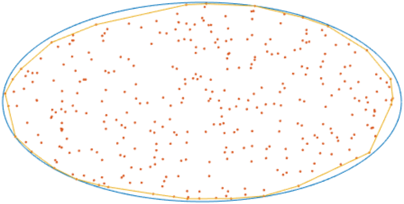
Typical and extremal results for random polytopes in smooth convex bodies

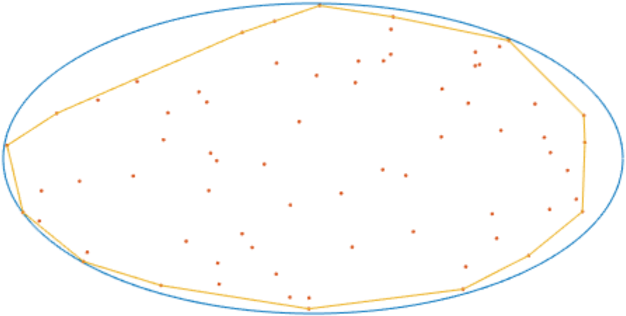


Pierre Calka, *J. E. Yukich*

28 September 2023, Salzburg
Conference on Convex Geometry and Geometric Probability

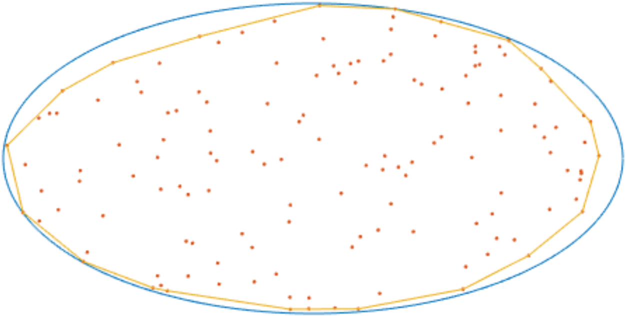
Random convex hull

K smooth convex body in $\mathbb{R}^d$
$X_1, \ldots, X_n$ n i.i.d. points uniformly distributed in K
K_n convex hull of $X_1, \ldots, X_n$



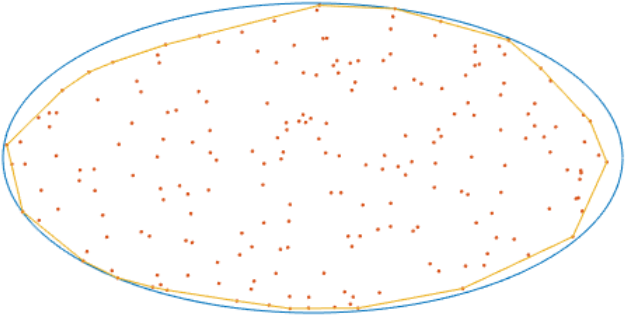
Random convex hull

K smooth convex body in $\mathbb{R}^d$
$X_1, \ldots, X_n$ n i.i.d. points uniformly distributed in K
K_n convex hull of $X_1, \ldots, X_n$



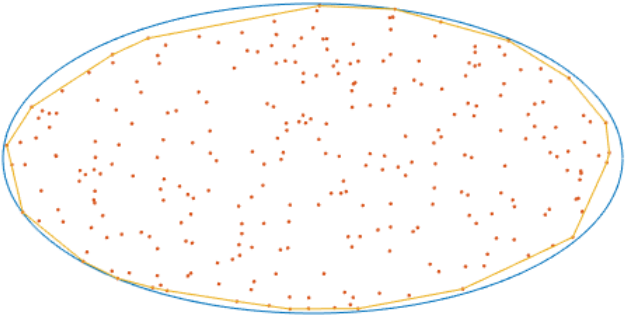
Random convex hull

K smooth convex body in $\mathbb{R}^d$
$X_1, \ldots, X_n$ n i.i.d. points uniformly distributed in K
K_n convex hull of $X_1, \ldots, X_n$



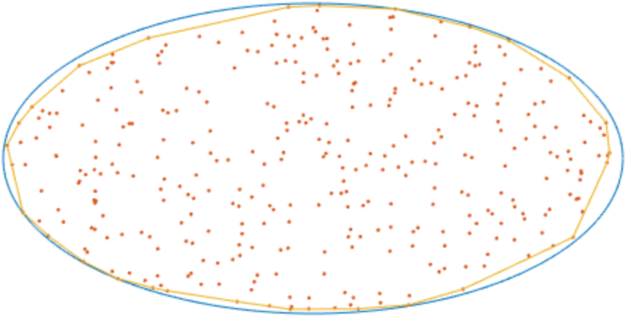
Random convex hull

K smooth convex body in $\mathbb{R}^d$
$X_1, \ldots, X_n$ n i.i.d. points uniformly distributed in K
K_n convex hull of $X_1, \ldots, X_n$



Random convex hull

K smooth convex body in $\mathbb{R}^d$
$X_1, \ldots, X_n$ n i.i.d. points uniformly distributed in K
K_n convex hull of $X_1, \ldots, X_n$



Known non-asymptotic results

$\mathsf{Vol}_d :=$ Lebesgue measure of $\mathbb{R}^d$, $f_k(\cdot) :=$ number of k-dimensional faces

Mean-value identities

$$\mathbb{E}f_0(K_n) = n\left(1 - \frac{\mathbb{E}(\mathsf{Vol}_d(K_{n-1}))}{\mathsf{Vol}_d(K)}\right)$$

References. Efron (1965), Buchta (2005)

Monotonicity results

K 2-dimensional, $\mathrm{Vol}_2(K) = 1$, $\mathcal{V}_n(K) = \mathbb{E}(\mathrm{Vol}_2(K_n))$

$$\mathcal{V}_n(\bigcirc) \leq \mathcal{V}_n(K) \leq \mathcal{V}_n(\triangle)$$

References. Blaschke (1917), Busemann (1953), Buchta (1983), Groemer (1973)

Explicit moment formulas

$K = \mathbb{B}^d$, $f_k(K_n) :=$ number of k-dimensional faces

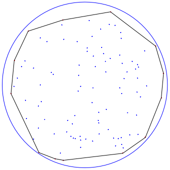
References. Kabluchko (2021)

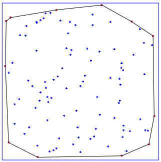
Known asymptotic results

Expectation estimates

$\mathrm{Vol}_d :=$ Lebesgue measure of $\mathbb{R}^d$, $f_k(\cdot) :=$ number of k-dimensional faces

		K smooth	K polytope
$\mathbb{E}(\mathrm{Vol}_d(K \setminus K_n))$	$\underset{n\to\infty}{\sim}$	$c_1(d,K)n^{-\frac{2}{d+1}}$	$c_1'(d,K)n^{-1}\log^{d-1}(n)$
$\mathbb{E}(f_k(K_n))$	$\underset{n\to\infty}{\sim}$	$c_2(d,k,K)n^{\frac{d-1}{d+1}}$	$c_2'(d,k,K)\log^{d-1}(n)$





References. Rényi & Sulanke (1963), Schneider & Wieacker (1980), Bárány (1989), Reitzner (2003)

Variance estimates, CLT and concentration

References. Reitzner (2005), Bárány & Reitzner (2010), PC & Yukich (2014), Vu (2005), Grote & Thäle (2018)

Typical facet

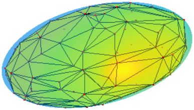
Notation K smooth convex body, $\kappa(\cdot) :=$ Gauss curvature along ∂K

$$Z_n := \mathbb{E}(f_{d-1}(K_n)) \sim c_d \Omega(K) n^{\frac{d-1}{d+1}}$$

where $\Omega(K) = \mathrm{Vol}_d(K)^{-\frac{d-1}{d+1}} \int_{\partial K} \kappa(z)^{\frac{1}{d+1}} \mathrm{d}z$

Reference. Raynaud (1970)

Facets of K_n Simplices a.s.



Typical facet $\mathcal{F}_n$ For every non-negative measurable function f,

$$\mathbb{E} f(\mathcal{F}_n) = \frac{1}{Z_n} \mathbb{E}\left(\sum_{F \in \{\text{facets of } K_n\}} f(F) \right)$$

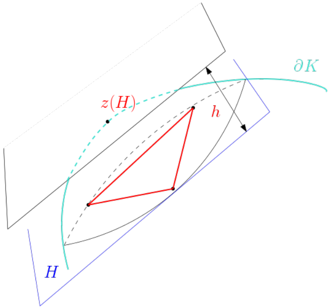
Asymptotically equivalent to a uniform choice in the facets of K_n

Reference. Bonnet & O'Reilly (2022)

Location of a facet

Height and z-value of a facet

Each facet is included in a section of K by a hyperplane H.



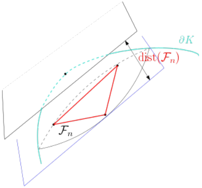
$z(H) :=$ support point in ∂K of the closest parallel hyperplane which is tangent to ∂K

$h :=$ distance from that tangent hyperplane to H

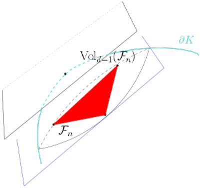
Notation $H_z(h) := H$, $\mathrm{dist}(F) := h$

Functionals of the typical facet

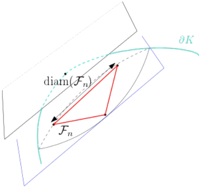
Height $\mathrm{dist}(\mathcal{F}_n)$



Volume $\mathrm{Vol}_{d-1}(\mathcal{F}_n)$



Diameter $\mathrm{diam}(\mathcal{F}_n)$



Aims

Regime of the convergence in distribution

Limit distributions for rescaled functionals of the typical facet

$$\mathbb{P}(n^{\alpha} f(\mathcal{F}_n) \geq t, z(\mathcal{F}_n) \in \cdot) \to \quad ??$$

Convergence in process and comparison to the KPZ universality class

Regime of the extremes

Limit measure when f_n is a properly rescaled functional

$$Z_n \mathbb{P}(f_n(\mathcal{F}_n) \in \cdot, z(\mathcal{F}_n) \in \cdot) \to \quad ??$$

Consequences for the limit extremal distributions

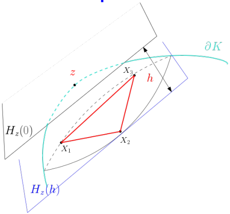
$$\mathbb{P}\Big(\max_{F \in \{\text{facets of } K_n\}} f_n(F) \leq t\Big) \to \quad ??$$

Plan

Plan

Rescaling: critical distance from the boundary

Volume of a cap of K associated with $(z, h) \in \partial K \times (0, \infty)$



$$\varphi_z(h) \sim \frac{c_d}{\sqrt{\kappa(z)}} h^{\frac{d+1}{2}} \text{ when } h \to 0$$

Probability of having a facet

$$\mathbb{P}(X_1, \cdots, X_d \text{ generate a facet}) = \left(1 - \frac{\varphi_z(h)}{\mathrm{Vol}_d(K)}\right)^{n-d} + O(e^{-cn})$$

$$n\varphi_z(h) = \Theta(1) \text{ when } h = n^{-\frac{2}{d+1}}$$

Rescaled functionals

Height	Volume	Diameter
$n^{\frac{2}{d+1}} \mathrm{dist}(\mathcal{F}_n)$	$n^{\frac{d-1}{d+1}} \mathrm{Vol}_{d-1}(\mathcal{F}_n)$	$n^{\frac{1}{d+1}} \mathrm{diam}(\mathcal{F}_n)$

Convergence in distribution of the rescaled functionals

✓ For any $t > 0$ and $B \in \mathcal{B}(\partial K)$, when $n \to \infty$,

$$\mathbb{P}(n^{\frac{2}{d+1}}\mathrm{dist}(\mathcal{F}_n) \geq t, z(\mathcal{F}_n) \in B)$$
$$\to c_d \int_B e^{-\beta_{d,z} t^{\frac{d+1}{2}}} \int_0^\infty e^{-w} \left(\beta_{d,z} t^{\frac{d+1}{2}} + w\right)^{d-1} \mathrm{d}w$$

where $\beta_{d,z} = \frac{\kappa_{d-1} 2^{\frac{d+1}{2}}}{(d+1)\sqrt{\kappa(z)}\mathrm{Vol}_d(K)}$ $\quad (\kappa_d = \mathrm{Vol}_d(\mathbb{B}^d))$

✓ Similar results for $n^{\frac{d-1}{d+1}}\mathrm{Vol}_{d-1}(\mathcal{F}_n)$ and $n^{\frac{1}{d+1}}\mathrm{diam}(\mathcal{F}_n)$

Particular case $\quad K = \mathbb{B}^2$, $n\mathrm{dist}(\mathcal{F}_n)$ typical height in $n\mathbb{B}^2$

$$\mathbb{P}(n^{-\frac{1}{3}}(n\mathrm{dist}(\mathcal{F}_n)) \geq t) \sim ct^{\frac{3}{2}} \exp\left(-\frac{4\sqrt{2}}{3\pi}t^{\frac{3}{2}}\right) \quad \text{when } n, t \to \infty$$

Plan

Rescaling

Exponent

f	dist	Vol_{d-1}	diam
α	$\frac{2}{d+1}$	$\frac{d-1}{d+1}$	$\frac{1}{d+1}$

Rescaled functional of the facet

$$f_n(\mathcal{F}_n) := a_0^{-1} n f(\mathcal{F}_n)^{\frac{1}{\alpha}} - (a_1 \log n + a_2 \log\log n + a_3)$$

with a_i, $0 \leq i \leq 3$, positive constants depending on d and K

Associated function along ∂K

f	dist	Vol_{d-1}	diam
$\psi \propto$	$\sqrt{\kappa(z)}$	$\kappa(z)^{-\frac{1}{d-1}}$	$c_{z,1}^{-\frac{d+1}{2}} \sqrt{\kappa(z)}$

where $\kappa(\cdot)$ Gaussian curvature on ∂K and $c_{z,1}$ first principal curvature

$$a_0 = \frac{d+1}{\kappa_{d-1}} \mathrm{Vol}_d(K) \max_{z \in \partial K} \psi(z), \quad a_1 = \frac{d-1}{d+1}$$

Associated measure and asymptotic support

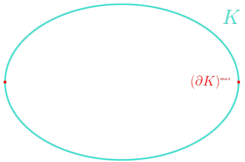
Associated measure

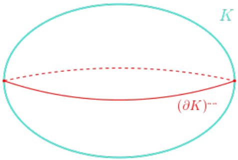
$$\mu_n(A \times B) := Z_n \mathbb{P}(f_n(\mathcal{F}_n) \in A, z(\mathcal{F}_n) \in B), \quad A \in \mathcal{B}(\mathbb{R}),\ B \in \mathcal{B}(\partial K)$$

where $Z_n = \mathbb{E}(f_{d-1}(K_n)) \sim c_d \Omega(K) n^{\frac{d-1}{d+1}}$

Asymptotic support on ∂K $\qquad (\partial K)^{\text{max}} := \text{argmax}\ \psi$

Example 1 $\qquad$ When $f = \text{dist}$, $(\partial K)^{\text{max}} = \text{argmax}\ \kappa$





Example 2 $\qquad$ When $f = \text{Vol}_{d-1}$, $(\partial K)^{\text{max}} = \text{argmin}\ \kappa$

Convergence in the regime of the extremes

$\mu_n(A \times B) = Z_n \mathbb{P}(f_n(\mathcal{F}_n) \in A, z(\mathcal{F}_n) \in B)$

For $\tau \in \mathbb{R}$ and admissible K,

$$\mu_n(\cdot \times \cdot \cap ((\tau, \infty) \times \partial K)) \to \mathbf{1}_{(\tau,\infty)}(x) e^{-x} \mathrm{d}x \times \nu_{(\partial K)^{\max}}$$

where $\nu_{(\partial K)^{\max}}$ is a probability measure with support in $(\partial K)^{\max}$.

$\dim((\partial K)^{\max}) = d - 1$ Convergence in total variation

When $f = \text{dist}$ or Vol_{d-1}, $\nu_{(\partial K)^{\max}} = \text{Unif}_{(\partial K)^{\max}}$

$\dim((\partial K)^{\max}) < d - 1$ Weak convergence

When $f = \text{dist}$ or Vol_{d-1}, $\nu_{(\partial K)^{\max}}$ has a density $\propto (\det(A_z))^{-\frac{1}{2}}$, A_z being the Hessian matrix of κ at z.

Plan

Strategy

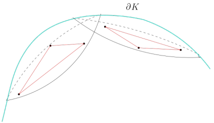
Aim when $f = \text{dist}$ or Vol_{d-1}, $\max_{F \in \{\text{facets of } K_n\}} f_n(F) \xrightarrow{D}$??

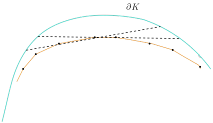
Prerequisite Convergence of $Z_n \mathbb{P}(f_n(\mathcal{F}_n) \geq \tau)$ to $e^{-\tau}$

The joint measures $Z_n \mathbb{P}(f_n(\mathcal{F}_n) \in \cdot, z(\mathcal{F}_n) \in \cdot)$ tell the distribution of the location of the exceedances.

Mixing conditions

Maximum of the volumes	Maximum of the heights
Facets asymptotically independent	Exceedances by blocks $\hookrightarrow$ Use of the *blocking method*



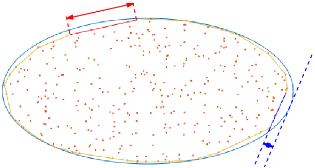


Maxima of the facet heights and volumes

$f_n(\mathcal{F}_n) = a_0^{-1} n f(\mathcal{F}_n)^{\frac{1}{\alpha}} - (a_1 \log n + a_2 \log\log n + a_3)$

When $f = \text{dist}$ or Vol_{d-1}, $\max_{F \in \{\text{facets of } K_n\}} f_n(F) \xrightarrow{D} G$

where G is a Gumbel variable, i.e. $\mathbb{P}(G \leq t) = e^{-e^{-t}}$, $t \in \mathbb{R}$.



Hausdorff distance

When $f = \text{dist}$, $\max_{F \in \{\text{Facets of } K_n\}} f_n(F)$ and the rescaled version of $d_H(K_n, K)$ have same limit distribution.

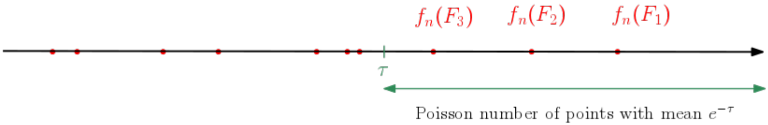
Reference. Bárány (1989), Bräker, Hsing & Bingham (1998)

Poisson approximation in the case $f = \mathrm{Vol}_{d-1}$

$$f_n(\mathcal{F}_n) = a_0^{-1} n Vol_{d-1}(\mathcal{F}_n)^{\frac{d+1}{d-1}} - (a_1 \log n + a_2 \log\log n + a_3)$$

✓ Poisson convergence for the Kantorovich-Rubinstein distance of the point process of $\{f_n(F), F \in \{\text{Facets of } K_n\}\}$

✓ Poisson convergence of the point process of the couples $(f_n(F), z(F))$ when $\dim((\partial K)^{\max}) = d-1$



References. O. Bobrowski, M. Schulte, D. Yogeshwaran (2021)

Plan

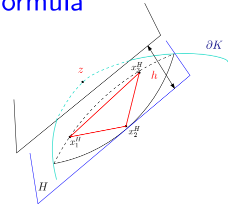
Integral representation and change of variables

Integral formula

$$Z_n \mathbb{P}(f_n(\mathcal{F}_n) \in A, z(\mathcal{F}_n) \in B) = \binom{n}{d} \int_{K^d} \mathbf{1}(\cdots) \left(1 - \frac{\varphi_z(h)}{\mathrm{Vol}_d(K)}\right)^{n-d} \frac{\mathrm{d}x_1 \ldots \mathrm{d}x_d}{\mathrm{Vol}_d(K)^d} + O(e^{-cn})$$

where $\varphi_z(h)$:= volume of the cap of K associated with (z, h)

Blaschke-Petkantschin formula



$$\mathrm{d}x_1 \cdots \mathrm{d}x_d = (d-1)! \left[\mathrm{Vol}_{d-1}(\mathrm{Conv}(x_1^H, \cdots, x_d^H)) \mathrm{d}x_1^H \cdots \mathrm{d}x_d^H\right] \mathrm{d}H$$

Change with (z, h) coordinates

$$\mathrm{d}H = \kappa(z) \mathrm{d}z \mathrm{d}h$$

Asymptotic stochastic equivalence

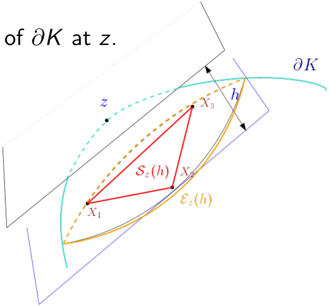
- The position $(z(\mathcal{F}_n), h(\mathcal{F}_n))$ of the facet is chosen with density proportional to $\kappa(z)\left(1 - \frac{\varphi_z(h)}{\mathrm{Vol}_d(K)}\right)^{n-d} \varphi_z'(h)^{d+1}$.

- Conditional on $(z(\mathcal{F}_n), h(\mathcal{F}_n)) = (z, h)$, the d vertices $X_1, \cdots, X_d$ in $H_z(h) \cap K$ are chosen as i.d. points in the osculating ellipsoid $\mathcal{E}_z(h)$ with density proportional to $\mathrm{Vol}_{d-1}(\mathrm{Conv}(x_1, \cdots, x_d))$.

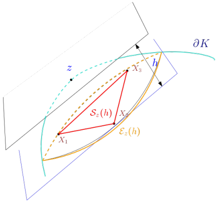
$$\mathcal{E}_z(h) : \frac{1}{2} \sum_{i=1}^{d-1} c_{z,i} (x^{(i)})^2 \leq h \qquad \text{in the proper frame}$$

where $c_{z,1}, \cdots, c_{z,d-1}$ are the principal curvatures of ∂K at z.

- $\mathcal{F}_n = \mathcal{S}_z(h) = \mathrm{Conv}(X_1, \cdots, X_d)$



New integral representation and additional arguments



$$Z_n \mathbb{P}(f_n(\mathcal{F}_n) \in A, z(\mathcal{F}_n) \in B)$$
$$\sim \frac{n^d}{d!} \int_{z \in B} \int_0^{\ell_z} \kappa(z) \left(1 - \frac{\varphi_z(h)}{\mathrm{Vol}_d(K)}\right)^{n-d} (\varphi_z'(h))^{d+1} \mathbb{P}(f_n(\mathcal{S}_z(h)) \in A) \mathrm{d}h \mathrm{d}z$$

- Negligibility of the event $\{d(z(\mathcal{F}_n), (\partial K)^{\max}) > \varepsilon\}$
- Two-term expansion of $\varphi_z(h)$ (and of $\psi(z)$ near a maximum)
- Distribution tail of $\mathrm{Vol}_{d-1}(\mathcal{S}_z(h))$ or $\mathrm{diam}(\mathcal{S}_z(h))$
- Calibration of the constants accordingly

Thank you for your attention!