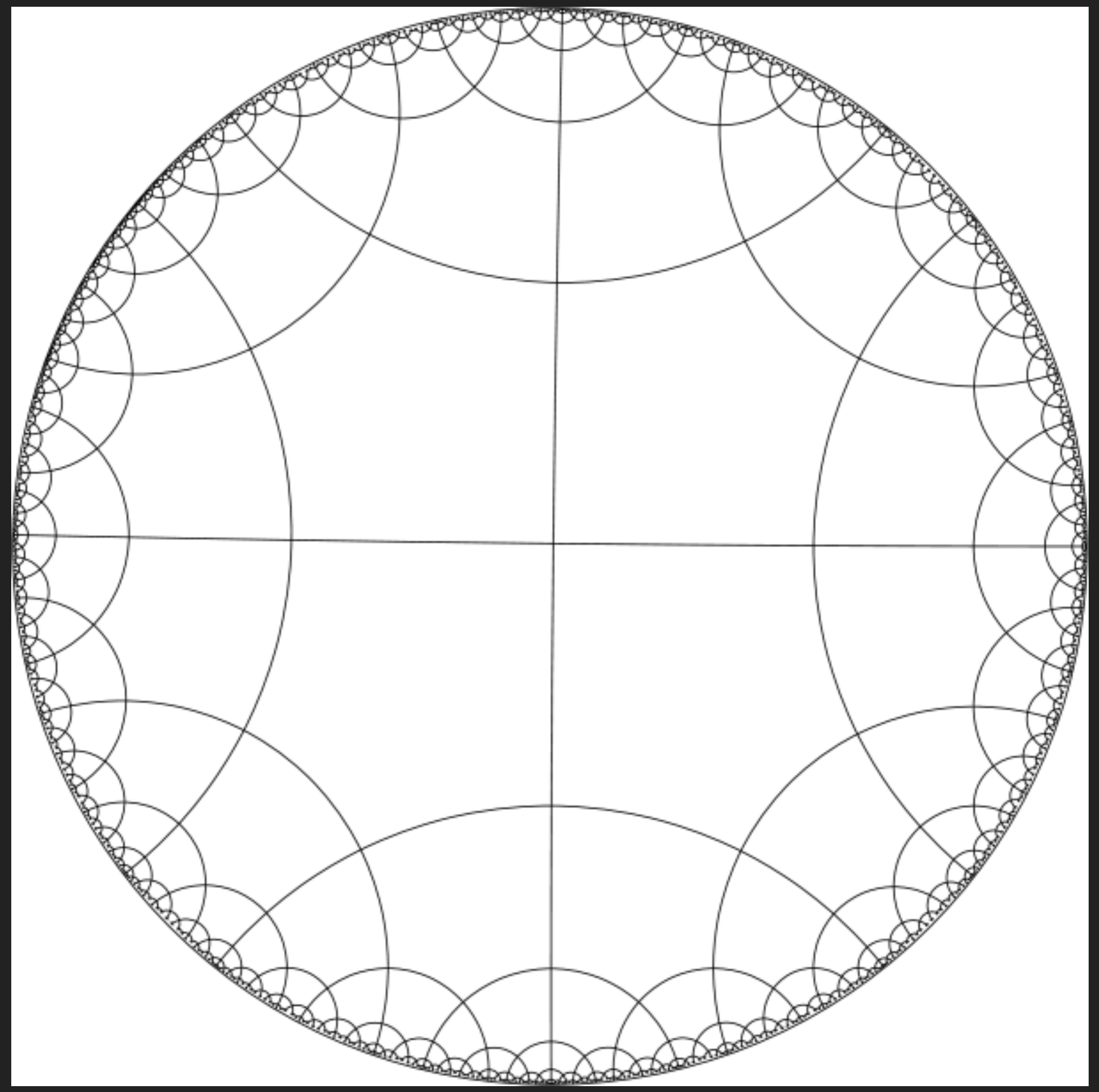
  or circular arcs that intersect the boundary of B^d orthogonally

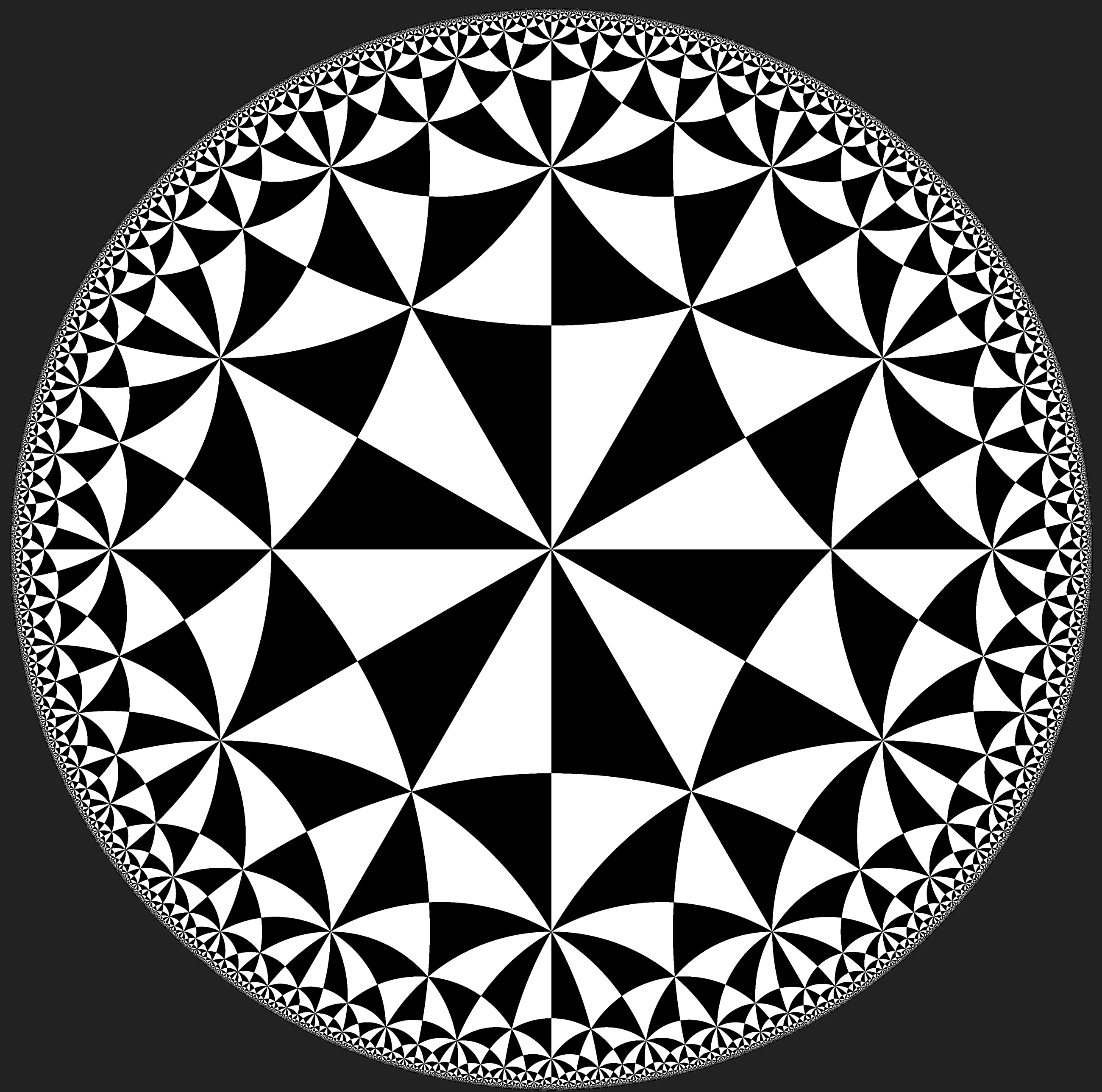
- Volume and surface growth:

$$\mathscr{H}^d(B^d_R) = \omega_d \int_0^R \sinh^{d-1} s \, \mathrm{d}s = \Theta(e^{(d-1)R})$$
$$\mathscr{H}^d(\mathbb{S}^{d-1}_R) = \omega_d \sinh^{d-1} R = \Theta(e^{(d-1)R})$$

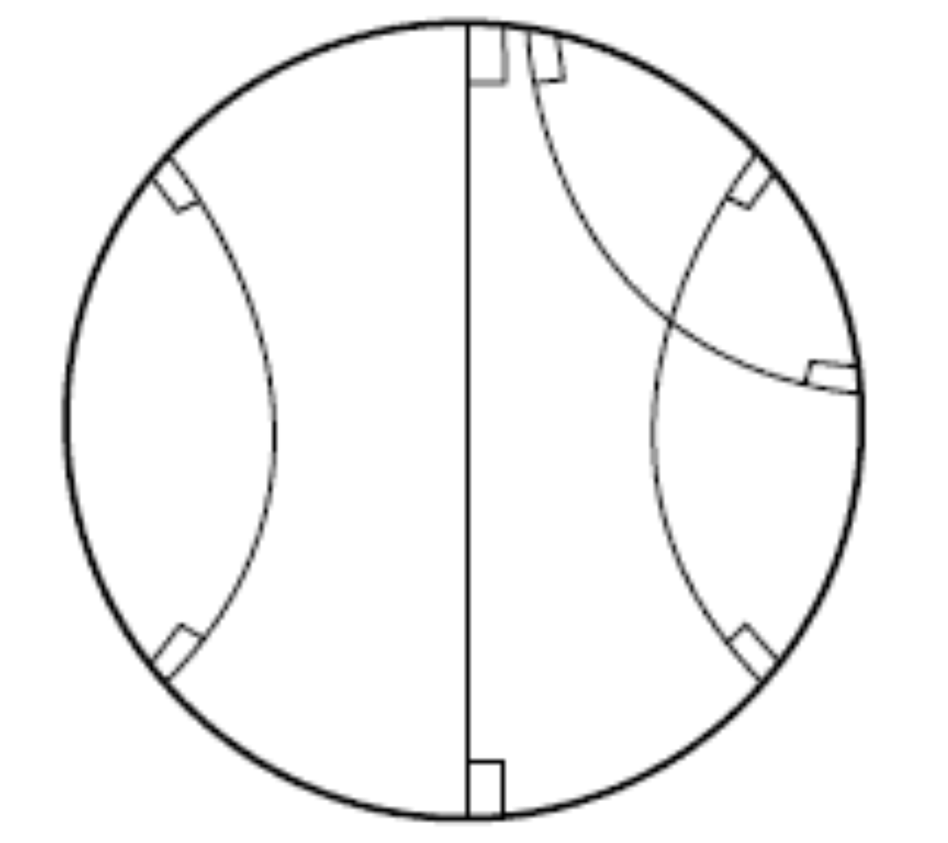
Euclidean case

$$\mathscr{H}^d(B^d_R) = \omega_d \int_0^R s^{d-1} \, \mathrm{d}s = \Theta(R^d)$$
$$\mathscr{H}^d(\mathbb{S}^{d-1}_R) = \omega_d R^{d-1} = \Theta(R^{d-1})$$

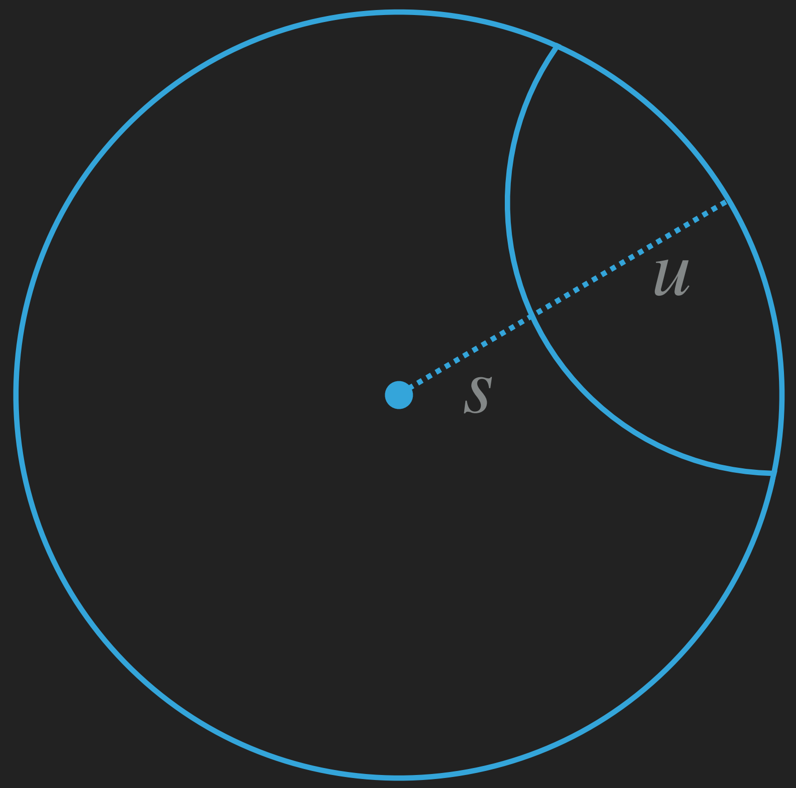




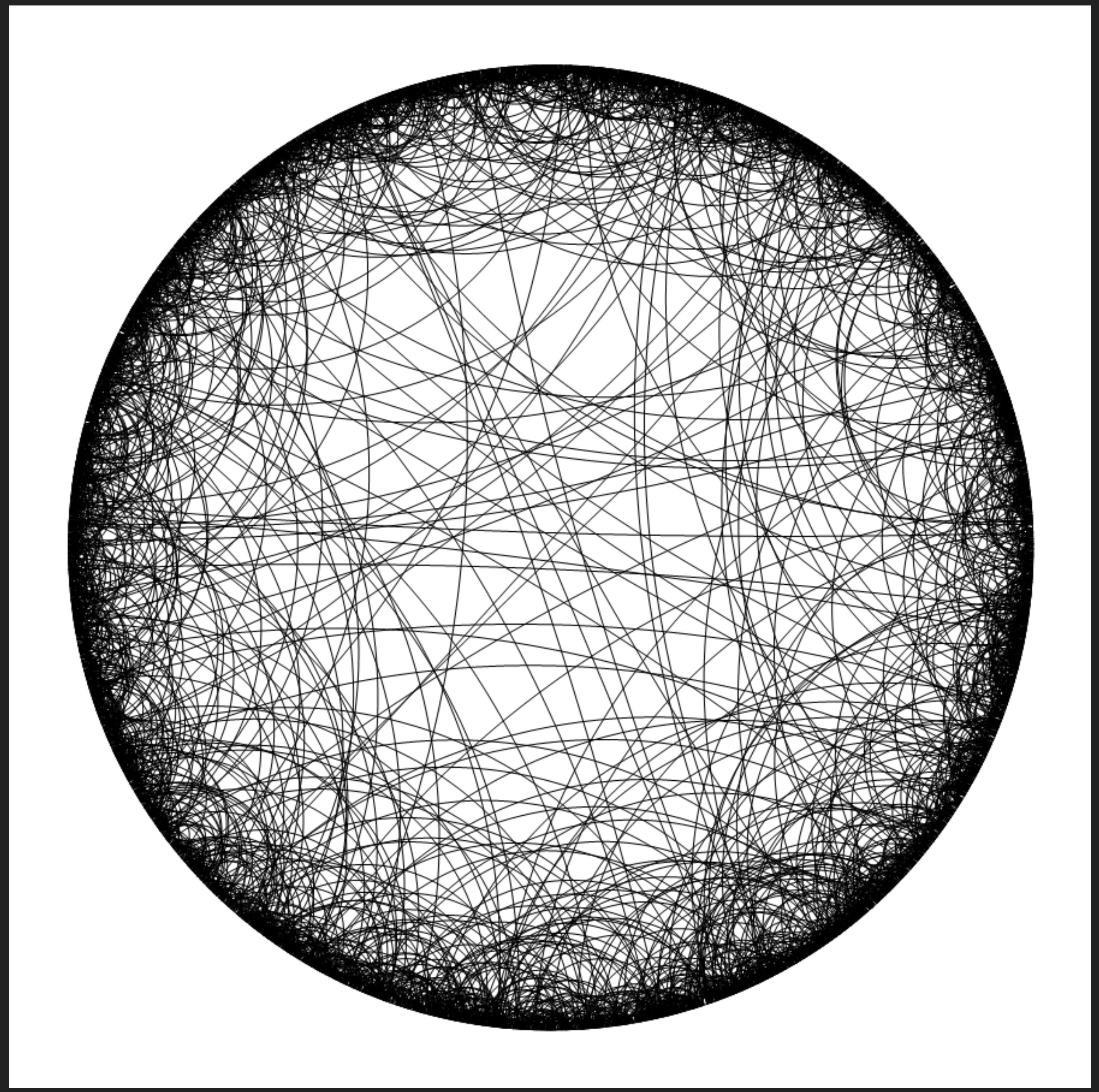
POISSON GEODESIC HYPERPLANES



- Geodesic hyperplane in $\mathbb{H}^d$ = totally geodesic hypersurface
- In the conformal ball model: Euclidean hyperplanes through the centre or spheres orthogonal to the boundary



- $\mathbb{A}_0$ = space of geodesic hyperplanes in $\mathbb{H}^d$
- parametrization: $H = H(u, s)$
- invariant measure (Santaló): $\mathrm{d}\Lambda_0 = \cosh^{d-1} s\, \mathrm{d}s \mathrm{d}u$
- η_0 = Poisson process on $\mathbb{A}_0$ with intensity measure Λ_0



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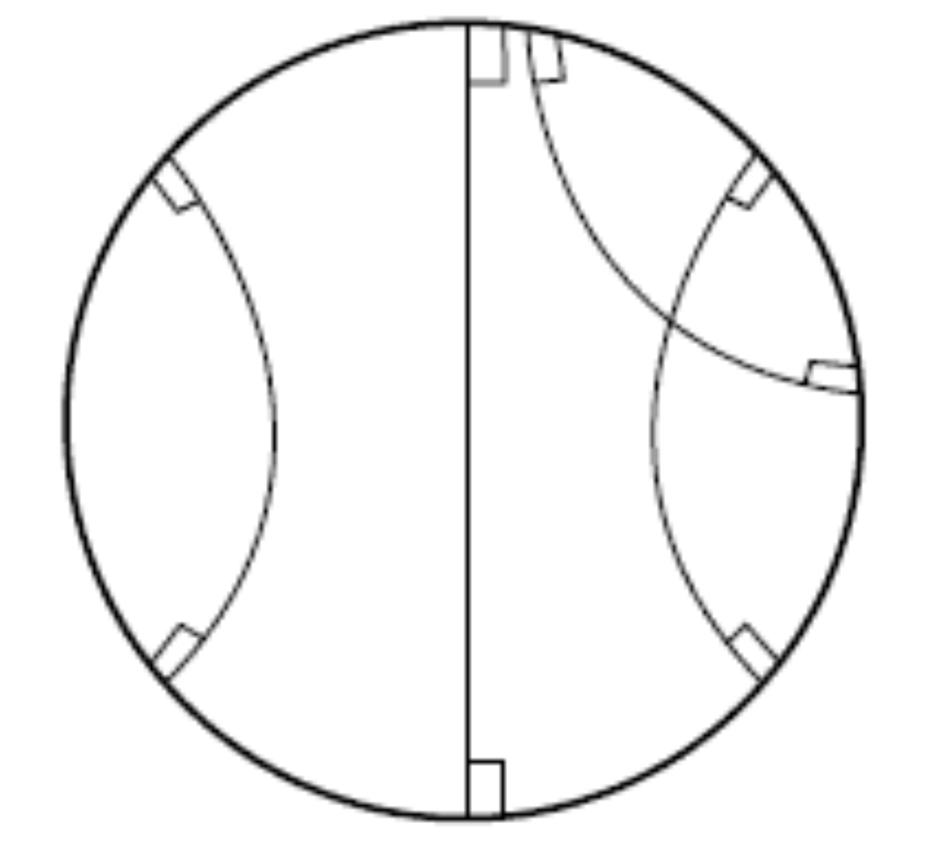
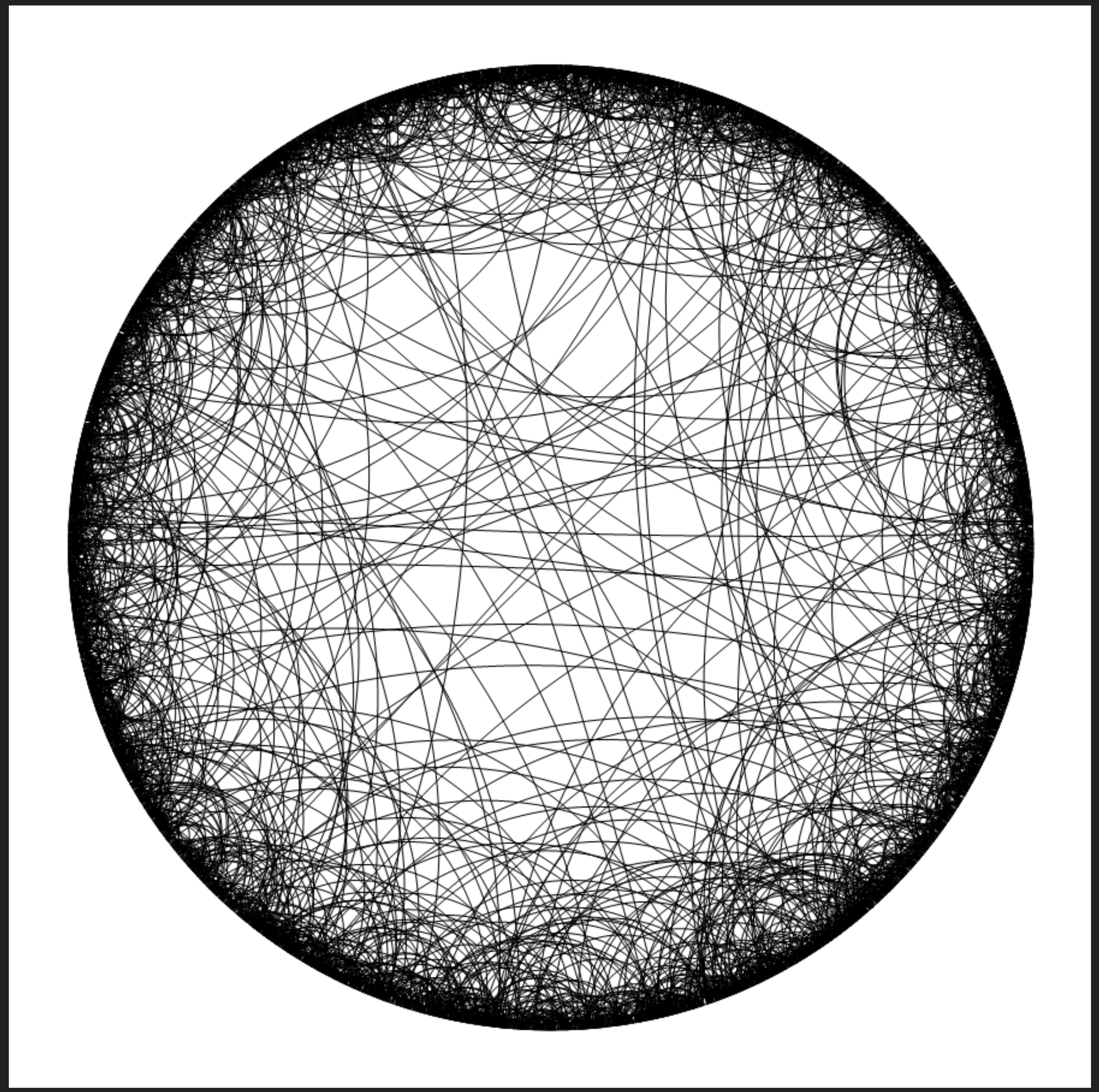


- Surface functional $S_R := \mathscr{H}^{d-1}\Big(\bigcup_{H\in\eta_0} H \cap B_R\Big)$
- Question: distributional behavior as $R \to \infty$?



Theorem (Herold, Hug, T. 2019 PTRF)

$$\frac{S_R - \mathbb{E}S_R}{\sqrt{\operatorname{Var} S_R}} \xrightarrow{D} \mathscr{N}(0,1) \quad \text{if and only if} \quad d \le 3$$

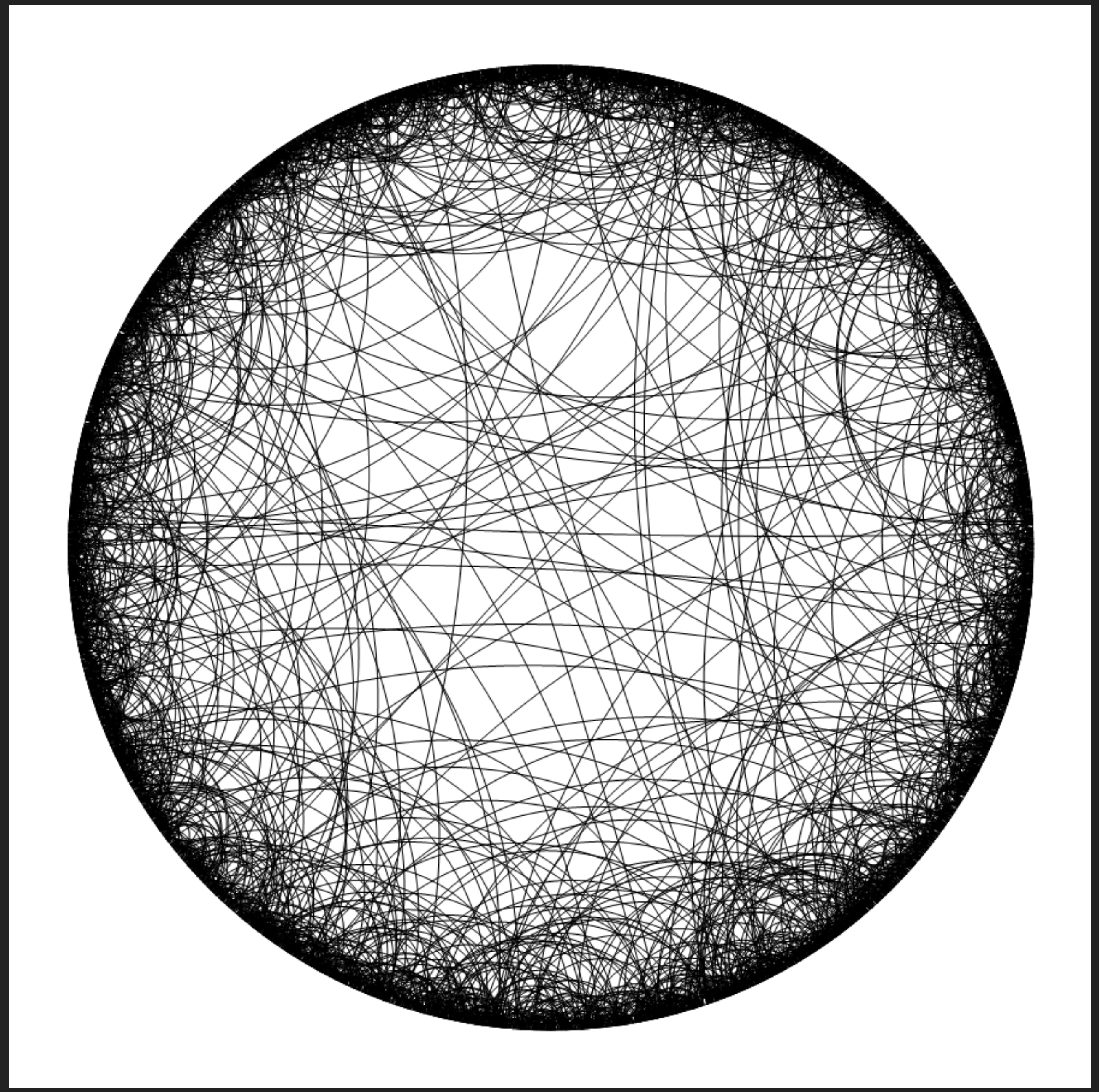
POISSON GEODESIC HYPERPLANES

Theorem (Herold, Hug, T. 2019 PTRF)

$$\frac{S_R - \mathbb{E}S_R}{\sqrt{\operatorname{Var} S_R}} \xrightarrow{D} \mathcal{N}(0,1) \quad \text{if and only if} \quad d \leq 3$$

Some words about the proof:

- Basic technique: Malliavin-Stein method
- $d = 2$: easy
- $d = 3$: not easy, depends on a full classification of a set of partitions with prescribed properties ($\approx$ 1 weekend on the RUB HPC cluster)
- $d \geq 4$: show that the 4th cumulants are uniformly bounded from below



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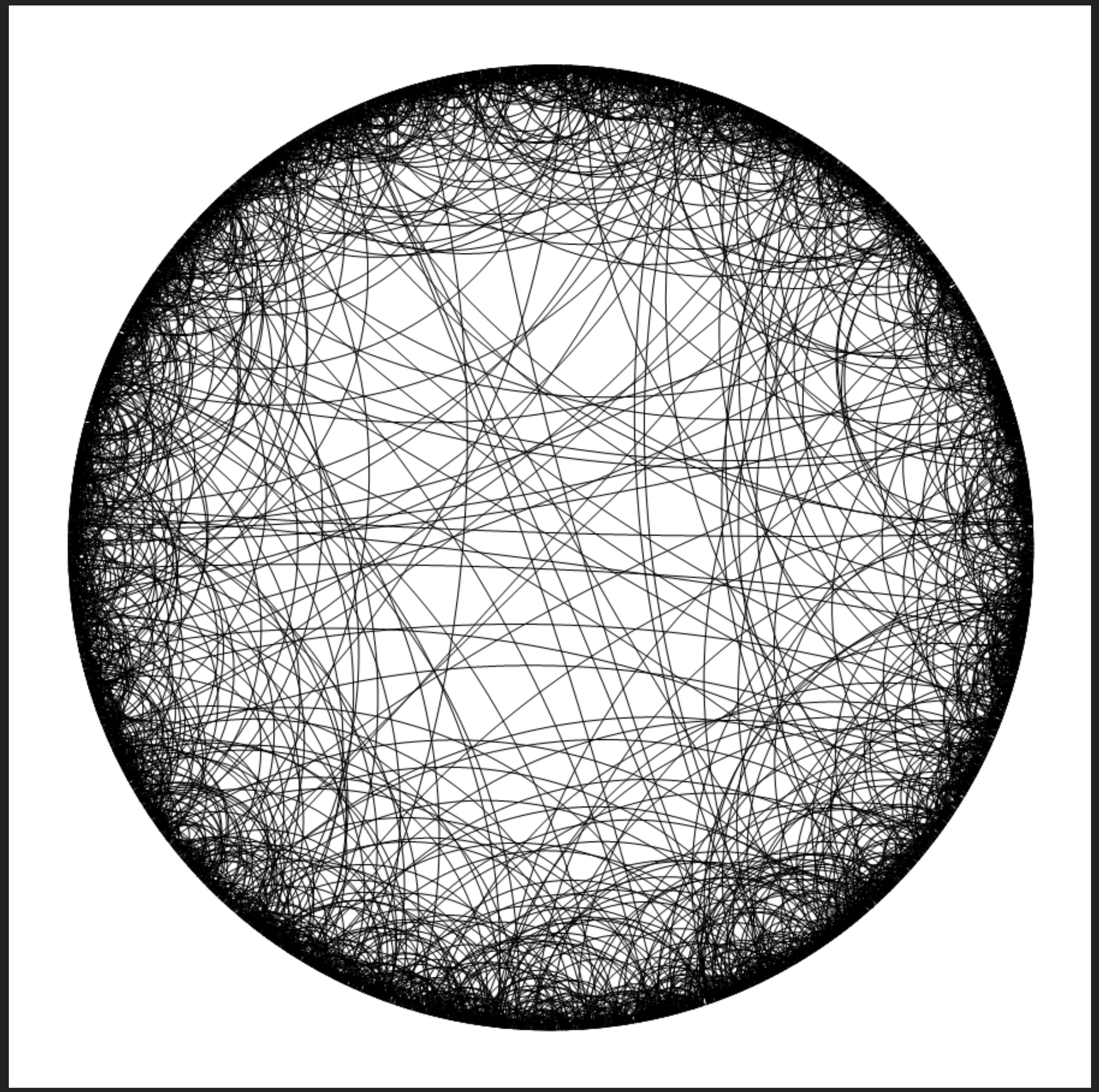
Natural follow-up questions:

- What are hyperplanes in hyperbolic space?
- What happens for $d \geq 4$?



TOTALLY UMBILIC HYPERSURFACES

- Let $\Sigma \subset (M, g)$ be a hypersurface

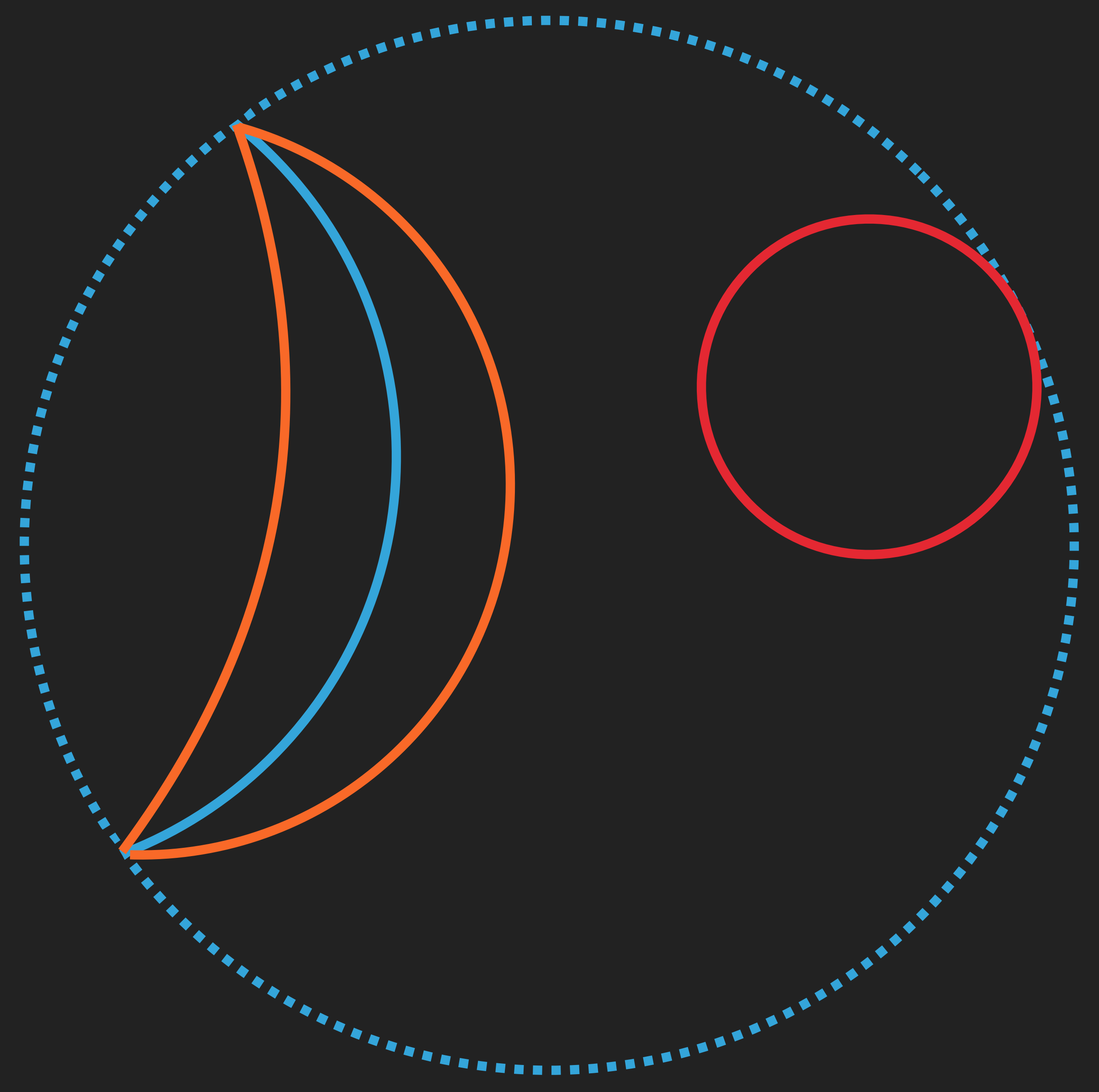
- The second fundamental form at $x \in \Sigma$

$$B : T_x\Sigma \times T_x\Sigma \to \mathbb{R}, \qquad B(v, v) = \kappa_{\gamma_v}(x)$$

- Σ is totally umbilic if at each point $x \in \Sigma$, $B = \lambda g$

  If M has constant curvature, λ is constant on Σ

- In particular, a totally umbilic hypersurface with $\lambda = 0$ is totally geodesic



sphere intersecting the boundary at an angle θ, where $\cos\theta = \lambda$

Euclidean space $\mathbb{R}^d$		Hyperbolic space $\mathbb{H}^d$	
$\lambda = 0$	hyperplanes	$\lambda = 0$	genuine geodesic hyperplanes
$\lambda > 0$	spheres	$\lambda \in (0,1)$	equidistants from genuine geodesic hyperplanes
		$\lambda > 1$	hyperbolic spheres

TOTALLY UMBILIC HYPERSURFACES

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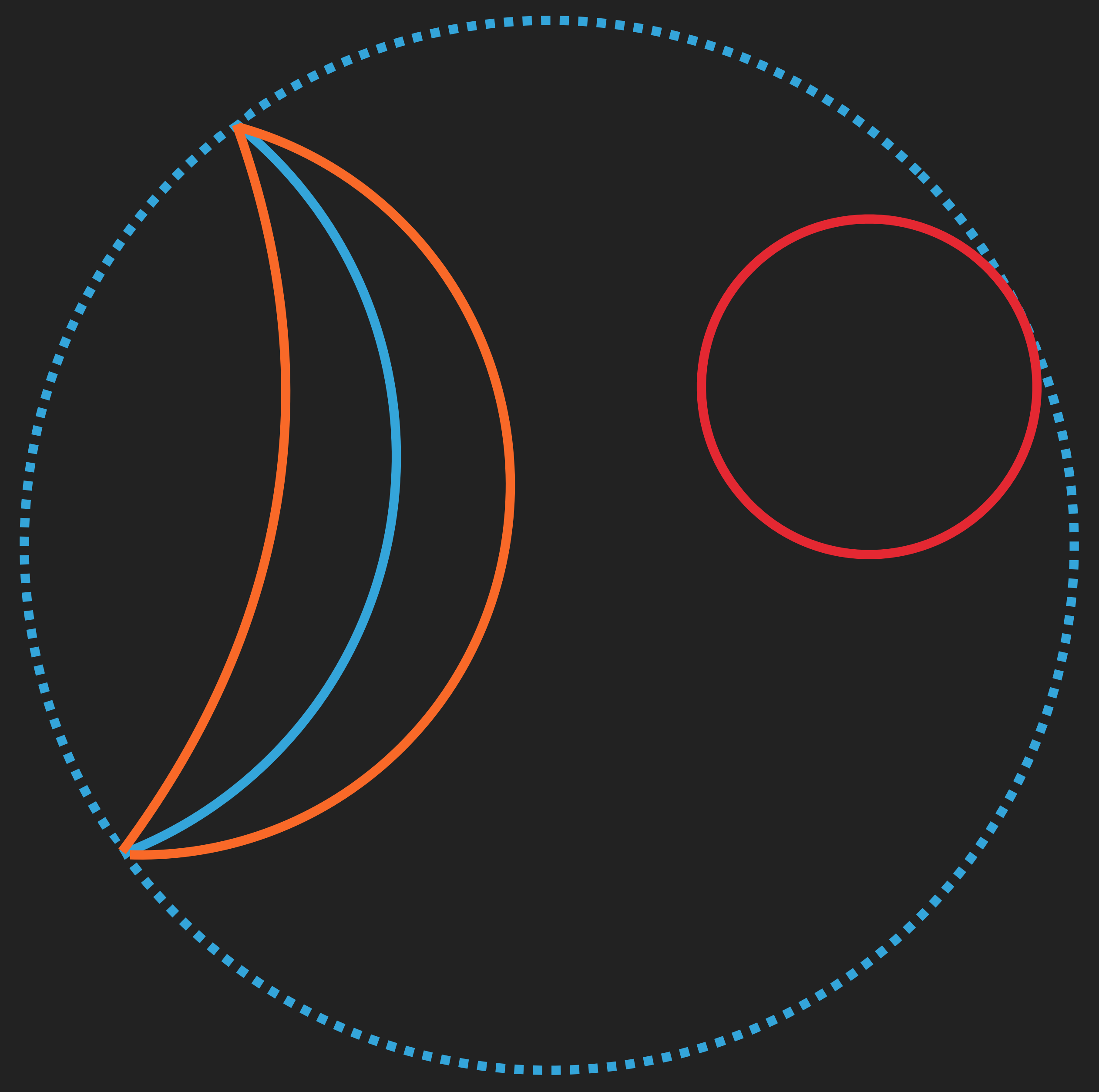


sphere intersecting the boundary at an angle θ, where $\cos\theta = \lambda$

sphere intersecting the boundary at a single point

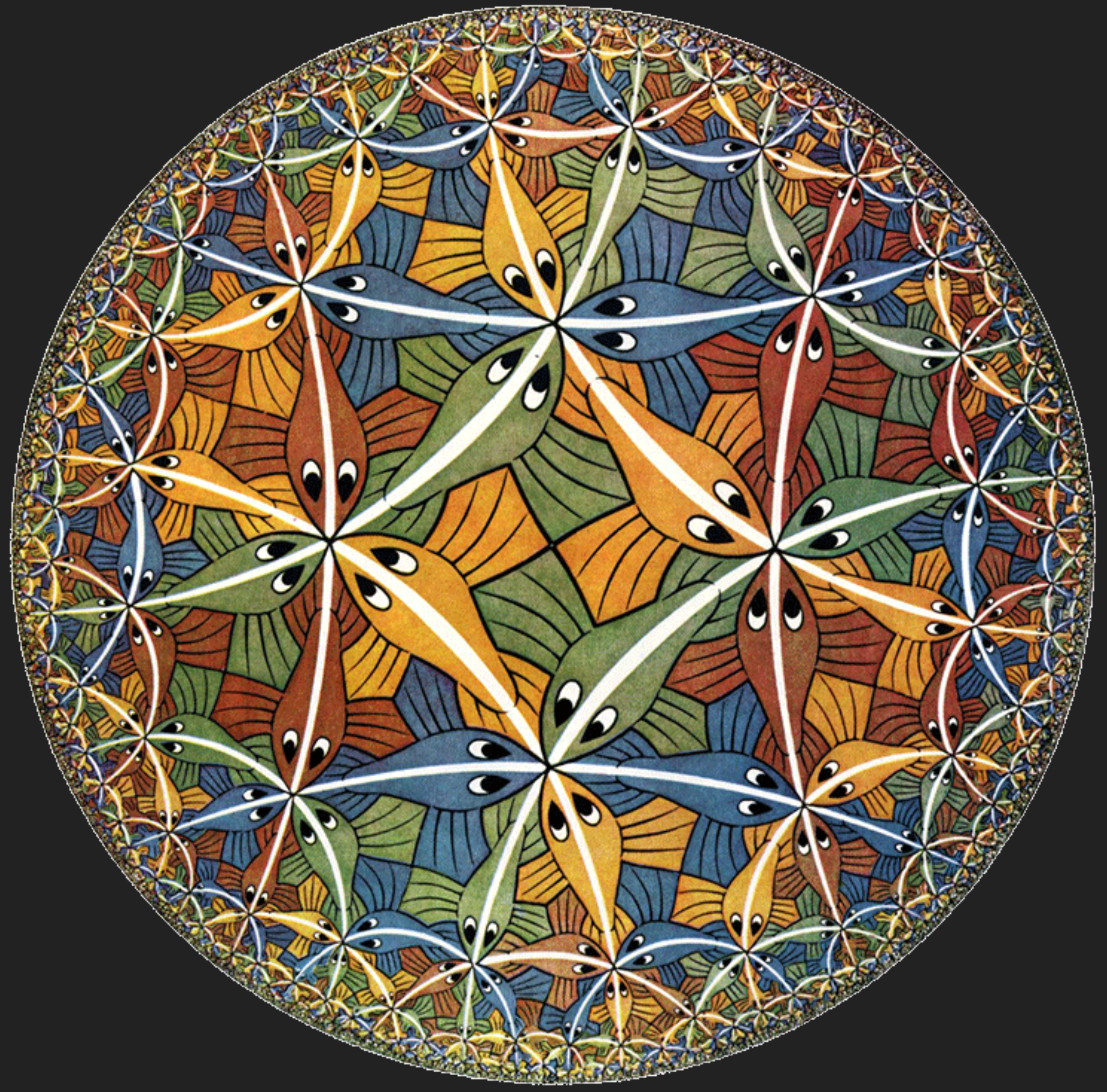
Definition (Solanes)

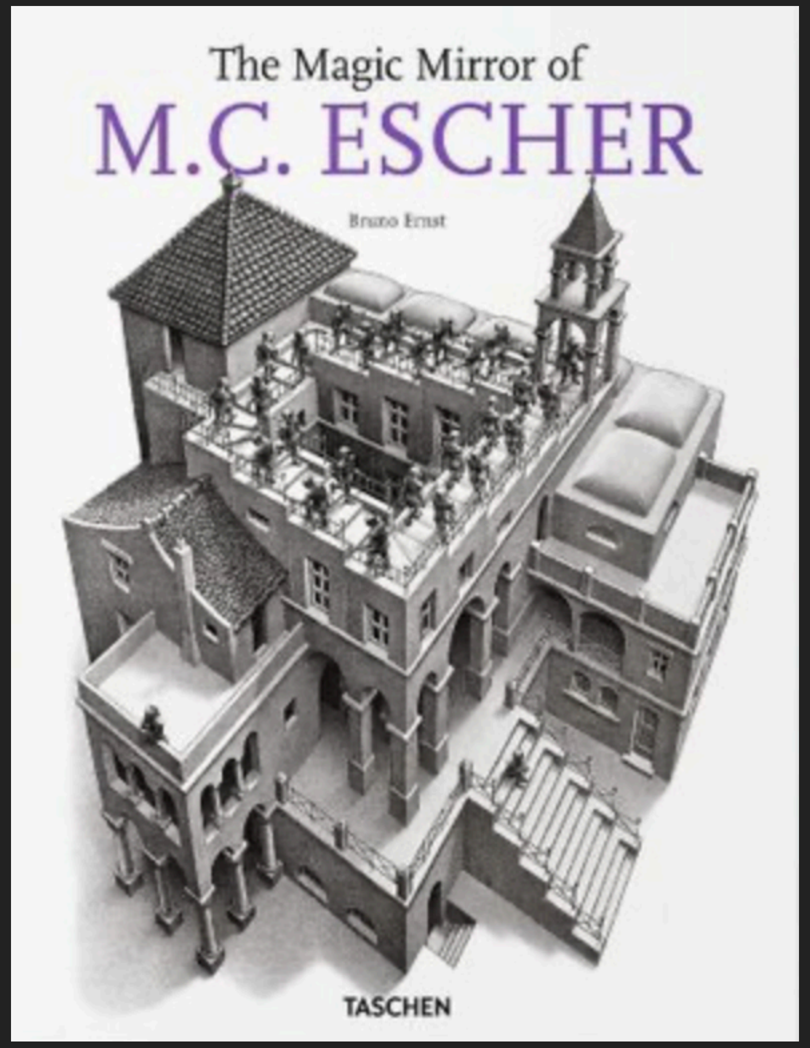
A λ-geodesic hyperplane is a totally umbilic hypersurface with $\lambda \in [0,1]$.

Hyperbolic space $\mathbb{H}^d$	
$\lambda = 0$	genuine geodesic hyperplanes
$\lambda \in (0,1)$	equidistants from genuine geodesic hyperplanes
$\lambda = 1$	horospheres
$\lambda > 1$	hyperbolic spheres

INTERLUDE: COXETER AND ESCHER

- M.C. Escher's Circle Limit III depicts a tessellation of the hyperbolic plane
- Escher: the fish „… shoot up perpendicularly from the boundary …"
- B. Ernst (The Magic Mirror of Escher): some arcs are not „… placed at right angles to the circumference (as they ought to be)"





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- Coxeter (in a publication in Leonardo):

Leonardo, Vol. 12, pp. 19–25.
Pergamon Press, 1979. Printed in Great Britain.

THE NON-EUCLIDEAN SYMMETRY OF ESCHER'S PICTURE 'CIRCLE LIMIT III'*

H. S. M. Coxeter**

Abstract—*Of all Escher's pictures with a mathematical background, the most sophisticated is his* 1959 *woodcut,* Circle Limit III, *which uses four colours in addition to black and white. Queues of fishes of each colour are swimming along white arcs that cut the peripheral circle at a certain angle. After discussing the kind of symmetry that is involved and the underlying regular tessellations (so cleverly disguised), the author explains why the above-mentioned angle is not* 90° *but* 80°.

I. INTRODUCTION

I first met Escher [1] in September 1954, when an exhibition of his work was sponsored by the International Congress of Mathematicians, meeting that year in Amsterdam. Throughout the previous 17 years he had been making designs in which a drawing of some animal (such as a fish or a reptile or a bird) is repeated as on wallpaper, with two remarkable innovations: the basic unit (usually a single animal, or one half of a symmetrical animal or two different animals juxtaposed) is repeated not only by translations but also by other isometries (or *congruent transformations*): rotations, reflections or glide-reflections [2]; and the replicas ingeniously fit together so that there are no interstices. In the language of mathematics (a subject in which Escher resolutely claimed to be 'absolutely innocent of training or knowledge'), the basic unit is a *fundamental region* for a *symmetry group*.

In a letter of December 1958 he wrote: 'Did I ever thank you for sending me . . . "A Symposium on Symmetry"? I was so pleased with this booklet and proud of the two reproductions of my plane patterns!

'Though the text of your article on "Crystal Symmetry and its Generalizations" [3] is much too learned for a simple, self-made plane pattern-man like me, some of the text-illustrations and especially Figure 7, page 11, gave me quite a shock.

'Since a long time I am interested in patterns with "motives" getting smaller and smaller till they reach the limit of infinite smallness. The question is relatively simple if the limit is a point in the centre of a pattern. Also a line-limit is not new to me, but I was never able to make a pattern in which each "blot" is getting smaller gradually from a centre towards the outside circle-limit, as shows your Figure 7 [reproduced here as Fig. 1]. I tried to find out how this figure was geometrically constructed, but I succeeded only in finding the centres and radii of the largest inner-circles. If you could give me a simple explanation how to construct the following circles, whose centres approach gradually from the outside till they reach the limit, I should be immensely pleased and very thankful to you! Are there other systems besides this one to reach a circle-limit?

'Nevertheless I used your model for a large woodcut (of which I executed only a sector of 120° in wood, which I printed 3 times). I am sending you a copy of it. . . .'.

This was his picture 'Circle Limit I', concerning which he wrote on another occasion [4]: 'The largest animal figures are now located in the centre, and the limit of the infinitely many and infinitely small is

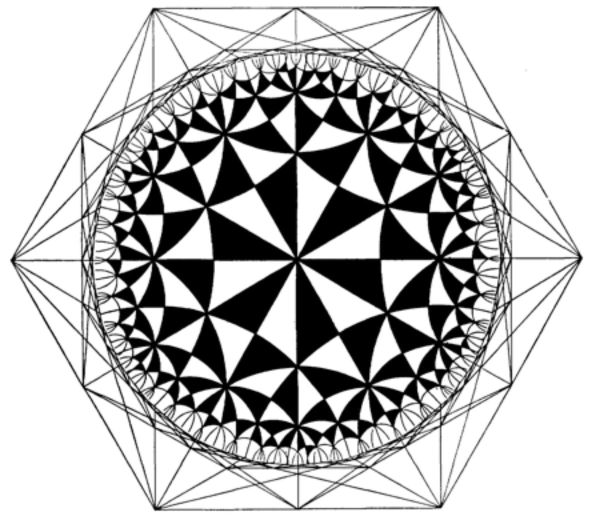


Fig. 1. Pattern whose symmetry group is (6, 4, 2) *(with scaffolding). Two adjacent triangles (one white and one black) form a fundamental region.*

*This article is based on a lecture given in May 1978 at the University of Siena, Italy, by request of the Dept. of Mathematics there.

**Mathematician, University of Toronto, Toronto M5S 1A1, Canada. (Received 18 Sept. 1978)

INTERLUDE: COXETER AND ESCHER

- M.C. Escher's Circle Limit III depicts a tessellation of the hyperbolic plane
- Escher: the fish „… shoot up perpendicularly from the bc
- B. Ernst (The Magic Mirror of Escher): some arcs are not „… placed at right angles to the circumference (as they ought to be)"
- Coxeter (in a publication in Leonardo): the arcs along with Esher's fishes swim are equidistants. He computed $\lambda = \dfrac{2^{1/4} - 2^{-1/4}}{2}$ and deduced $\theta \approx 80°$

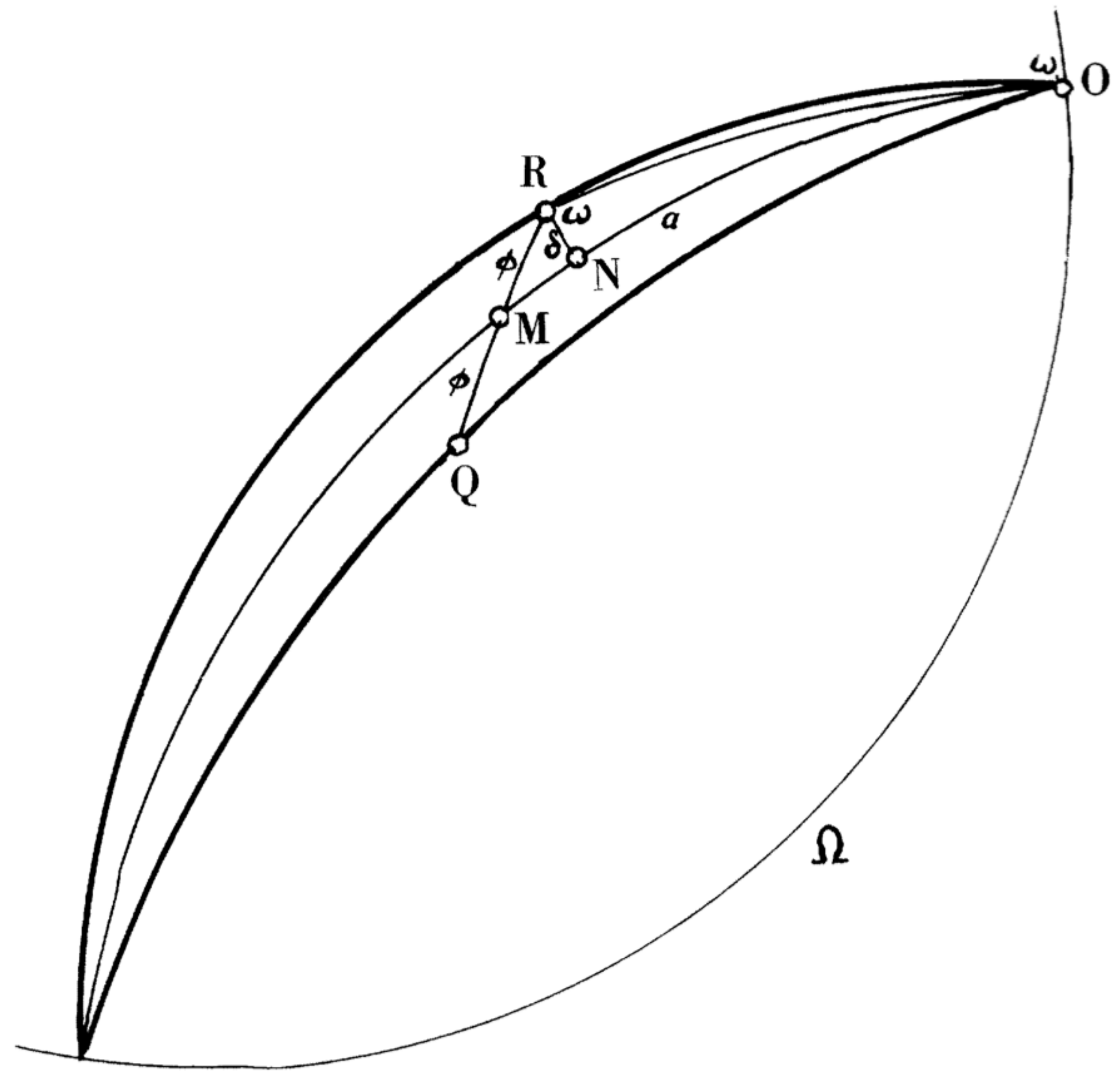


Fig. 6. The angle of parallelism $\omega = \Pi(\delta)$.

$$\tanh \delta = \tanh \phi \cos(\pi/3) = \tfrac{1}{2} \tanh . \phi$$

By equation (1), $\tanh \delta = \cos \omega$, and by equation (2),

$$\cosh^2 \phi = (2+\sqrt{2})/3,$$

$$\tanh^2 \varphi = 1 - \operatorname{sech}^2 \phi = 1 - 3/(2+\sqrt{2}) = (3-2\sqrt{2})/\sqrt{2},$$

$$\tanh \phi = (\sqrt{2} - 1)/\sqrt[4]{2} = 2^{1/4} - 2^{-1/4}.$$

Hence, $\boxed{\tanh \delta = (2^{1/4} - 2^{-1/4})/2}$ and

$$\omega = \Pi(\delta) = \arccos (2^{1/4} - 2^{-1/4})/2 = \arccos 0{\cdot}17417 = \boxed{79°58'}.$$

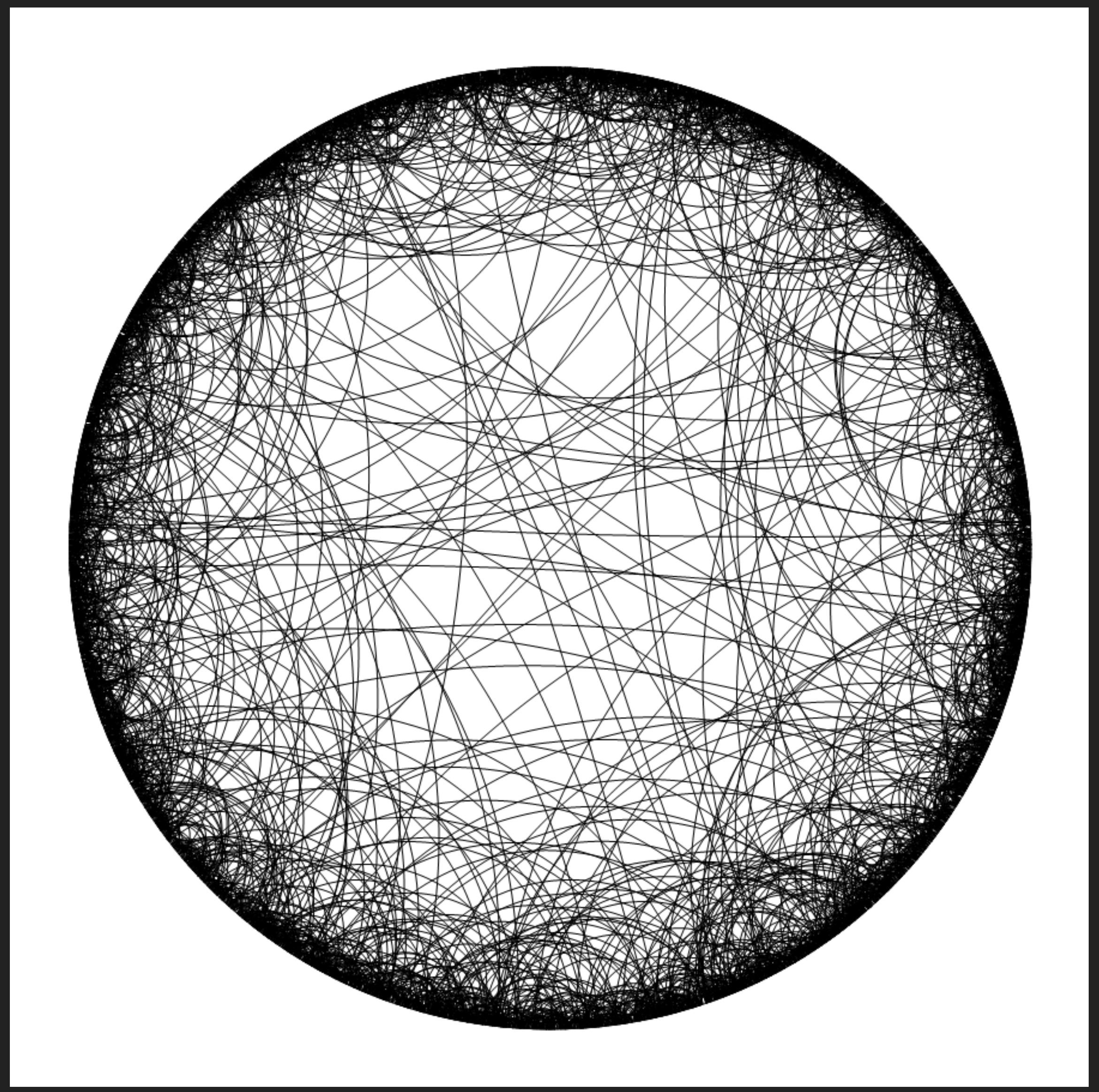
Escher's integrity is revealed in the fact that he drew this angle correctly even though he apparently believed that it 'ought' to be 90°.

whose angles are 60°. But this is absurd, as a triangle with angles 60° would be Euclidean, not hyperbolic. The above discussion explains the paradox by showing that Escher's 'lines' are not straight: each is one branch of an equidistant-curve.

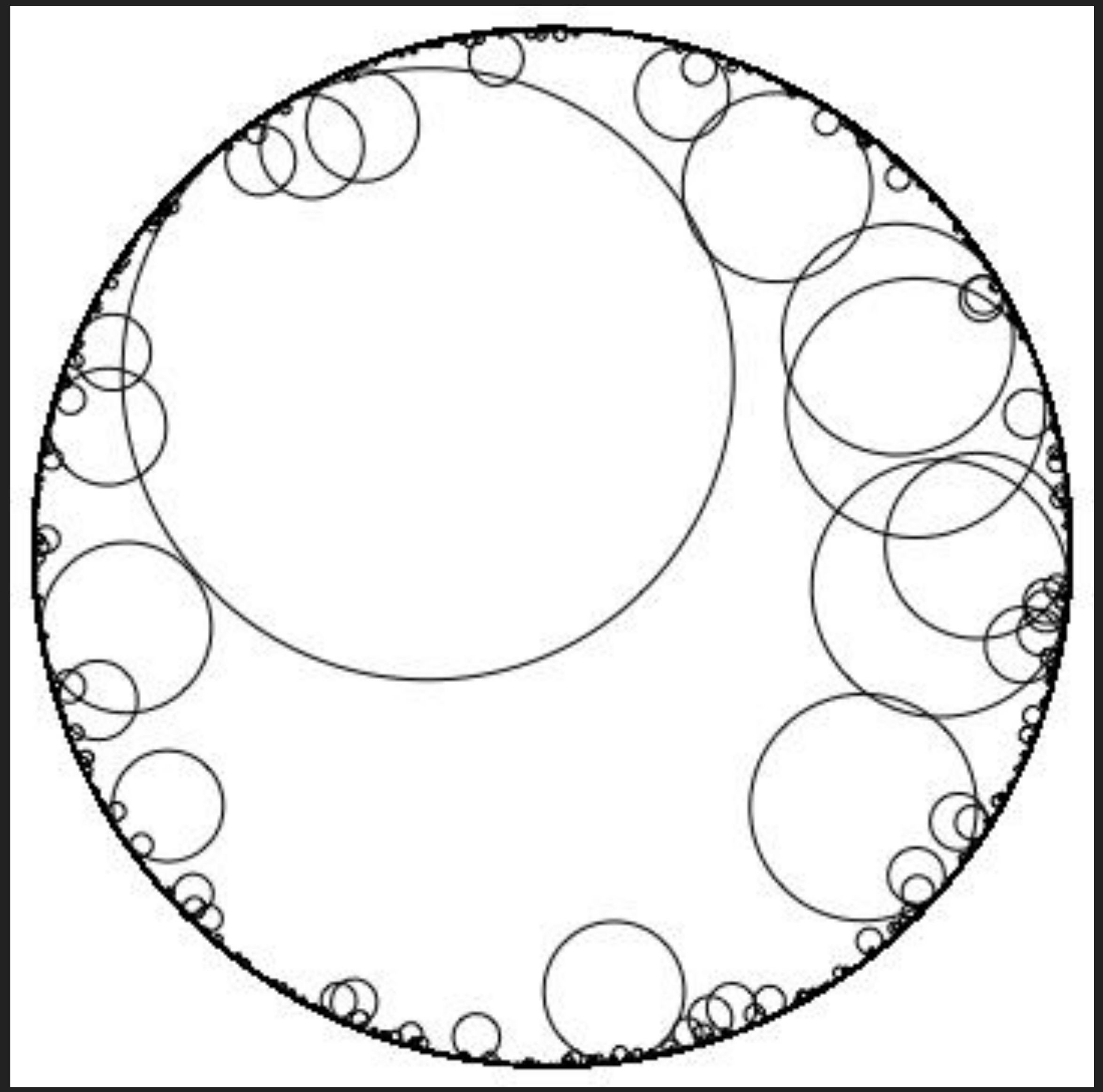
POISSON λ-GEODESIC HYPERPLANES

- $\mathbb{A}_\lambda$: space of λ-geodesic hyperplanes
- invariant measure (Santaló, Gallego, Naveira, Solanes):

  $\mathrm{d}\Lambda_\lambda = (\cosh s - \lambda \sinh s)^{d-1}\,\mathrm{d}s\mathrm{d}u$
- η_λ: Poisson process on the space $\mathbb{A}_\lambda$ with intensity measure $\mathrm{d}\Lambda_\lambda$
- Surface functional $S_{R,\lambda} := \mathscr{H}^{d-1}\Big(\bigcup_{H\in\eta_\lambda} H \cap B_R\Big)$
- Question: distributional behavior as $R \to \infty$?



$\lambda = 0$



$\lambda = 1$

POISSON λ-GEODESIC HYPERPLANES

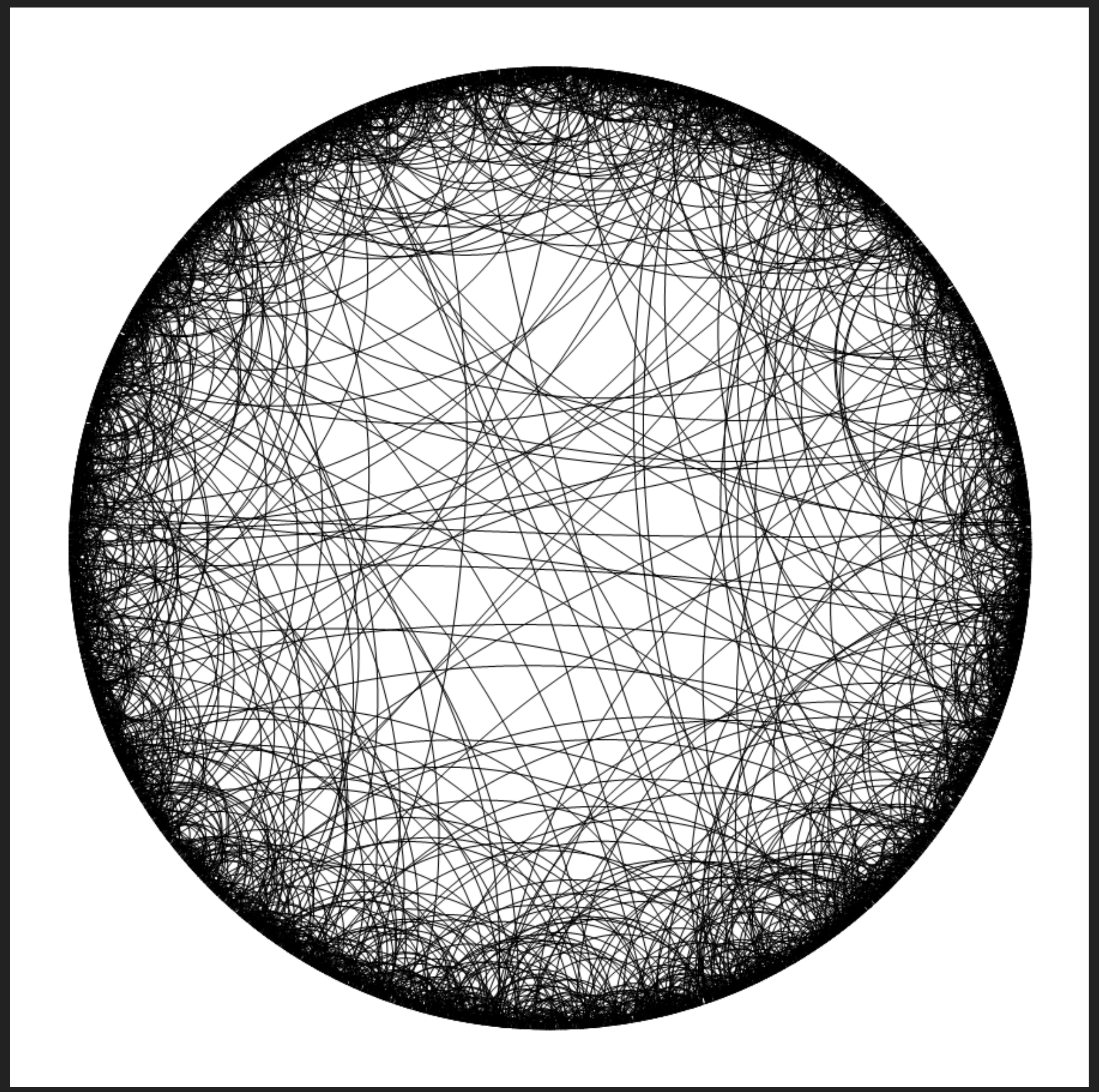
- Surface functional $S_{R,\lambda} := \mathscr{H}^{d-1}\Big(\bigcup_{H \in \eta_\lambda} H \cap B_R\Big)$

Lemma: $\mathbb{E}S_{R,\lambda} = \mathscr{H}^d(B_R)$

$$\mathbb{E}S_{R,\lambda} = \mathbb{E}\sum_{H\in\eta_\lambda} \mathscr{H}^{d-1}(H\cap B_R)$$

$$= \int_{\mathbb{A}_\lambda} \mathscr{H}^{d-1}(H\cap B_R)\,\Lambda_\lambda(\mathrm{d}H) \qquad \text{Mecke equation}$$

$$= \mathscr{H}^d(B_R) \qquad \text{Crofton formula for } \lambda\text{-geodesic hyperplanes}$$



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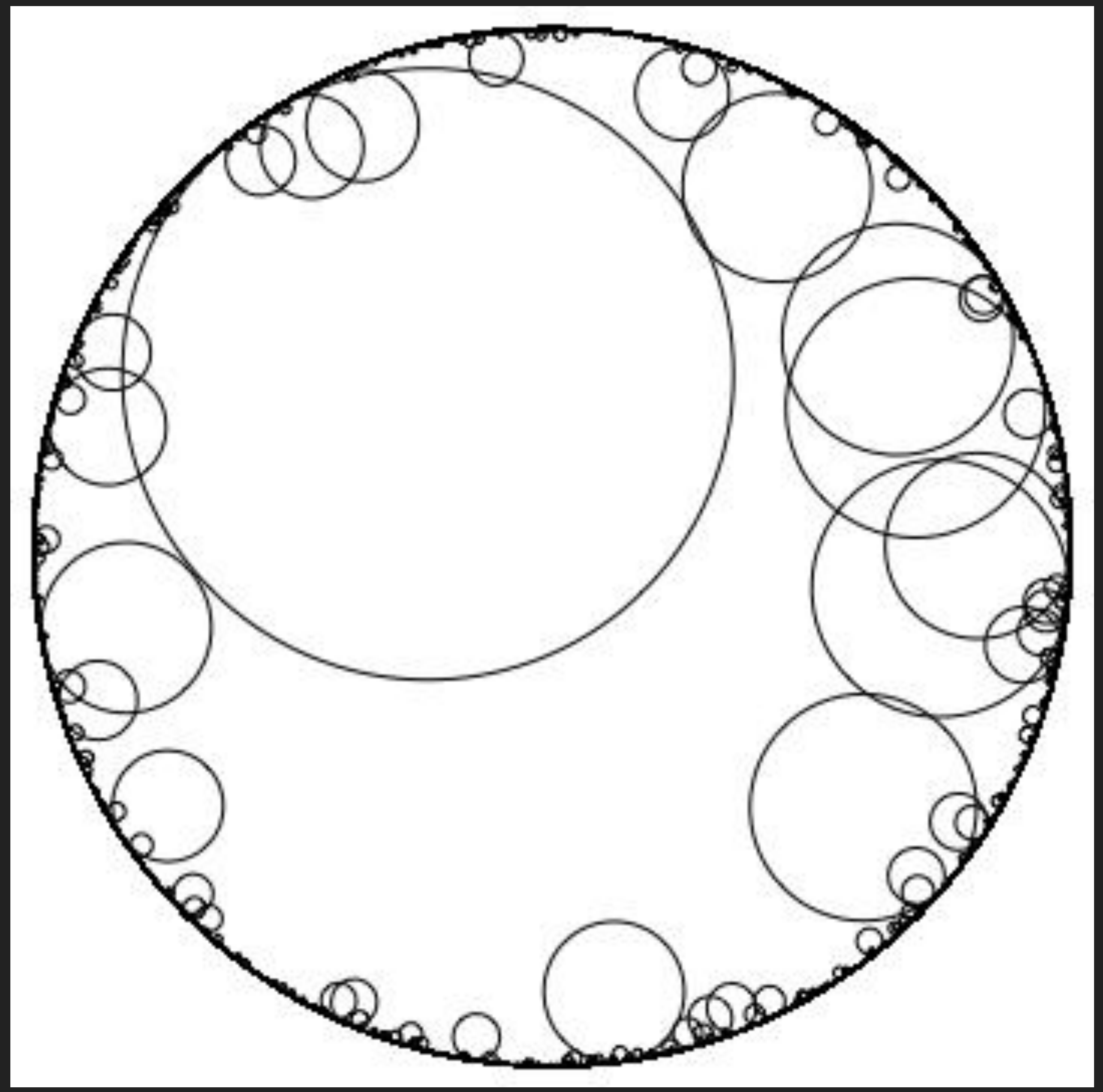

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Lemma: $\mathbb{E} S_{R,\lambda} = \mathscr{H}^d(B_R)$

Lemma:

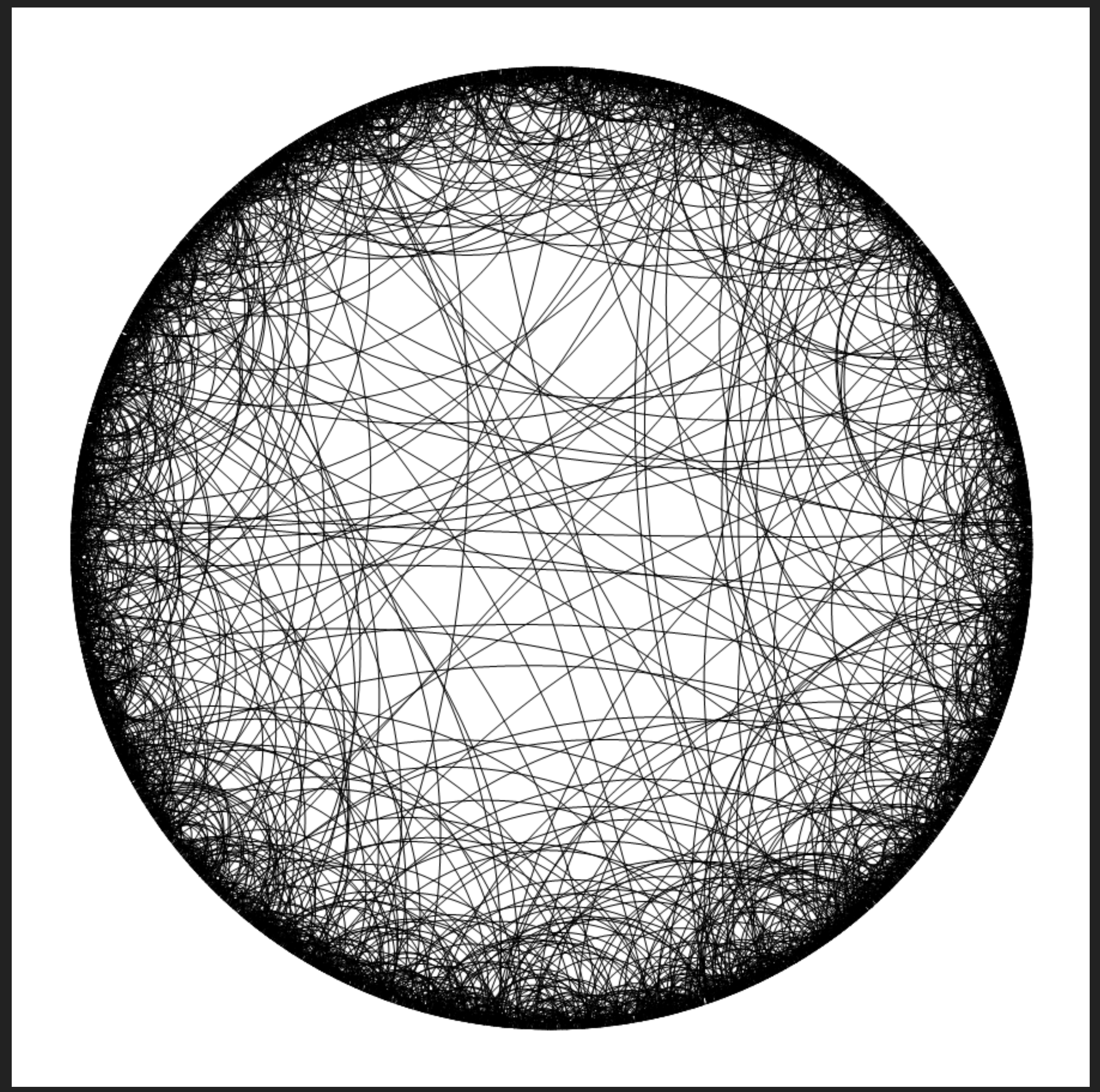
$$\operatorname{Var} S_{R,\lambda} \asymp \begin{cases} e^R & : \lambda < 1 \text{ and } d = 2 \\ Re^{2R} & : \lambda < 1 \text{ and } d = 3 \\ e^{2(d-1)R} & : \lambda < 1 \text{ and } d \geq 4 \\ Re^{(d-1)R} & : \lambda = 1 \text{ and } d \geq 2 \end{cases}$$

Expectation

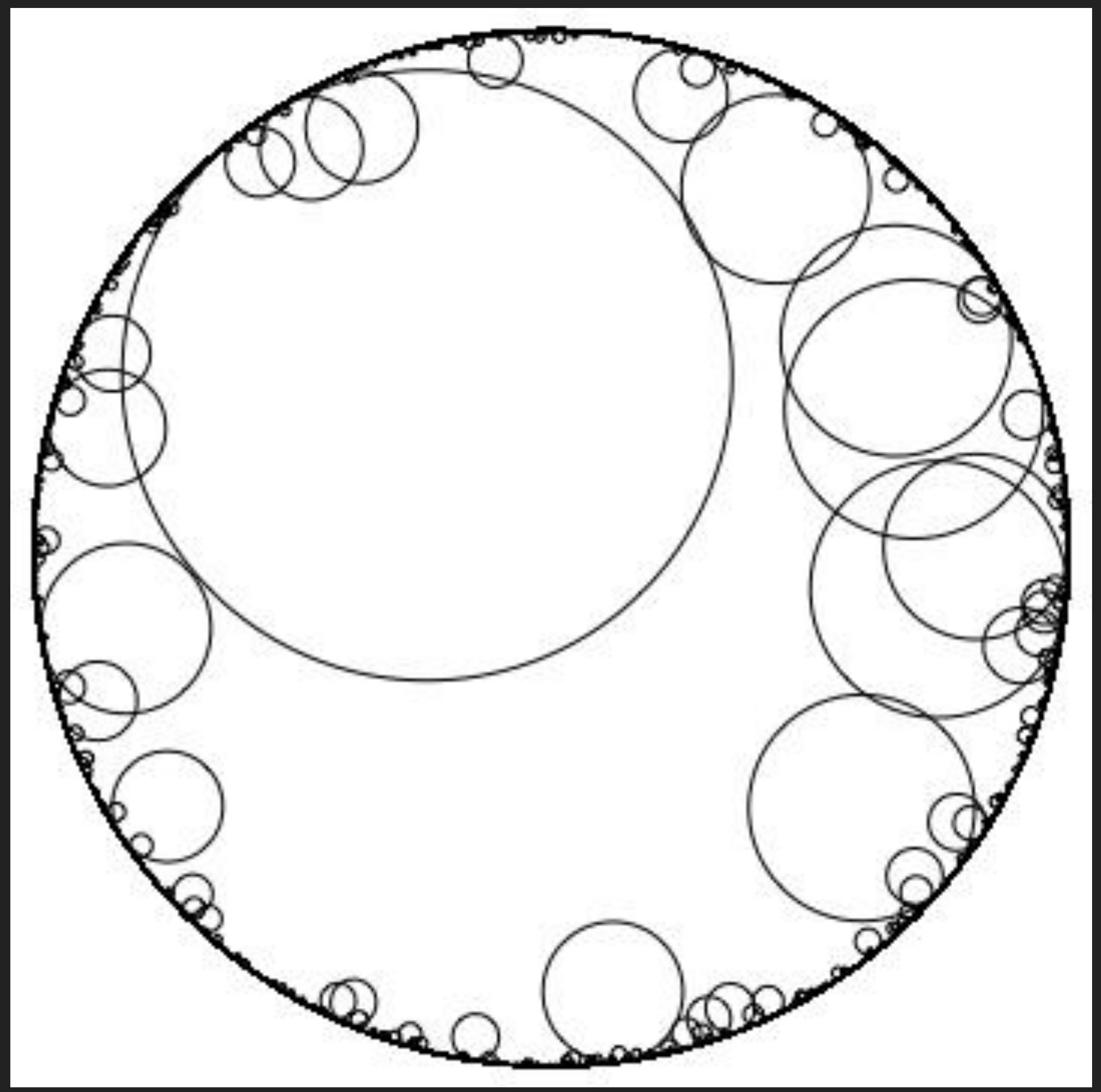
$$\int_{\mathbb{A}_\lambda} \mathscr{H}^{d-1}(H \cap B_R)\, \Lambda_\lambda(\mathrm{d}H)$$

Variance

$$\int_{\mathbb{A}_\lambda} \mathscr{H}^{d-1}(H \cap B_R)^2\, \Lambda_\lambda(\mathrm{d}H)$$



$\lambda = 0$



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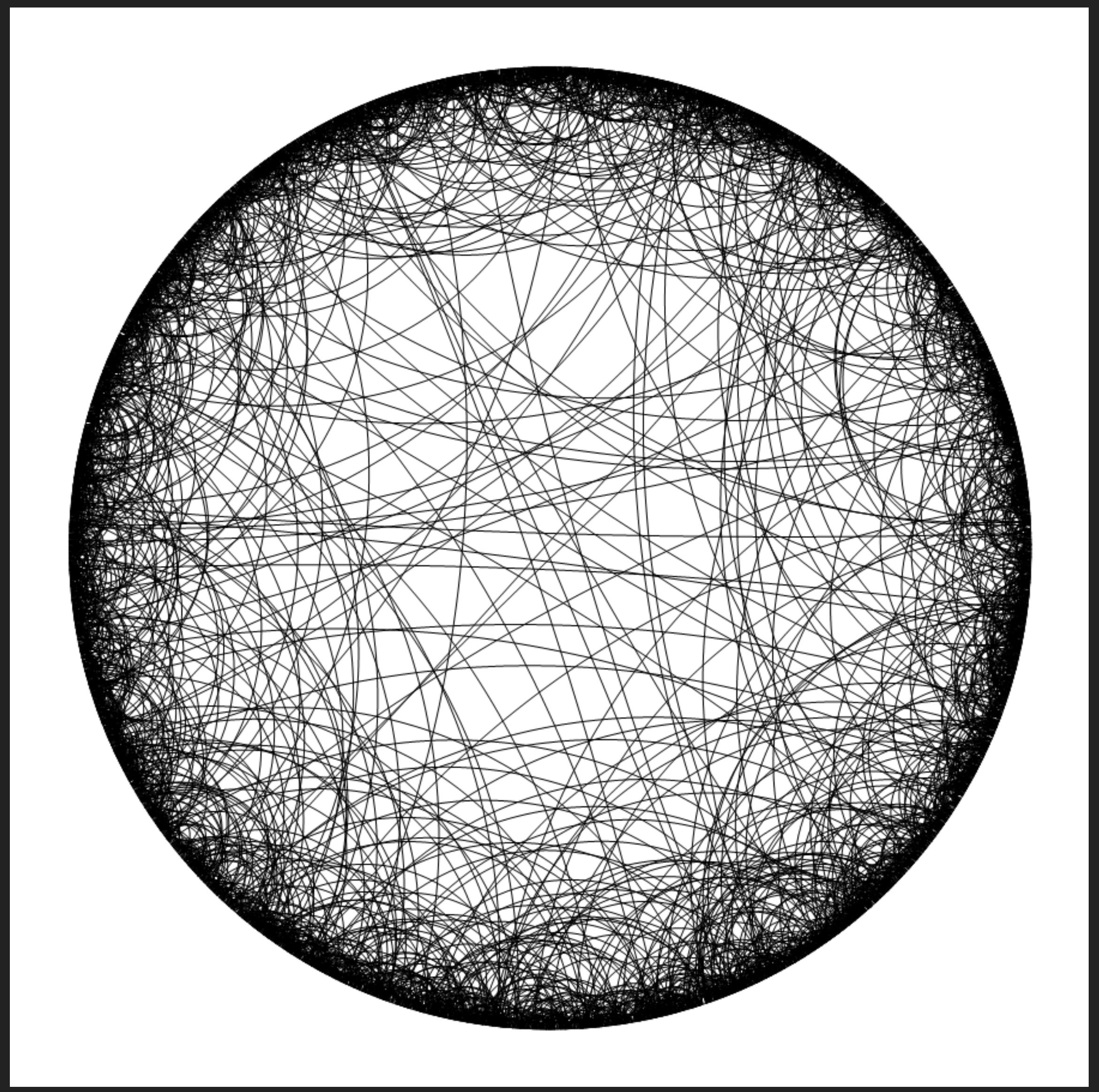
Lemma: $\mathbb{E}S_{R,\lambda} = \mathscr{H}^d(B_R)$

Lemma:

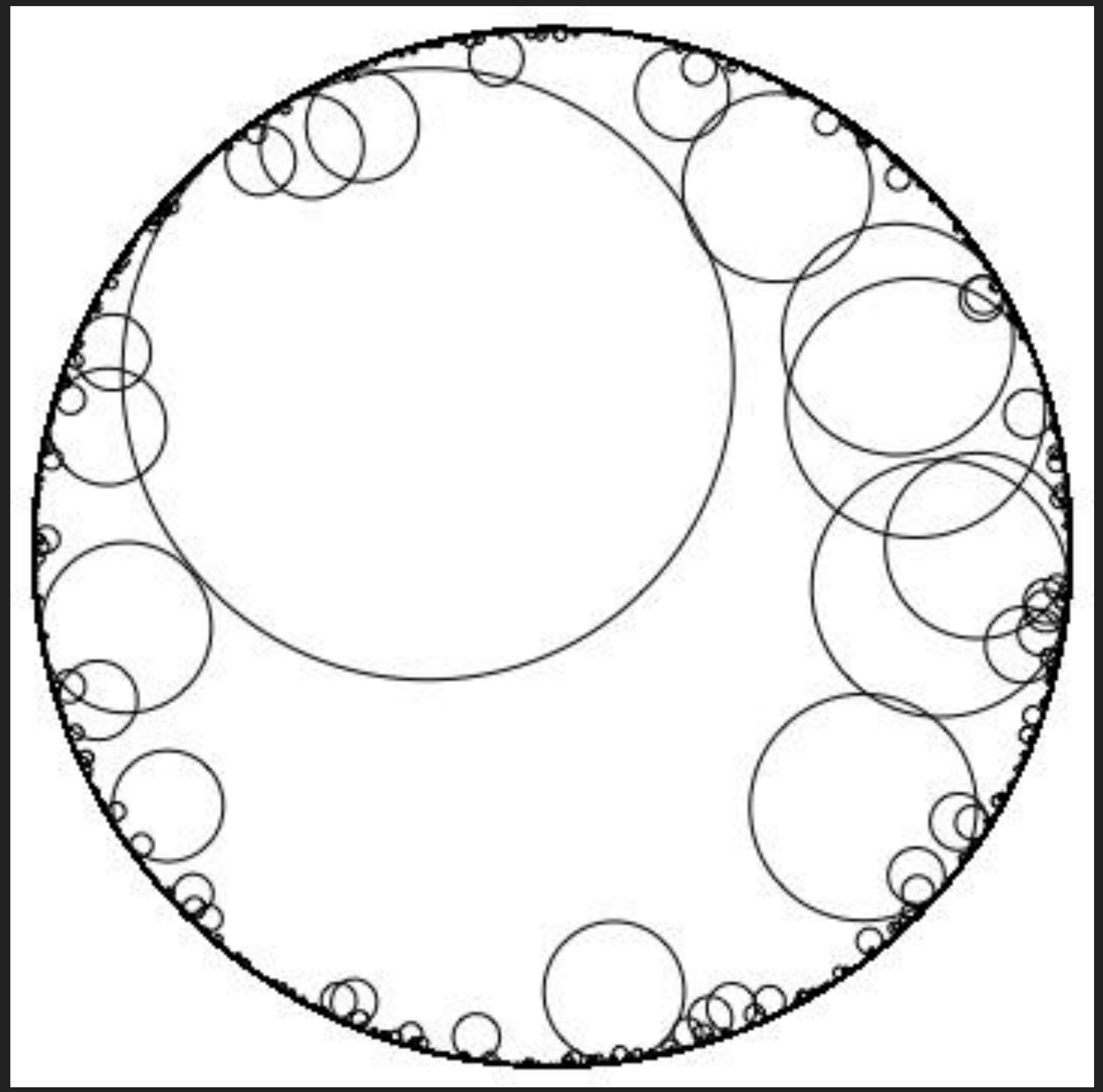
$$\operatorname{Var} S_{R,\lambda} \asymp \begin{cases} e^R & : \lambda < 1 \text{ and } d = 2 \\ Re^{2R} & : \lambda < 1 \text{ and } d = 3 \\ e^{2(d-1)R} & : \lambda < 1 \text{ and } d \geq 4 \\ Re^{(d-1)R} & : \lambda = 1 \text{ and } d \geq 2 \end{cases}$$

Theorem (Kabluchko, Rosen, T. 2023 IJM):

Suppose that $0 \leq \lambda < 1$ and $d \leq 3$. Then $\dfrac{S_R - \mathbb{E}S_R}{\sqrt{\operatorname{Var} S_R}} \xrightarrow{D} \mathscr{N}(0,1)$.



$\lambda = 0$



$\lambda = 1$

POISSON λ-GEODESIC HYPERPLANES

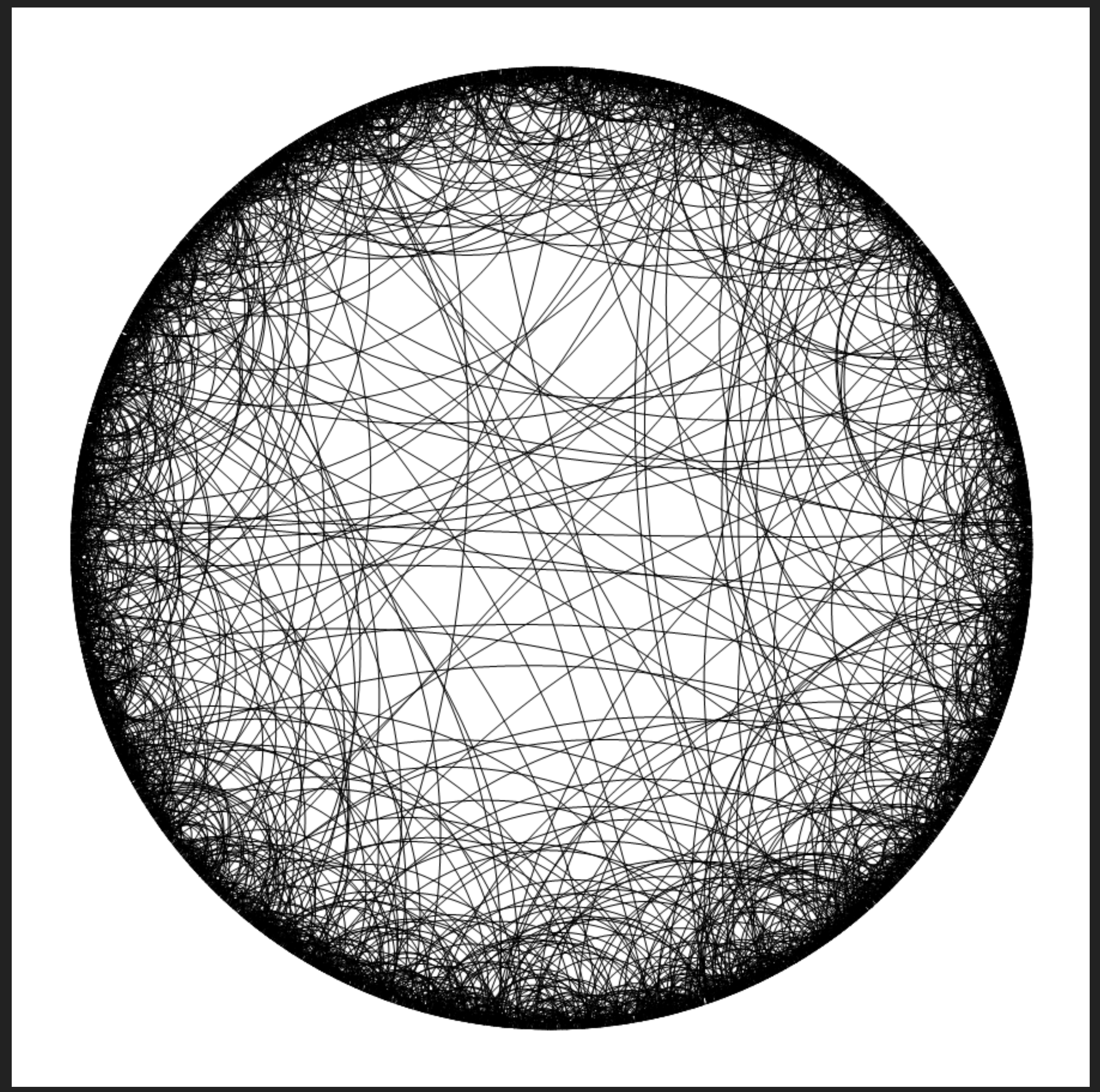
Theorem (Kabluchko, Rosen, T. 2023 IJM):

Suppose that $0 \le \lambda < 1$ and $d \le 3$. Then
$$\frac{S_R - \mathbb{E} S_R}{\sqrt{\operatorname{Var} S_R}} \xrightarrow{D} \mathcal{N}(0,1)\,.$$

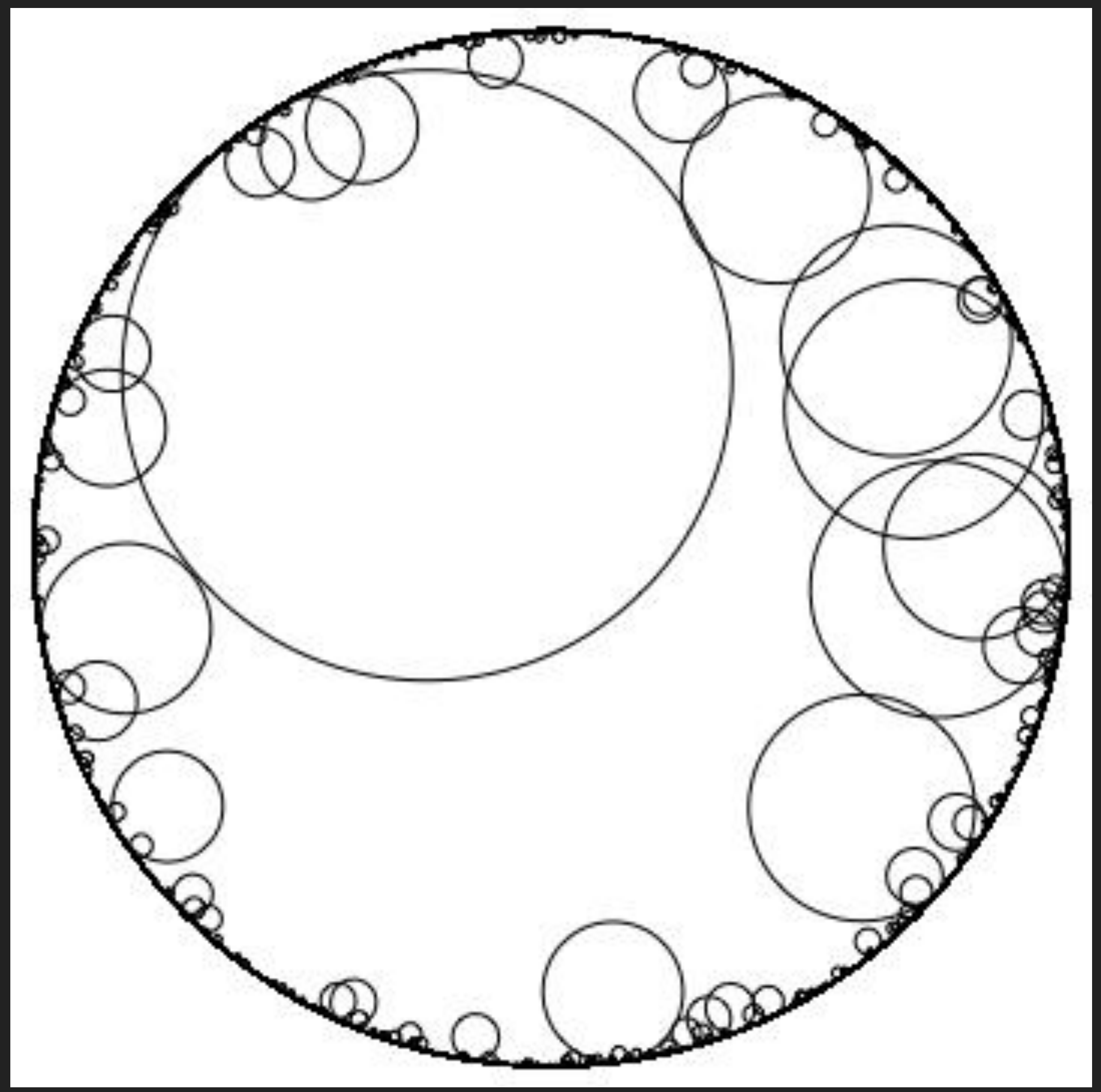
Theorem (Kabluchko, Rosen, T. 2023 IJM):

Suppose that $0 \le \lambda < 1$ and $d \ge 4$. Then
$$\frac{S_R - \mathbb{E} S_R}{e^{(d-2)R}} \xrightarrow{D} Z$$

- Z infinitely divisible
- non-Gaussian
- $\kappa_m(Z) = \sqrt{\pi}(1-\lambda^2)^{\frac{d-1}{2}} \dfrac{\Gamma\left(\frac{(d-2)\ell - (d-1)}{2}\right)}{\Gamma\left(\frac{(d-2)(\ell-1)}{2}\right)}\,.$



$\lambda = 0$



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POISSON λ-GEODESIC HYPERPLANES

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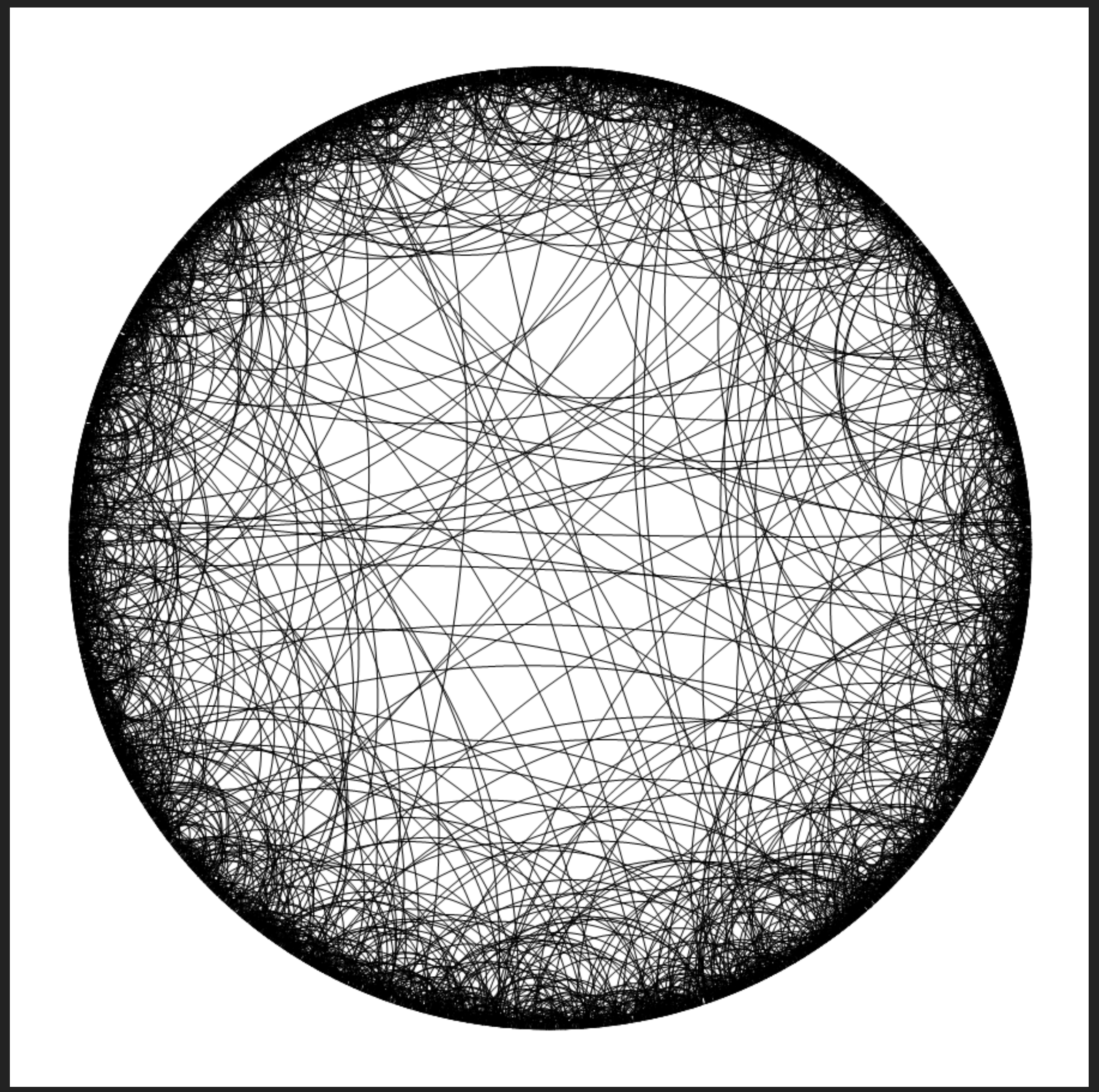
Theorem (Kabluchko, Rosen, T. 2023 IJM):

Suppose that $0 \leq \lambda < 1$ and $d \geq 4$. Then $\dfrac{S_R - \mathbb{E}S_R}{e^{(d-2)R}} \xrightarrow{D} Z$

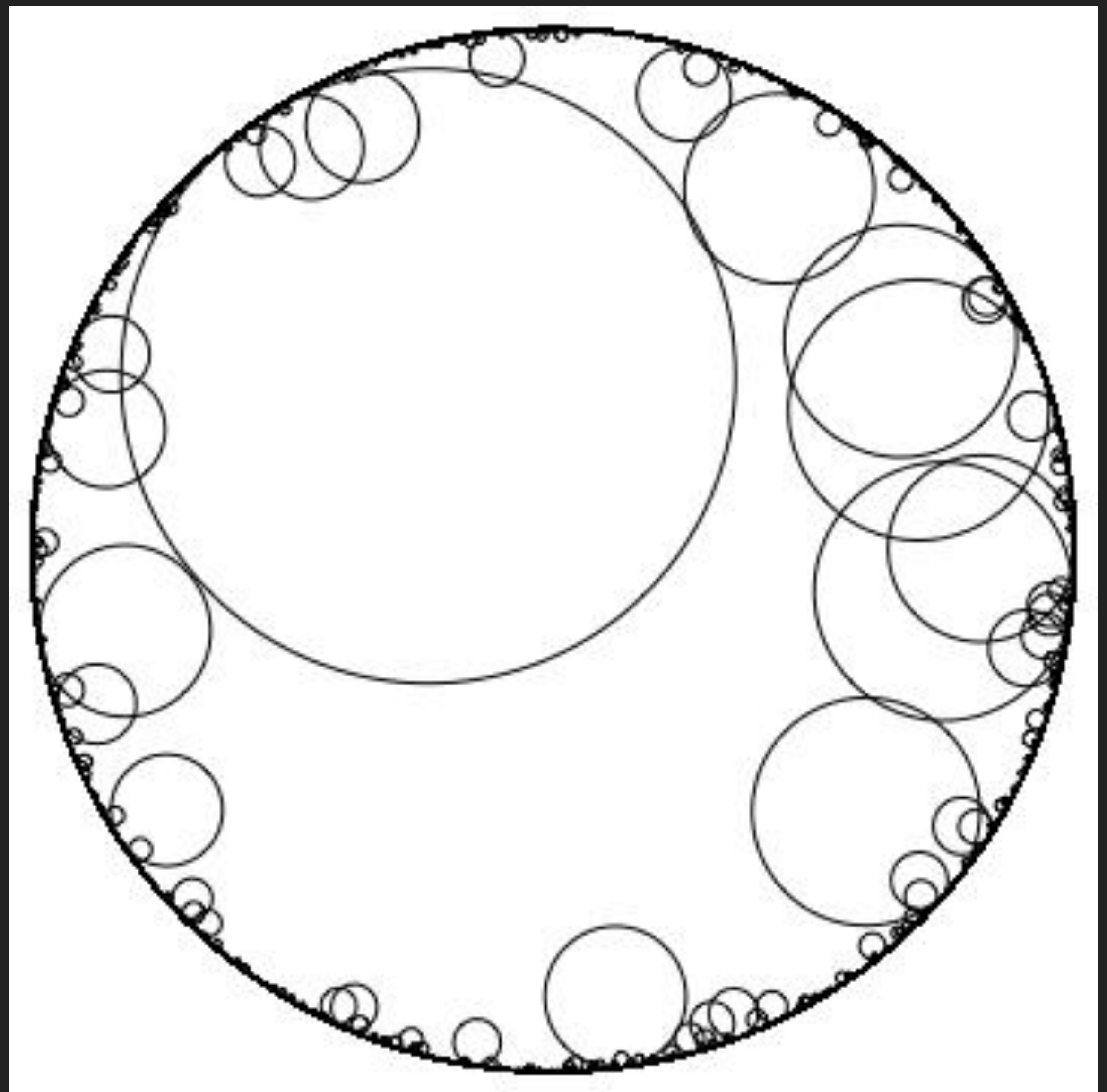
- Z infinitely divisible
- non-Gaussian

Theorem (Kabluchko, Rosen, T. 2023 IJM):

Suppose that $\lambda = 1$ and $d \geq 2$. Then $\dfrac{S_R - \mathbb{E}S_R}{\sqrt{\operatorname{Var} S_R}} \xrightarrow{D} \mathcal{N}\left(0, \dfrac{1}{2}\right)$.



$\lambda = 0$



$\lambda = 1$

SOME IDEAS ABOUT THE PROOF ($0 \leq \lambda < 1$)

Theorem (Kabluchko, Rosen, T. 2023 IJM):

Suppose that $0 \leq \lambda < 1$ and $d \geq 4$. Then $\dfrac{S_R - \mathbb{E}S_R}{e^{(d-2)R}} \xrightarrow{D} Z$

- Z infinitely divisible
- non-Gaussian

Define $F_R := \dfrac{S_R - \mathbb{E}S_R}{e^{(d-2)R}}$

1. Consider the characteristic function of F_R:

$$\Psi_R(t) := \mathbb{E}e^{itF_R} = \exp\left(\int_{-R}^{R} [e^{itg_R(s)} - 1 - itg_R(s)]p_\lambda(s)\,\mathrm{d}s\right)$$

$$g_R(s) = e^{-(d-2)R}\mathscr{H}^{d-1}(H(s) \cap B_R)$$

$$p_\lambda(s) = (\cosh s - \lambda \sinh s)^{d-1}$$

2. Try to interchange limit and integral

Key geometric lemma

$$g_R(s) \longrightarrow C_{d,\lambda} \cosh^{-(d-2)}(s - \Delta)$$

$$\Delta = \operatorname{artanh}(\lambda)$$

3. Justify step 2 by dominated convergence

   This is possible exactly if $d \geq 4$

4. Observe that by the Lévy-Khintchin formula the random variable Z is infinitely divisible without Gaussian component

OUTLOOK

What is special about $d = 2$, $d = 3$ and $d \geq 4$?

Consider k-dimensional totally geodesic subspaces.
Intersect m of them $\longrightarrow$ intersection process of oder m.
Look at the sum of the $[d - m(d-k)]$-volumes.

Theorem (Betken, Hug, T. 2023 SPA):

$2k < d$: Then $m = 1$ and CLT holds with usual rate

$2k = d$: Then $m \in \{1,2\}$ and CLT holds with usual rate

$2k = d + 1$: Then $m \in \{1,2,3\}$ and CLT holds with slower rate

$2k > d + 1$: CLT breaks down

Open problem: Characterize the limit distribution for intersection processes.

The bigger picture: Stochastic geometry in non-Euclidean geometries

- Random polytopes
- Random tessellations
- Random graphs

THANK YOU!