Unimodular Continuum Spaces

Ali Khezeli

INRIA Paris

Conference on Convex Geometry and Geometric Probability,
Salzburg, September 2023

Outline

1. The Mass Transport Principle in Various Subjects
2. Unimodular Continumm Spaces
3. Properties of Unimodular Spaces
4. Palm Theory
5. Amenability

Outline

1. The Mass Transport Principle in Various Subjects
2. Unimodular Continumm Spaces
3. Properties of Unimodular Spaces
4. Palm Theory
5. Amenability

Introduction

Connection between various fields:

- Stationary point processes,
- Unimodular random graphs,
- Unimodular discrete spaces,
- Stationary random measures,
- Scaling limits,
- Borel equivalence relations.

Key property: The **mass transport principle (MTP)**.

Introduction

Connection between various fields:

- Stationary point processes,
- Unimodular random graphs,
- Unimodular discrete spaces,
- Stationary random measures,
- Scaling limits,
- Borel equivalence relations.

Key property: The **mass transport principle (MTP)**.

1. Point Processes

- Φ: A stationary point process on $\mathbb{R}^d$.
  - i.e., a random discrete subset of $\mathbb{R}^d$,
  - s.th., $\forall t \in \mathbb{R}^d : \Phi + t \sim \Phi$.
- The **Palm version** of Φ:
  - $\Phi_0 := \Phi$ conditioned on containing 0,
  - or Φ seen from a *typical point of* Φ.
  - Formally:

$$\mathbb{E}\left[h(\Phi_0)\right] = \frac{1}{\lambda}\mathbb{E}\left[\sum_{x\in\Phi\cap[0,1]^d} h(\Phi - x)\right].$$

- Heuristically, for a translation-invariant function $g(\Phi, x)$,

$$\mathbb{E}\left[g(\Phi_0, 0)\right] \longleftrightarrow \sum_{x\in\Phi} g(\Phi, x).$$

1. Point Processes

- Φ: A stationary point process on $\mathbb{R}^d$.
  - i.e., a random discrete subset of $\mathbb{R}^d$,
  - s.th., $\forall t \in \mathbb{R}^d : \Phi + t \sim \Phi$.
- The **Palm version** of Φ:
  - $\Phi_0 :=$ Φ conditioned on containing 0,
  - or Φ seen from a *typical point of* Φ.
  - Formally:

$$\mathbb{E}\left[h(\Phi_0)\right] = \frac{1}{\lambda}\mathbb{E}\left[\sum_{x\in\Phi\cap[0,1]^d} h(\Phi - x)\right].$$

- Heuristically, for a translation-invariant function $g(\Phi, x)$,

$$\mathbb{E}\left[g(\Phi_0, 0)\right] \longleftrightarrow \sum_{x\in\Phi} g(\Phi, x).$$

1. Point Processes

- Φ: A stationary point process on $\mathbb{R}^d$.
  - i.e., a random discrete subset of $\mathbb{R}^d$,
  - s.th., $\forall t \in \mathbb{R}^d : \Phi + t \sim \Phi$.
- The **Palm version** of Φ:
  - $\Phi_0 := \Phi$ conditioned on containing 0,
  - or Φ seen from a *typical point of* Φ.
  - Formally:

$$\mathbb{E}\left[h(\Phi_0)\right] = \frac{1}{\lambda}\mathbb{E}\left[\sum_{x \in \Phi \cap [0,1]^d} h(\Phi - x)\right].$$

- Heuristically, for a translation-invariant function $g(\Phi, x)$,

$$\mathbb{E}\left[g(\Phi_0, 0)\right] \longleftrightarrow \sum_{x \in \Phi} g(\Phi, x).$$

- **Mecke's formula**:
  For all measurable functions $h(\Phi_0, x) \geqslant 0$ (for $x \in \mathbb{R}^d$):
  $\mathbb{E}\left[\sum_{x\in\Phi_0} h(\Phi_0, x)\right] = \mathbb{E}\left[\sum_{x\in\Phi_0} h(\Phi_0 - x, -x)\right]$.
- Let $g(\Phi_0, x, y) := h(\Phi_0 - x, y - x) \Rightarrow$

Theorem (MTP)

For all measurable functions $g(\Phi_0, x, y) \geqslant 0$ that are translation-invariant:

$$\mathbb{E}\left[\sum_{x\in\Phi_0} g(\Phi_0, 0, x)\right] = \mathbb{E}\left[\sum_{x\in\Phi_0} g(\Phi_0, x, 0)\right].$$

2. Unimodular Graphs

- $\mathcal{G}_*$: The space of all rooted graphs (G, o) $(o \in V(G))$ up to isomorphisms.
- $[\boldsymbol{G}, \boldsymbol{o}]$: A random rooted graph.
- It is called **unimodular** if

$$\mathbb{E}\left[\sum_{x \in \boldsymbol{G}} g(\boldsymbol{G}, \boldsymbol{o}, x)\right] = \mathbb{E}\left[\sum_{x \in \boldsymbol{G}} g(\boldsymbol{G}, x, \boldsymbol{o})\right] \qquad \text{(MTP)}$$

  for all measurable functions $g(G, x, y) \geqslant 0$ (for $x, y \in V(G)$) that are isometry-invariant.
- **Example**:
  1. Every finite graph G with a uniformly-random root $\boldsymbol{o} \in V(G)$.
  2. Cayley graphs.
  3. **Example**: Any graph constructed *equivariantly* on (the Palm version of) a stationary point process.

2. Unimodular Graphs

- $\mathcal{G}_*$: The space of all rooted graphs (G, o) $(o \in V(G))$ up to isomorphisms.
- $[\boldsymbol{G}, \boldsymbol{o}]$: A random rooted graph.
- It is called **unimodular** if

$$\mathbb{E}\left[\sum_{x \in \boldsymbol{G}} g(\boldsymbol{G}, \boldsymbol{o}, x)\right] = \mathbb{E}\left[\sum_{x \in \boldsymbol{G}} g(\boldsymbol{G}, x, \boldsymbol{o})\right] \qquad \text{(MTP)}$$

  for all measurable functions $g(G, x, y) \geqslant 0$ (for $x, y \in V(G)$) that are isometry-invariant.
- **Example**:
  1. Every finite graph G with a uniformly-random root $\boldsymbol{o} \in V(G)$.
  2. Cayley graphs.
  3. **Example**: Any graph constructed *equivariantly* on (the Palm version of) a stationary point process.

2. Unimodular Graphs

- $\mathcal{G}_*$: The space of all rooted graphs (G, o) $(o \in V(G))$ up to isomorphisms.
- $[\boldsymbol{G}, \boldsymbol{o}]$: A random rooted graph.
- It is called **unimodular** if

$$\mathbb{E}\left[\sum_{x \in \boldsymbol{G}} g(\boldsymbol{G}, \boldsymbol{o}, x)\right] = \mathbb{E}\left[\sum_{x \in \boldsymbol{G}} g(\boldsymbol{G}, x, \boldsymbol{o})\right] \qquad \text{(MTP)}$$

  for all measurable functions $g(G, x, y) \geqslant 0$ (for $x, y \in V(G)$) that are isometry-invariant.
- **Example**:
  1. Every finite graph G with a uniformly-random root $\boldsymbol{o} \in V(G)$.
  2. Cayley graphs.
  3. **Example**: Any graph constructed *equivariantly* on (the Palm version of) a stationary point process.

2. Unimodular Graphs

- $\mathcal{G}_*$: The space of all rooted graphs (G, o) $(o \in V(G))$ up to isomorphisms.
- $[\boldsymbol{G}, \boldsymbol{o}]$: A random rooted graph.
- It is called **unimodular** if

$$\mathbb{E}\left[\sum_{x \in \boldsymbol{G}} g(\boldsymbol{G}, \boldsymbol{o}, x)\right] = \mathbb{E}\left[\sum_{x \in \boldsymbol{G}} g(\boldsymbol{G}, x, \boldsymbol{o})\right] \qquad \text{(MTP)}$$

  for all measurable functions $g(G, x, y) \geqslant 0$ (for $x, y \in V(G)$) that are isometry-invariant.
- **Example**:
  1. Every finite graph G with a uniformly-random root $\boldsymbol{o} \in V(G)$.
  2. Cayley graphs.
  3. **Example**: Any graph constructed *equivariantly* on (the Palm version of) a stationary point process.

3. Unimodular Discrete Spaces

- $[\boldsymbol{D}, \boldsymbol{o}]$: A random rooted discrete metric space.
  - $\boldsymbol{D}$ should be *boundedly-finite*.
- It is called **unimodular** if for all measurable functions $g(D, x, y) \geqslant 0$ (for $x, y \in D$) that are isometry-invariant,

$$\mathbb{E}\left[\sum_{x \in \boldsymbol{D}} g(\boldsymbol{D}, \boldsymbol{o}, x)\right] = \mathbb{E}\left[\sum_{x \in \boldsymbol{D}} g(\boldsymbol{D}, x, \boldsymbol{o})\right]. \qquad \text{(MTP)}$$

- (Almost-) Unification of:
  - Unimodular graphs,
  - Palm version of stationary point processes,
  - Point-stationary point processes.

3. Unimodular Discrete Spaces

- $[\boldsymbol{D}, \boldsymbol{o}]$: A random rooted discrete metric space.
  - $\boldsymbol{D}$ should be *boundedly-finite*.
- It is called **unimodular** if for all measurable functions $g(D, x, y) \geqslant 0$ (for $x, y \in D$) that are isometry-invariant,

$$\mathbb{E}\left[\sum_{x \in \boldsymbol{D}} g(\boldsymbol{D}, \boldsymbol{o}, x)\right] = \mathbb{E}\left[\sum_{x \in \boldsymbol{D}} g(\boldsymbol{D}, x, \boldsymbol{o})\right]. \qquad \text{(MTP)}$$

- (Almost-) Unification of:
  - Unimodular graphs,
  - Palm version of stationary point processes,
  - Point-stationary point processes.

4. Random Measures

- Φ: A stationary random measure on $\mathbb{R}^d$.
  - i.e., $\forall t \in \mathbb{R}^d : \Phi + t \sim \Phi$.
- **Example**: Every point process is a random measure.
- Φ_0: The **Palm version** of Φ,
  - or Φ seen from a *typical point*.
  - Heuristically:

$$\mathbb{E}\left[g(\Phi_0, 0)\right] \longleftrightarrow \int g(\Phi, x) d\Phi(x).$$

Theorem (MTP)

For all measurable functions $g(\Phi_0, x, y) \geqslant 0$ that are translation-invariant:

$$\mathbb{E}\left[\int g(\Phi_0, 0, x) d\Phi_0(x)\right] = \mathbb{E}\left[\int g(\Phi_0, x, 0) d\Phi_0(x)\right].$$

- This equation characterizes **mass-stationary** random measures.

4. Random Measures

- Φ: A stationary random measure on $\mathbb{R}^d$.
  - i.e., $\forall t \in \mathbb{R}^d : \Phi + t \sim \Phi$.
- **Example**: Every point process is a random measure.
- Φ_0: The **Palm version** of Φ,
  - or Φ seen from a *typical point*.
  - Heuristically:

$$\mathbb{E}\left[g(\Phi_0, 0)\right] \longleftrightarrow \int g(\Phi, x) d\Phi(x).$$

Theorem (MTP)

For all measurable functions $g(\Phi_0, x, y) \geqslant 0$ that are translation-invariant:

$$\mathbb{E}\left[\int g(\Phi_0, 0, x) d\Phi_0(x)\right] = \mathbb{E}\left[\int g(\Phi_0, x, 0) d\Phi_0(x)\right].$$

- This equation characterizes **mass-stationary** random measures.

4. Random Measures

- Φ: A stationary random measure on $\mathbb{R}^d$.
  - i.e., $\forall t \in \mathbb{R}^d : \Phi + t \sim \Phi$.
- **Example**: Every point process is a random measure.
- Φ_0: The **Palm version** of Φ,
  - or Φ seen from a *typical point*.
  - Heuristically:

$$\mathbb{E}\left[g(\Phi_0, 0)\right] \longleftrightarrow \int g(\Phi, x) d\Phi(x).$$

Theorem (MTP)

For all measurable functions $g(\Phi_0, x, y) \geqslant 0$ that are translation-invariant:

$$\mathbb{E}\left[\int g(\Phi_0, 0, x) d\Phi_0(x)\right] = \mathbb{E}\left[\int g(\Phi_0, x, 0) d\Phi_0(x)\right].$$

- This equation characterizes **mass-stationary** random measures.

4. Random Measures

- Φ: A stationary random measure on $\mathbb{R}^d$.
  - i.e., $\forall t \in \mathbb{R}^d : \Phi + t \sim \Phi$.
- **Example**: Every point process is a random measure.
- Φ_0: The **Palm version** of Φ,
  - or Φ seen from a *typical point*.
  - Heuristically:

$$\mathbb{E}\left[g(\Phi_0, 0)\right] \longleftrightarrow \int g(\Phi, x) d\Phi(x).$$

Theorem (MTP)

For all measurable functions $g(\Phi_0, x, y) \geqslant 0$ that are translation-invariant:

$$\mathbb{E}\left[\int g(\Phi_0, 0, x) d\Phi_0(x)\right] = \mathbb{E}\left[\int g(\Phi_0, x, 0) d\Phi_0(x)\right].$$

- This equation characterizes **mass-stationary** random measures.

5. Scaling limits

- Assume $[\boldsymbol{G}_n, \boldsymbol{o}_n, \boldsymbol{\mu}_n]$ is such that
  - $\boldsymbol{G}_n$: A finite metric space,
  - $\boldsymbol{o}_n \in G_n$ chosen uniformly at random,
  - $\boldsymbol{\mu}_n$: The counting measure on G_n.
- Assume $[\epsilon_n G_n, o_n, \delta_n \mu_n]$ converges weakly.
- Example:
  - $\mathbb{Z}^d \Rightarrow \mathbb{R}^d$.
  - Random trees $\Rightarrow$ Brownian continuum random tree.
  - Zeros of simple random walk $\Rightarrow$ Zeros of Brownian motion.
  - Cayley graph $\Rightarrow$ A locally-compact group.
- We will see that there exists an MTP for the scaling limit.

5. Scaling limits

- Assume $[\boldsymbol{G}_n, \boldsymbol{o}_n, \boldsymbol{\mu}_n]$ is such that
  - $\boldsymbol{G}_n$: A finite metric space,
  - $\boldsymbol{o}_n \in G_n$ chosen uniformly at random,
  - $\boldsymbol{\mu}_n$: The counting measure on G_n.
- Assume $[\epsilon_n G_n, o_n, \delta_n \mu_n]$ converges weakly.
- Example:
  - $\mathbb{Z}^d \Rightarrow \mathbb{R}^d$.
  - Random trees $\Rightarrow$ Brownian continuum random tree.
  - Zeros of simple random walk $\Rightarrow$ Zeros of Brownian motion.
  - Cayley graph $\Rightarrow$ A locally-compact group.
- We will see that there exists an MTP for the scaling limit.

5. Scaling limits

- Assume $[\boldsymbol{G}_n, \boldsymbol{o}_n, \boldsymbol{\mu}_n]$ is such that
  - $\boldsymbol{G}_n$: A finite metric space,
  - $\boldsymbol{o}_n \in G_n$ chosen uniformly at random,
  - $\boldsymbol{\mu}_n$: The counting measure on G_n.
- Assume $[\epsilon_n G_n, o_n, \delta_n \mu_n]$ converges weakly.
- Example:
  - $\mathbb{Z}^d \Rightarrow \mathbb{R}^d$.
  - Random trees $\Rightarrow$ Brownian continuum random tree.
  - Zeros of simple random walk $\Rightarrow$ Zeros of Brownian motion.
  - Cayley graph $\Rightarrow$ A locally-compact group.
- We will see that there exists an MTP for the scaling limit.

5. Scaling limits

- Assume $[\boldsymbol{G}_n, \boldsymbol{o}_n, \boldsymbol{\mu}_n]$ is such that
  - $\boldsymbol{G}_n$: A finite metric space,
  - $\boldsymbol{o}_n \in G_n$ chosen uniformly at random,
  - $\boldsymbol{\mu}_n$: The counting measure on G_n.
- Assume $[\epsilon_n G_n, o_n, \delta_n \mu_n]$ converges weakly.
- Example:
  - $\mathbb{Z}^d \Rightarrow \mathbb{R}^d$.
  - Random trees $\Rightarrow$ Brownian continuum random tree.
  - Zeros of simple random walk $\Rightarrow$ Zeros of Brownian motion.
  - Cayley graph $\Rightarrow$ A locally-compact group.
- We will see that there exists an MTP for the scaling limit.

5. Scaling limits

- Assume $[\boldsymbol{G}_n, \boldsymbol{o}_n, \boldsymbol{\mu}_n]$ is such that
  - $\boldsymbol{G}_n$: A finite metric space,
  - $\boldsymbol{o}_n \in G_n$ chosen uniformly at random,
  - $\boldsymbol{\mu}_n$: The counting measure on G_n.
- Assume $[\epsilon_n G_n, o_n, \delta_n \mu_n]$ converges weakly.
- Example:
  - $\mathbb{Z}^d \Rightarrow \mathbb{R}^d$.
  - Random trees $\Rightarrow$ Brownian continuum random tree.
  - Zeros of simple random walk $\Rightarrow$ Zeros of Brownian motion.
  - Cayley graph $\Rightarrow$ A locally-compact group.
- We will see that there exists an MTP for the scaling limit.

5. Scaling limits

- Assume $[\boldsymbol{G}_n, \boldsymbol{o}_n, \boldsymbol{\mu}_n]$ is such that
  - $\boldsymbol{G}_n$: A finite metric space,
  - $\boldsymbol{o}_n \in G_n$ chosen uniformly at random,
  - $\boldsymbol{\mu}_n$: The counting measure on G_n.
- Assume $[\epsilon_n G_n, o_n, \delta_n \mu_n]$ converges weakly.
- Example:
  - $\mathbb{Z}^d \Rightarrow \mathbb{R}^d$.
  - Random trees $\Rightarrow$ Brownian continuum random tree.
  - Zeros of simple random walk $\Rightarrow$ Zeros of Brownian motion.
  - Cayley graph $\Rightarrow$ A locally-compact group.
- We will see that there exists an MTP for the scaling limit.

The Goals

Our goals:

- A unification of the various versions of the MTP.
- Generalizing Palm theory in order to use for studying the dimension of scaling limits.

Outline

1. The Mass Transport Principle in Various Subjects
2. Unimodular Continumm Spaces
3. Properties of Unimodular Spaces
4. Palm Theory
5. Amenability

Random Continuum Spaces

- $\mathcal{M}_*$:= The space of all (X, o, μ) , where:
  - X is a metric space (and is *boundedly-compact*),
  - $o \in X$ (the root),
  - μ is a measure on X (and is *boundedly-finite*).
- $\mathcal{M}_*$ is a Polish space (with the GHP metric).
- A **random rmm space** (rooted measured metric space): A random element $[\boldsymbol{X}, \boldsymbol{o}, \boldsymbol{\mu}]$ in $\mathcal{M}_*$.

$$\mathbb{E}\left[f(\boldsymbol{X}, \boldsymbol{o}, \boldsymbol{\mu})\right] = \int_{\mathcal{M}_*} f([X, o, \mu]) d\mathbb{P}([X, o, \mu]).$$

Random Continuum Spaces

- $\mathcal{M}_*$:= The space of all (X, o, μ) , where:
  - X is a metric space (and is *boundedly-compact*),
  - $o \in X$ (the root),
  - μ is a measure on X (and is *boundedly-finite*).
- $\mathcal{M}_*$ is a Polish space (with the GHP metric).
- A **random rmm space** (rooted measured metric space): A random element $[\boldsymbol{X}, \boldsymbol{o}, \boldsymbol{\mu}]$ in $\mathcal{M}_*$.

$$\mathbb{E}\left[f(\boldsymbol{X}, \boldsymbol{o}, \boldsymbol{\mu})\right] = \int_{\mathcal{M}_*} f([X, o, \mu]) d\mathbb{P}([X, o, \mu]).$$

Random Continuum Spaces

- $\mathcal{M}_*$:= The space of all (X, o, μ) , where:
  - X is a metric space (and is *boundedly-compact*),
  - $o \in X$ (the root),
  - μ is a measure on X (and is *boundedly-finite*).
- $\mathcal{M}_*$ is a Polish space (with the GHP metric).
- A **random rmm space** (rooted measured metric space):
  A random element $[\boldsymbol{X}, \boldsymbol{o}, \boldsymbol{\mu}]$ in $\mathcal{M}_*$.

$$\mathbb{E}\left[f(\boldsymbol{X}, \boldsymbol{o}, \boldsymbol{\mu})\right] = \int_{\mathcal{M}_*} f([X, o, \mu]) d\mathbb{P}([X, o, \mu]).$$

Unimodular Continuum Spaces

- $\mathcal{M}_{**}$:= The space of all (X, o, p, μ).
  - $p \in X$ is called the second root.
- $[\boldsymbol{X}, \boldsymbol{o}, \boldsymbol{\mu}]$: A random rmm space.

Definition

$[\boldsymbol{X}, \boldsymbol{o}, \boldsymbol{\mu}]$ is a **unimodular random rmm space** if for all g:

$$\mathbb{E}\left[\int_{\boldsymbol{X}} g(\boldsymbol{o}, x) d\boldsymbol{\mu}(x)\right] = \mathbb{E}\left[\int_{\boldsymbol{X}} g(x, \boldsymbol{o}) d\boldsymbol{\mu}(x)\right],$$

where $g(\boldsymbol{o}, x) := g(\boldsymbol{X}, \boldsymbol{o}, x, \boldsymbol{\mu})$ and $g : \mathcal{M}_{**} \to \mathbb{R}^{\geqslant 0}$ is measurable.

$$\mathbb{E}\left[g^{+}(\boldsymbol{o})\right] = \mathbb{E}\left[g^{-}(\boldsymbol{o})\right]$$

Unimodular Continuum Spaces

- $\mathcal{M}_{**}$:= The space of all (X, o, p, μ).
  - $p \in X$ is called the second root.
- $[\boldsymbol{X}, \boldsymbol{o}, \boldsymbol{\mu}]$: A random rmm space.

Definition

$[\boldsymbol{X}, \boldsymbol{o}, \boldsymbol{\mu}]$ is a **unimodular random rmm space** if for all g:

$$\mathbb{E}\left[\int_{\boldsymbol{X}} g(\boldsymbol{o}, x) d\boldsymbol{\mu}(x)\right] = \mathbb{E}\left[\int_{\boldsymbol{X}} g(x, \boldsymbol{o}) d\boldsymbol{\mu}(x)\right],$$

where $g(\boldsymbol{o}, x) := g(\boldsymbol{X}, \boldsymbol{o}, x, \boldsymbol{\mu})$ and $g : \mathcal{M}_{**} \to \mathbb{R}^{\geq 0}$ is measurable.

$$\mathbb{E}\left[g^{+}(\boldsymbol{o})\right] = \mathbb{E}\left[g^{-}(\boldsymbol{o})\right]$$

Trivial Examples

- When $\boldsymbol{\mu} = 0$.
- When $\boldsymbol{\mu} = \delta_{\boldsymbol{o}}$.
- Compact spaces:
  - $[\boldsymbol{X}, \boldsymbol{\mu}]$: Any random compact measured metric space,
  - $\boldsymbol{o} \in \boldsymbol{X}$ random with distribution proportional to $\boldsymbol{\mu}$,
  - Then $[\boldsymbol{X}, \boldsymbol{o}, \boldsymbol{\mu}]$ is unimodular.
- Compact unimodular spaces are re-rooting invariant.
- In general, heuristically, the root is a *typical point*:

$$\mathbb{E}\left[h(\boldsymbol{o})\right] \longleftrightarrow \int_{\boldsymbol{X}} h(y)\boldsymbol{\mu}(dy).$$

Trivial Examples

- When $\boldsymbol{\mu} = 0$.
- When $\boldsymbol{\mu} = \delta_{\boldsymbol{o}}$.
- Compact spaces:
  - $[\boldsymbol{X}, \boldsymbol{\mu}]$: Any random compact measured metric space,
  - $\boldsymbol{o} \in \boldsymbol{X}$ random with distribution proportional to $\boldsymbol{\mu}$,
  - Then $[\boldsymbol{X}, \boldsymbol{o}, \boldsymbol{\mu}]$ is unimodular.
- Compact unimodular spaces are re-rooting invariant.
- In general, heuristically, the root is a *typical point*:

$$\mathbb{E}\left[h(\boldsymbol{o})\right] \longleftrightarrow \int_{\boldsymbol{X}} h(y)\boldsymbol{\mu}(dy).$$

Examples

- (Palm version of) Stationary point processes,
  - $[\Phi_0, 0, \mathrm{counting}(\Phi_0)]$.
  - $[\mathbb{R}^d, 0, \mathrm{counting}(\Phi_0)]$. ($\rightarrow$ no need to have $\mathrm{supp}(\boldsymbol{\mu}) = \boldsymbol{X}$)
- Point-stationary point processes,
- Unimodular random graphs,
- Unimodular discrete spaces,
- (Palm version of) Stationary random measures,
- Mass-stationary random measures.
- Unimodular random manifolds (Abért and Biringer, 22).

Examples

- (Palm version of) Stationary point processes,
  - $[\Phi_0, 0, \text{counting}(\Phi_0)]$.
  - $[\mathbb{R}^d, 0, \text{counting}(\Phi_0)]$. ($\rightarrow$ no need to have $\text{supp}(\boldsymbol{\mu}) = \boldsymbol{X}$)
- Point-stationary point processes,
- Unimodular random graphs,
- Unimodular discrete spaces,
- (Palm version of) Stationary random measures,
- Mass-stationary random measures.
- Unimodular random manifolds (Abért and Biringer, 22).

Examples: Weak Limits

Lemma

Any weak limit of a sequence of unimodular spaces is unimodular.

Corollary

Scaling limits are unimodular (under the assumptions already mentioned).

Corollary

All compact scaling limits have the re-rooting invariance property:
If $\boldsymbol{o}' \in \boldsymbol{X}$ is random with distribution proportional to μ, then $[\boldsymbol{X}, \boldsymbol{o}', \mu] \sim [\boldsymbol{X}, \boldsymbol{o}, \mu]$.

Examples: Weak Limits

Lemma

Any weak limit of a sequence of unimodular spaces is unimodular.

Corollary

Scaling limits are unimodular (under the assumptions already mentioned).

Corollary

All compact scaling limits have the re-rooting invariance property: If $\boldsymbol{o}' \in \boldsymbol{X}$ is random with distribution proportional to μ, then $[\boldsymbol{X}, \boldsymbol{o}', \mu] \sim [\boldsymbol{X}, \boldsymbol{o}, \mu]$.

Examples: Weak Limits

Lemma

Any weak limit of a sequence of unimodular spaces is unimodular.

Corollary

Scaling limits are unimodular (under the assumptions already mentioned).

Corollary

All compact scaling limits have the re-rooting invariance property*:*
If $\boldsymbol{o}' \in \boldsymbol{X}$ *is random with distribution proportional to* $\boldsymbol{\mu}$*, then*
$[\boldsymbol{X}, \boldsymbol{o}', \boldsymbol{\mu}] \sim [\boldsymbol{X}, \boldsymbol{o}, \boldsymbol{\mu}]$.

Examples: Symmetric Spaces

Some symmetric spaces are unimodular:

- $[\mathbb{R}^d, 0, \mathrm{Leb}]$.
- Every *unimodular* topological group (i.e., when the left and right Haar measures are equal).
- $[\mathbb{H}^d, o, \mathrm{vol}]$.
- Every symmetric metric space (or manifold) with a unimodular symmetry group (e.g., $\mathbb{H}^n$ or $\mathbb{S}^n$),
  - or having an action of a unimodular group that is transitive and measure preserving.

Examples: Symmetric Spaces

Some symmetric spaces are unimodular:

- $[\mathbb{R}^d, 0, \mathrm{Leb}]$.
- Every *unimodular* topological group (i.e., when the left and right Haar measures are equal).
- $[\mathbb{H}^d, o, \mathrm{vol}]$.
- Every symmetric metric space (or manifold) with a unimodular symmetry group (e.g., $\mathbb{H}^n$ or $\mathbb{S}^n$),
    - or having an action of a unimodular group that is transitive and measure preserving.

Example: Deterministic Spaces

- (X, μ): deterministic.
- When can we find a random $\boldsymbol{o} \in X$ s.th. $[X, \boldsymbol{o}, \mu]$ is unimodular?
- **Example:** Quasi-transitive graphs.
- **Theorem:** ... (a necessary and sufficient condition in terms of $\mathrm{Aut}(X, \mu)$).
- **Example:** A horoball.

Example: Deterministic Spaces

- (X, μ): deterministic.
- When can we find a random $\boldsymbol{o} \in X$ s.th. $[X, \boldsymbol{o}, \mu]$ is unimodular?
- **Example:** Quasi-transitive graphs.
- **Theorem:** ... (a necessary and sufficient condition in terms of $\mathrm{Aut}(X, \mu)$).
- **Example:** A horoball.

Outline

1. The Mass Transport Principle in Various Subjects
2. Unimodular Continumm Spaces
3. Properties of Unimodular Spaces
4. Palm Theory
5. Amenability

Subset Selection

Two equivalent definitions:

1. If $A \subseteq \mathcal{M}_*$ is measurable, then $S := S(X, \mu) := \{y \in X : (X, y, \mu) \in A\}$ is called a **factor subset**.
2. A **factor subset** is a map $(X, \mu) \mapsto S(X, \mu) \subseteq X$ such that it is isometry-equivariant and $A := \{(X, y, \mu) : y \in S(X, \mu)\}$ is measurable.

Lemma (Everything Happens at the Root)

If $[\boldsymbol{X}, \boldsymbol{o}, \boldsymbol{\mu}]$ is unimodular and S is a factor subset, then:

$$\boldsymbol{o} \in S \text{ a.s.} \quad \Longleftrightarrow \quad \boldsymbol{\mu}(\boldsymbol{X} \setminus S) = 0 \text{ a.s.},$$
$$\mathbb{P}[\boldsymbol{\mu}(S) > 0] > 0 \quad \Longleftrightarrow \quad \mathbb{P}[\boldsymbol{o} \in S] > 0.$$

Corollary

$\boldsymbol{o} \in \mathrm{supp}(\boldsymbol{\mu})$ *a.s.*

Subset Selection

Two equivalent definitions:

1. If $A \subseteq \mathcal{M}_*$ is measurable, then $S := S(X, \mu) := \{y \in X : (X, y, \mu) \in A\}$ is called a **factor subset**.
2. A **factor subset** is a map $(X, \mu) \mapsto S(X, \mu) \subseteq X$ such that it is isometry-equivariant and $A := \{(X, y, \mu) : y \in S(X, \mu)\}$ is measurable.

Lemma (Everything Happens at the Root)

If $[\boldsymbol{X}, \boldsymbol{o}, \boldsymbol{\mu}]$ is unimodular and S is a factor subset, then:

$$\begin{aligned} \boldsymbol{o} \in S \text{ a.s.} \quad &\Longleftrightarrow \quad \boldsymbol{\mu}(X \setminus S) = 0 \text{ a.s.} \ , \\ \mathbb{P}\left[\boldsymbol{\mu}(S) > 0\right] > 0 \quad &\Longleftrightarrow \quad \mathbb{P}\left[\boldsymbol{o} \in S\right] > 0. \end{aligned}$$

Corollary

$\boldsymbol{o} \in \mathrm{supp}(\boldsymbol{\mu})$ *a.s.*

Root-Change

- $[\boldsymbol{X}, \boldsymbol{o}, \boldsymbol{\mu}]$: unimodular
- Assume for each (X, o, μ), a probability measure $\alpha = \alpha_{(X,o,\mu)}$ on X is given (isometry-equivariant with some measurability property).
- Let $\boldsymbol{o}' \in \boldsymbol{X}$ be chosen with distribution α.

Lemma

1. *$[\boldsymbol{X}, \boldsymbol{o}', \boldsymbol{\mu}] \sim [\boldsymbol{X}, \boldsymbol{o}, \boldsymbol{\mu}]$ if $\boldsymbol{\mu}$ is a stationary measure for the Markovian kernel on $\boldsymbol{X}$.*
2. *This holds if $f(o, x)$ is the density of α w.r.t. μ at x and $f^-(\boldsymbol{o}) = 1$ a.s., where $f^-(\boldsymbol{o}) := \int_{\boldsymbol{X}} f(y, \boldsymbol{o})\boldsymbol{\mu}(dy)$.*
3. *If $f(o, x)$ is the density of α w.r.t. μ at x (if exists), then the density of $[\boldsymbol{X}, \boldsymbol{o}', \boldsymbol{\mu}]$ w.r.t. $[\boldsymbol{X}, \boldsymbol{o}, \boldsymbol{\mu}]$ is $f^-(\boldsymbol{o})$.*

- This generalizes Mecke's theorem (invariance under bijective point-shifts).

Root-Change

- $[\boldsymbol{X}, \boldsymbol{o}, \boldsymbol{\mu}]$: unimodular
- Assume for each (X, o, μ), a probability measure $\alpha = \alpha_{(X,o,\mu)}$ on X is given (isometry-equivariant with some measurable property).
- Let $\boldsymbol{o}' \in \boldsymbol{X}$ be chosen with distribution α.

Lemma

1. *$[\boldsymbol{X}, \boldsymbol{o}', \boldsymbol{\mu}] \sim [\boldsymbol{X}, \boldsymbol{o}, \boldsymbol{\mu}]$ if $\boldsymbol{\mu}$ is a stationary measure for the Markovian kernel on $\boldsymbol{X}$.*
2. *This holds if $f(o, x)$ is the density of α w.r.t. μ at x and $f^-(\boldsymbol{o}) = 1$ a.s., where $f^-(\boldsymbol{o}) := \int_{\boldsymbol{X}} f(y, \boldsymbol{o})\boldsymbol{\mu}(dy)$.*
3. *If $f(o, x)$ is the density of α w.r.t. μ at x (if exists), then the density of $[\boldsymbol{X}, \boldsymbol{o}', \boldsymbol{\mu}]$ w.r.t. $[\boldsymbol{X}, \boldsymbol{o}, \boldsymbol{\mu}]$ is $f^-(\boldsymbol{o})$.*

- This generalizes Mecke's theorem (invariance under bijective point-shifts).

Random Walk

- Fix h_0 such that $h_0^+(\cdot) = 1$ and $h > 0$.

$$h(x, y) := \int_X \frac{h_0(x, z) h_0(y, z)}{h_0^-(z)} d\mu(z).$$

- So, $h^+(\cdot) = h^-(\cdot) = 1$.
- Let $[\boldsymbol{X}, \boldsymbol{o}, \boldsymbol{\mu}]$ be random.
- Define a random walk $(\boldsymbol{x}_n)_n$ on $\boldsymbol{X}$ such that $\boldsymbol{x}_0 = \boldsymbol{o}$ and $\boldsymbol{x}_{n+1} \sim h(\boldsymbol{x}_n, \cdot)\boldsymbol{\mu}$.

Theorem

$[\boldsymbol{X}, \boldsymbol{o}, \boldsymbol{\mu}]$ is unimodular if and only if $(\boldsymbol{x}_n)_n$ is stationary and reversible; i.e.,

$$[\boldsymbol{X}, \boldsymbol{x}_1, \boldsymbol{\mu}, (\boldsymbol{x}_{n+1})_n] \sim [\boldsymbol{X}, \boldsymbol{o}, \boldsymbol{\mu}, (\boldsymbol{x}_n)],$$
$$[\boldsymbol{X}, \boldsymbol{o}, \boldsymbol{\mu}, (\boldsymbol{x}_{-n})_n] \sim [\boldsymbol{X}, \boldsymbol{o}, \boldsymbol{\mu}, (\boldsymbol{x}_n)].$$

Random Walk

- Fix h_0 such that $h_0^+(\cdot) = 1$ and $h > 0$.

$$h(x,y) := \int_X \frac{h_0(x,z)h_0(y,z)}{h_0^-(z)} d\mu(z).$$

- So, $h^+(\cdot) = h^-(\cdot) = 1$.
- Let $[\boldsymbol{X}, \boldsymbol{o}, \boldsymbol{\mu}]$ be random.
- Define a random walk $(\boldsymbol{x}_n)_n$ on $\boldsymbol{X}$ such that $\boldsymbol{x}_0 = \boldsymbol{o}$ and $\boldsymbol{x}_{n+1} \sim h(\boldsymbol{x}_n, \cdot)\boldsymbol{\mu}$.

Theorem

$[\boldsymbol{X}, \boldsymbol{o}, \boldsymbol{\mu}]$ is unimodular if and only if $(\boldsymbol{x}_n)_n$ is stationary and reversible; i.e.,

$$[\boldsymbol{X}, \boldsymbol{x}_1, \boldsymbol{\mu}, (\boldsymbol{x}_{n+1})_n] \sim [\boldsymbol{X}, \boldsymbol{o}, \boldsymbol{\mu}, (\boldsymbol{x}_n)],$$
$$[\boldsymbol{X}, \boldsymbol{o}, \boldsymbol{\mu}, (\boldsymbol{x}_{-n})_n] \sim [\boldsymbol{X}, \boldsymbol{o}, \boldsymbol{\mu}, (\boldsymbol{x}_n)].$$

Random Walk

- Fix h_0 such that $h_0^+(\cdot) = 1$ and $h > 0$.

$$h(x, y) := \int_X \frac{h_0(x, z) h_0(y, z)}{h_0^-(z)} d\mu(z).$$

- So, $h^+(\cdot) = h^-(\cdot) = 1$.
- Let $[\boldsymbol{X}, \boldsymbol{o}, \boldsymbol{\mu}]$ be random.
- Define a random walk $(\boldsymbol{x}_n)_n$ on $\boldsymbol{X}$ such that $\boldsymbol{x}_0 = \boldsymbol{o}$ and $\boldsymbol{x}_{n+1} \sim h(\boldsymbol{x}_n, \cdot)\boldsymbol{\mu}$.

Theorem

$[\boldsymbol{X}, \boldsymbol{o}, \boldsymbol{\mu}]$ *is unimodular if and only if* $(\boldsymbol{x}_n)_n$ *is stationary and reversible; i.e.,*

$$[\boldsymbol{X}, \boldsymbol{x}_1, \boldsymbol{\mu}, (\boldsymbol{x}_{n+1})_n] \sim [\boldsymbol{X}, \boldsymbol{o}, \boldsymbol{\mu}, (\boldsymbol{x}_n)],$$
$$[\boldsymbol{X}, \boldsymbol{o}, \boldsymbol{\mu}, (\boldsymbol{x}_{-n})_n] \sim [\boldsymbol{X}, \boldsymbol{o}, \boldsymbol{\mu}, (\boldsymbol{x}_n)].$$

Ergodicity

- An event A is **(root-change-) invariant** if $[X, o, \mu] \in A \Rightarrow [X, y, \mu] \in A, \forall y \in X$.
- **Definition:** A unimodular rmm space $[\boldsymbol{X}, \boldsymbol{o}, \boldsymbol{\mu}]$ is ergodic when $\mathbb{P}[A] \in \{0, 1\}$ for every invariant event A.

Theorem (Ergodic Decomposition)

(i) *$[\boldsymbol{X}, \boldsymbol{o}, \boldsymbol{\mu}]$ is ergodic if and only if the random walk $(\boldsymbol{x}_n)$ is ergodic.*

(ii) *Every unimodular probability measure can be uniquely written as a mixture of ergodic probability measures.*

Ergodicity

- An event A is **(root-change-) invariant** if $[X, o, \mu] \in A \Rightarrow [X, y, \mu] \in A, \forall y \in X$.
- **Definition:** A unimodular rmm space $[\boldsymbol{X}, \boldsymbol{o}, \boldsymbol{\mu}]$ is ergodic when $\mathbb{P}[A] \in \{0, 1\}$ for every invariant event A.

Theorem (Ergodic Decomposition)

1. *$[\boldsymbol{X}, \boldsymbol{o}, \boldsymbol{\mu}]$ is ergodic if and only if the random walk $(\boldsymbol{x}_n)$ is ergodic.*
2. *Every unimodular probability measure can be uniquely written as a mixture of ergodic probability measures.*

Ergodicity

- An event A is **(root-change-) invariant** if $[X, o, \mu] \in A \Rightarrow [X, y, \mu] \in A, \forall y \in X$.
- **Definition:** A unimodular rmm space $[\boldsymbol{X}, \boldsymbol{o}, \boldsymbol{\mu}]$ is ergodic when $\mathbb{P}[A] \in \{0, 1\}$ for every invariant event A.

Theorem (Ergodic Decomposition)

(i) *$[\boldsymbol{X}, \boldsymbol{o}, \boldsymbol{\mu}]$ is ergodic if and only if the random walk $(\boldsymbol{x}_n)$ is ergodic.*

(ii) *Every unimodular probability measure can be uniquely written as a mixture of ergodic probability measures.*

Outline

1. The Mass Transport Principle in Various Subjects
2. Unimodular Continumm Spaces
3. Properties of Unimodular Spaces
4. Palm Theory
5. Amenability

Random Measures

- $[\boldsymbol{X}, \boldsymbol{o}, \boldsymbol{\mu}, \Phi]$ is a random element in $\mathcal{M}_*^2$, where

$$\mathcal{M}_*^2 := \{(X, o, \mu, \varphi) : \varphi \text{ is a measure on } X\}.$$

- Assume $[\boldsymbol{X}, \boldsymbol{o}, \boldsymbol{\mu}, \Phi]$ is unimodular; i.e., the MTP holds when g depends on Φ as well.

Theorem (Invariant Distintegration)

The conditional distribution given $[\boldsymbol{X}, \boldsymbol{o}, \boldsymbol{\mu}]$ has a version which does not depend on $\boldsymbol{o}$.

Non-dependence on the root $\leftrightarrow$ stationarity.

Definition

We say that Φ is an **equivariant random measure on** $[\boldsymbol{X}, \boldsymbol{o}, \boldsymbol{\mu}]$.

Random Measures

- $[\boldsymbol{X}, \boldsymbol{o}, \boldsymbol{\mu}, \Phi]$ is a random element in $\mathcal{M}_*^2$, where

$$\mathcal{M}_*^2 := \{(X, o, \mu, \varphi) : \varphi \text{ is a measure on } X\}.$$

- Assume $[\boldsymbol{X}, \boldsymbol{o}, \boldsymbol{\mu}, \Phi]$ is unimodular; i.e., the MTP holds when g depends on Φ as well.

Theorem (Invariant Distintegration)

The conditional distribution given $[\boldsymbol{X}, \boldsymbol{o}, \boldsymbol{\mu}]$ has a version which does not depend on $\boldsymbol{o}$.

Non-dependence on the root $\leftrightarrow$ stationarity.

Definition

We say that Φ is an **equivariant random measure on** $[\boldsymbol{X}, \boldsymbol{o}, \boldsymbol{\mu}]$.

Random Measures

- $[\boldsymbol{X}, \boldsymbol{o}, \boldsymbol{\mu}, \Phi]$ is a random element in $\mathcal{M}_*^2$, where

$$\mathcal{M}_*^2 := \{(X, o, \mu, \varphi) : \varphi \text{ is a measure on } X\}.$$

- Assume $[\boldsymbol{X}, \boldsymbol{o}, \boldsymbol{\mu}, \Phi]$ is unimodular; i.e., the MTP holds when g depends on Φ as well.

Theorem (Invariant Distintegration)

The conditional distribution given $[\boldsymbol{X}, \boldsymbol{o}, \boldsymbol{\mu}]$ has a version which does not depend on $\boldsymbol{o}$.

Non-dependence on the root $\leftrightarrow$ stationarity.

Definition

We say that Φ is an **equivariant random measure on** $[\boldsymbol{X}, \boldsymbol{o}, \boldsymbol{\mu}]$.

Random Measures

- $[\boldsymbol{X}, \boldsymbol{o}, \boldsymbol{\mu}, \Phi]$ is a random element in $\mathcal{M}^2_*$, where

$$\mathcal{M}^2_* := \{(X, o, \mu, \varphi) : \varphi \text{ is a measure on } X\}.$$

- Assume $[\boldsymbol{X}, \boldsymbol{o}, \boldsymbol{\mu}, \Phi]$ is unimodular; i.e., the MTP holds when g depends on Φ as well.

Theorem (Invariant Distintegration)

The conditional distribution given $[\boldsymbol{X}, \boldsymbol{o}, \boldsymbol{\mu}]$ has a version which does not depend on $\boldsymbol{o}$.

Non-dependence on the root $\leftrightarrow$ stationarity.

Definition

We say that Φ is an **equivariant random measure on** $[\boldsymbol{X}, \boldsymbol{o}, \boldsymbol{\mu}]$.

Examples

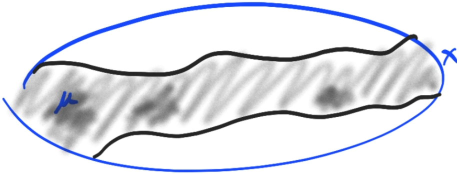
1. $\Phi = \mu$ or any factor of (X, μ).
2. The **intensity measure** of Φ is also an equivariant (factor) measure: $\lambda(X, \mu) := \mathbb{E}\left[\Phi(X, \mu)\right]$.
3. $\Phi :=$ the **Poisson point process** with intensity measure μ.

Examples

1. $\Phi = \mu$ or any factor of (X, μ).
2. The **intensity measure** of Φ is also an equivariant (factor) measure: $\lambda(X, \mu) := \mathbb{E}\left[\Phi(X, \mu)\right]$.
3. $\Phi :=$ the **Poisson point process** with intensity measure μ.

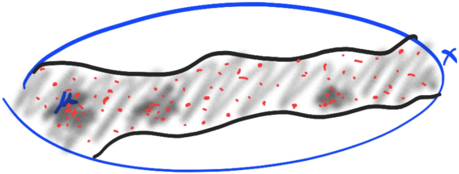
Examples

1. $\Phi = \mu$ or any factor of (X, μ).
2. The **intensity measure** of Φ is also an equivariant (factor) measure: $\lambda(X, \mu) := \mathbb{E}\left[\Phi(X, \mu)\right]$.
3. $\Phi :=$ the **Poisson point process** with intensity measure μ.



Examples

1. $\Phi = \mu$ or any factor of (X, μ).
2. The **intensity measure** of Φ is also an equivariant (factor) measure: $\lambda(X, \mu) := \mathbb{E}\left[\Phi(X, \mu)\right]$.
3. $\Phi :=$ the **Poisson point process** with intensity measure μ.



Towards Palm

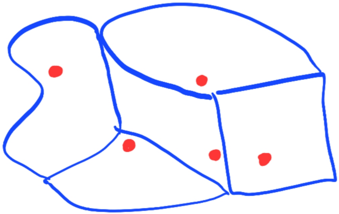
- There is no $[0,1]^d$ here!
  The classical definition does not generalize.
- Other definitions of Palm:
  1. via the Campbell measure.
  2. via a tessellation.

Towards Palm

- There is no $[0,1]^d$ here!
  The classical definition does not generalize.
- Other definitions of Palm:
  1. via the Campbell measure.
  2. via a tessellation.

Palm Via Tessellation

- Φ: A stationary point process in $\mathbb{R}^d$.
- **Equivariant tessellation**: Assigning a cell to each point of Φ equivariantly.



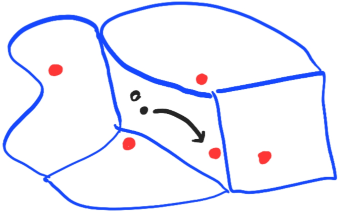
- **Fair tessellation**: When all cells have equal volumes.

Theorem

If the cell of $\boldsymbol{x} \in \Phi$ contains 0, then $\Phi - \boldsymbol{x} \sim \Phi_0$.

Palm Via Tessellation

- Φ: A stationary point process in $\mathbb{R}^d$.
- **Equivariant tessellation**: Assigning a cell to each point of Φ equivariantly.



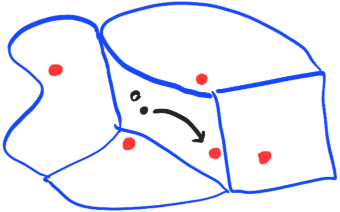
- **Fair tessellation**: When all cells have equal volumes.

Theorem

If the cell of $\boldsymbol{x} \in \Phi$ contains 0, then $\Phi - \boldsymbol{x} \sim \Phi_0$.

Palm Via Tessellation

- Φ: A stationary point process in $\mathbb{R}^d$.
- **Equivariant tessellation**: Assigning a cell to each point of Φ equivariantly.



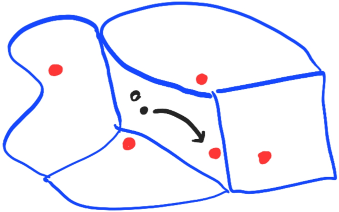
- **Fair tessellation**: When all cells have equal volumes.

Theorem

If the cell of $\boldsymbol{x} \in \Phi$ contains 0, then $\Phi - \boldsymbol{x} \sim \Phi_0$.

Palm Via Tessellation

- Φ: A stationary point process in $\mathbb{R}^d$.
- **Equivariant tessellation**: Assigning a cell to each point of Φ equivariantly.

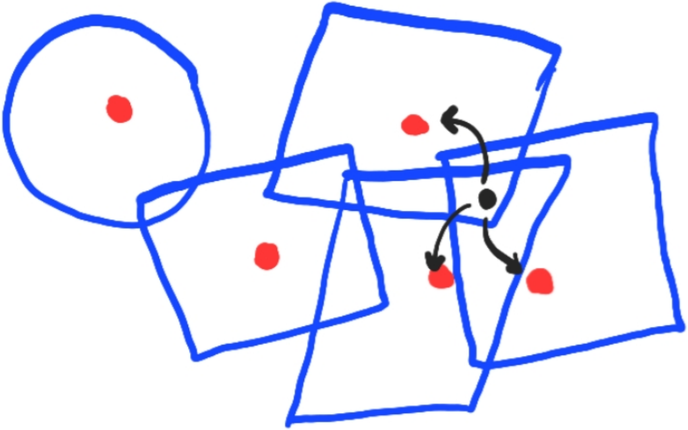


- **Fair tessellation**: When all cells have equal volumes.

Theorem

If the cell of $\boldsymbol{x} \in \Phi$ contains 0, then $\Phi - \boldsymbol{x} \sim \Phi_0$.

A Generalization



A Generalization

- Assume a function $h(x,y) \geqslant 0$ is a function depending on Φ (as a factor of Φ) such that

$$\forall y \in \Phi : h^{-}(y) := \int h(x,y)dx = 1.$$

- Example: Given a fair tessellation, let $h(x,y) := \lambda$ if $x \in \mathrm{cell}(y)$.

Theorem

Palm of Φ is obtained by a biasing and shifting the origin to a point of Φ chosen with distribution proportional to $h(0,\cdot)$; i.e.,

$$\mathbb{P}\left[\Phi_0 \in A\right] = \frac{1}{\lambda}\mathbb{E}\left[\sum_{y\in\Phi} 1_A(\Phi - y)h(0,y)\right],$$

where λ is the intensity of Φ.

A Generalization

- Assume a function $h(x, y) \geqslant 0$ is a function depending on Φ (as a factor of Φ) such that

$$\forall y \in \Phi : h^{-}(y) := \int h(x, y) dx = 1.$$

- Example: Given a fair tessellation, let $h(x, y) := \lambda$ if $x \in \mathrm{cell}(y)$.

Theorem

Palm of Φ is obtained by a biasing and shifting the origin to a point of Φ chosen with distribution proportional to $h(0, \cdot)$; i.e.,

$$\mathbb{P}[\Phi_0 \in A] = \frac{1}{\lambda}\mathbb{E}\left[\sum_{y \in \Phi} 1_A(\Phi - y) h(0, y)\right],$$

where λ is the intensity of Φ.

Palm on Unimodular Spaces

- Assume $h : \mathcal{M}^2_{**} \to \mathbb{R}^{\geqslant 0}$ is such that for all (X, y, μ, φ),

$$h^-(y) := \int_X h(x, y) d\mu(x) = 1 \quad (\text{if } \mu \neq 0).$$

- Bias and choose a new root $\sim h(\boldsymbol{o}, \cdot)\Phi$; i.e.,

Definition

Define a measure Q on $\mathcal{M}^2_*$ by:

$$Q(A) := \mathbb{E}\left[\int_{\boldsymbol{X}} 1_A(\boldsymbol{X}, y, \boldsymbol{\mu}, \Phi) h(\boldsymbol{o}, y) d\Phi(y)\right].$$

Define the **intensity** of Φ by $\lambda := |Q| = Q(\mathcal{M}^2_*)$.
Define the probability measure $\mathbb{P}_0 := \frac{1}{\lambda} Q$ (if $0 < \lambda < \infty$).
$\mathbb{P}_0$ is the distribution of the **Palm version**.

Palm on Unimodular Spaces

- Assume $h : \mathcal{M}^2_{**} \to \mathbb{R}^{\geqslant 0}$ is such that for all (X, y, μ, φ),

$$h^-(y) := \int_X h(x,y) d\mu(x) = 1 \quad (\text{if } \mu \neq 0).$$

- Bias and choose a new root $\sim h(\boldsymbol{o}, \cdot)\Phi$; i.e.,

Definition

Define a measure Q on $\mathcal{M}^2_*$ by:

$$Q(A) := \mathbb{E}\left[\int_{\boldsymbol{X}} 1_A(\boldsymbol{X}, y, \boldsymbol{\mu}, \Phi) h(\boldsymbol{o}, y) d\Phi(y)\right].$$

Define the **intensity** of Φ by $\lambda := |Q| = Q(\mathcal{M}^2_*)$.
Define the probability measure $\mathbb{P}_0 := \frac{1}{\lambda} Q$ (if $0 < \lambda < \infty$).
$\mathbb{P}_0$ is the distribution of the **Palm version**.

Palm Via the Campbell Measure

Theorem (Campbell Formula)

For all measurable functions $g \geqslant 0$ *on* $\mathcal{M}^2_{**}$, *by denoting* $g(x,y) := g(\boldsymbol{X}, x, y, \boldsymbol{\mu}, \Phi)$,

$$\mathbb{E}\left[\int_{\boldsymbol{X}} g(\boldsymbol{o}, y) d\Phi(y)\right] = \lambda \mathbb{E}_0 \left[\int_{\boldsymbol{X}} g(x, \boldsymbol{o}) d\boldsymbol{\mu}(x)\right].$$

In addition, $\mathbb{P}_0$ *is the unique probability measure on* $\mathcal{M}^2_{*}$ *with this property.*

- **Corollary**. Palm does not depend on the choice of h.

Unimodularity of Palm

- $[\boldsymbol{X}, \boldsymbol{o}, \boldsymbol{\mu}, \Phi]$ unimodular.

Lemma

Under $\mathbb{P}_0$, $[\boldsymbol{X}, \boldsymbol{o}, \Phi]$ is unimodular, and so is $[\boldsymbol{X}, \boldsymbol{o}, \Phi, \boldsymbol{\mu}]$.

Corollary

Under $\mathbb{P}_0$, the Palm of $\boldsymbol{\mu}$ (as random measure on $[\boldsymbol{X}, \boldsymbol{o}, \Phi]$) is $\mathbb{P}$.

Palm inversion = Palm

Unimodularity of Palm

- $[\boldsymbol{X}, \boldsymbol{o}, \boldsymbol{\mu}, \Phi]$ unimodular.

Lemma

Under $\mathbb{P}_0$, $[\boldsymbol{X}, \boldsymbol{o}, \Phi]$ is unimodular, and so is $[\boldsymbol{X}, \boldsymbol{o}, \Phi, \boldsymbol{\mu}]$.

Corollary

Under $\mathbb{P}_0$, the Palm of $\boldsymbol{\mu}$ (as random measure on $[\boldsymbol{X}, \boldsymbol{o}, \Phi]$) is $\mathbb{P}$.

Palm inversion = Palm

Unimodularity of Palm

- $[\boldsymbol{X}, \boldsymbol{o}, \boldsymbol{\mu}, \Phi]$ unimodular.

Lemma

Under $\mathbb{P}_0$, $[\boldsymbol{X}, \boldsymbol{o}, \Phi]$ is unimodular, and so is $[\boldsymbol{X}, \boldsymbol{o}, \Phi, \boldsymbol{\mu}]$.

Corollary

Under $\mathbb{P}_0$, the Palm of $\boldsymbol{\mu}$ (as random measure on $[\boldsymbol{X}, \boldsymbol{o}, \Phi]$) is $\mathbb{P}$.

Palm inversion = Palm

Examples

- If $\Phi = \boldsymbol{\mu}|_S$, where S is a factor subset,
  - Palm $=$ conditioning on $\boldsymbol{o} \in S$.
- If Φ is the Poisson point process with intensity measure $c\boldsymbol{\mu}$,
  - Palm version is $\Phi \cup \{\boldsymbol{o}\}$.
- Planar Duals:
  - $[\boldsymbol{G}, \boldsymbol{o}]$: a unimodular planar graph.
  - To make the dual $\boldsymbol{G}'$ of $\boldsymbol{G}$ unimodular:
    - $\boldsymbol{X} := \boldsymbol{G} \cup \boldsymbol{G}'$,
    - $\boldsymbol{\mu} :=$ the counting measure of $\boldsymbol{G}$,
    - $\Phi :=$ the counting measure of $\boldsymbol{G}'$,
    - it is enough to consider the Palm of Φ.
- Adding vertices and edges to a unimodular graph (unimodular extension) is an instance of Palm.

Outline

1. The Mass Transport Principle in Various Subjects
2. Unimodular Continumm Spaces
3. Properties of Unimodular Spaces
4. Palm Theory
5. Amenability

Amenability

- Let $[\boldsymbol{X}, \boldsymbol{o}, \boldsymbol{\mu}]$ be unimodular.

Theorem (Amenability)

The following are equivalent:

1. *There exists a local mean.*
2. *There exists an approximate mean.*
3. *Hyperfiniteness.*
4. *Folner condition.*

Mean

- To (almost) every (X, o, μ), assign a map $m : L^\infty(X, \mu) \to \mathbb{R}$ such that:
  - m is a positive linear functional.
  - m is isomorphism-invariant.
  - $\forall y \in X : m_{(X,o,\mu)} = m_{(X,y,\mu)}$.
  - Some measurability condition.
- **Definition:** This is called a **Local mean**.
- To (almost) every (X, o, μ), assign a sequence $\lambda_n : X \to \mathbb{R}^{\geqslant 0}$ such that:
  - λ_n is isomorphism-invariant and measurable.
  - $\forall y \in \boldsymbol{X} : \int_{\boldsymbol{X}} \lambda_n(y, \cdot) d\boldsymbol{\mu} = 1$ a.s.
  - $\forall y \in \boldsymbol{X} : ||\lambda_n(\boldsymbol{o}, \cdot) - \lambda_n(y, \cdot)||_1 \to 0$ a.s.
- **Definition:** This is called an **approximate mean**.

Mean

- To (almost) every (X, o, μ), assign a map $m : L^\infty(X, \mu) \to \mathbb{R}$ such that:
  - m is a positive linear functional.
  - m is isomorphism-invariant.
  - $\forall y \in X : m_{(X,o,\mu)} = m_{(X,y,\mu)}$.
  - Some measurability condition.
- **Definition:** This is called a **Local mean**.
- To (almost) every (X, o, μ), assign a sequence $\lambda_n : X \to \mathbb{R}^{\geqslant 0}$ such that:
  - λ_n is isomorphism-invariant and measurable.
  - $\forall y \in \boldsymbol{X} : \int_{\boldsymbol{X}} \lambda_n(y, \cdot) d\boldsymbol{\mu} = 1$ a.s.
  - $\forall y \in \boldsymbol{X} : ||\lambda_n(\boldsymbol{o}, \cdot) - \lambda_n(y, \cdot)||_1 \to 0$ a.s.
- **Definition:** This is called an **approximate mean**.

Hyperfiniteness

- To (almost) every (X, o, μ), assign a partition Π of X such that it is invariant, measurable, and every element of Π has finite mass w.r.t. μ.
- Allow Π to be random; e.g., depending on a random measure on (X, o, μ).
- **Definition:** This is called an **equivariant finite partition**.

Definition (Hyperfiniteness)

Three definitions:

1. $\exists$ nested equivariant finite partitions Π_n s.th. $\mathbb{P}\left[\bigcup_n \Pi_n(\boldsymbol{o}) = \boldsymbol{X}\right] = 1$.
2. $\exists$ nested equivariant finite partitions Π_n s.th. $\forall r < \infty : \mathbb{P}\left[\exists n : B_r(\boldsymbol{o}) \subseteq \Pi_n(\boldsymbol{o})\right] = 1$.
3. $\forall r < \infty, \forall \epsilon > 0, \exists$ an equivariant finite partition Π s.th. $\mathbb{P}\left[B_r(\boldsymbol{o}) \nsubseteq \Pi(\boldsymbol{o})\right] < \epsilon$.

Folner Condition

Definition

Two definitions:

1. $\forall r < \infty, \forall \epsilon > 0, \exists$, an equivariant finite partition Π s.th.

$$\mathbb{E}\left[\frac{\boldsymbol{\mu}(\partial_r \Pi(\boldsymbol{o}))}{\boldsymbol{\mu}(\Pi(\boldsymbol{o}))}\right] < \epsilon.$$

2. $\exists$ equivariant nested finite partitions Π_n s.th.

$$\forall r : \frac{\boldsymbol{\mu}(\partial_r \Pi_n(\boldsymbol{o}))}{\boldsymbol{\mu}(\Pi_n(\boldsymbol{o}))} \to 0, \quad a.s.$$

Proof Method

- Let Φ be the **marked Poisson point process** on $\boldsymbol{X}$ with intensity measure $\boldsymbol{\mu}$.
- Consider the Palm version of Φ.
- This gives a countable Borel equivalence relation R and the Palm distribution is an **invariant measure**.
- We prove that each definition is equivalent to the analogous definition for R.
- We use the amenability theorem for Borel equivalence relations.

Proof Method

- Let Φ be the **marked Poisson point process** on $\boldsymbol{X}$ with intensity measure $\boldsymbol{\mu}$.
- Consider the Palm version of Φ.
- This gives a countable Borel equivalence relation R and the Palm distribution is an **invariant measure**.
- We prove that each definition is equivalent to the analogous definition for R.
- We use the amenability theorem for Borel equivalence relations.

Proof Method

- Let Φ be the **marked Poisson point process** on $\boldsymbol{X}$ with intensity measure $\boldsymbol{\mu}$.
- Consider the Palm version of Φ.
- This gives a countable Borel equivalence relation R and the Palm distribution is an **invariant measure**.
- We prove that each definition is equivalent to the analogous definition for R.
- We use the amenability theorem for Borel equivalence relations.

Thank you!