Uniqueness of convex bodies by non-central sections in the plane

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Outline

Geometric Tomography

Geometric Tomography is an area of Mathematics dealing with properties of objects, in particular convex bodies or star bodies, based on the information concerning

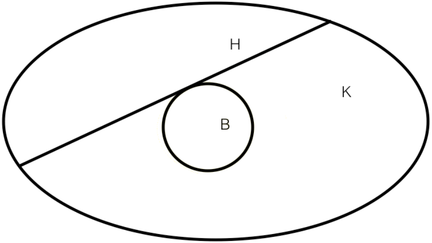
- sections $\begin{cases} \text{central sections,} \\ \text{maximal sections,} \\ \text{non-central sections,} \\ \text{conic sections;} \end{cases}$
- projections;
- etc.

	OS	Not OS
Central sections (volume)	True by Funk-Minkowski	False
Central sections (surface area)	Unknown	False by Yaskin and Ryabogin
Central sections (half sections)	True	False
Projections (volume)	True by Alexandrov	False
Projections (surface area)	True by Alexandrov	False
Conic sections (volume)	Determined by angles by Schneider	
Maximal sections (volume)	Klee problems, False by Nazarov, Ryabogin, and Zvavitch	
Non-central sections	Unknown	

Non-central sections

Problem (Barker and Larman)

Let K and L be convex bodies in $\mathbb{R}^n$ $(n \geq 2)$ that contain a Euclidean ball B in their interiors. If
$\mathrm{vol}_{n-1}(K \cap H) = \mathrm{vol}_{n-1}(L \cap H)$ for every hyperplane H that supports B, does it follow that $K = L$?



This problem is open even in $\mathbb{R}^2$. Some particular cases are known to be true.

- A body in $\mathbb{R}^2$ all of whose chords supporting a disk inside have the same length must itself be a disk.
- This problem has a positive answer in the class of convex polytopes in $\mathbb{R}^n$ by Yaskin.

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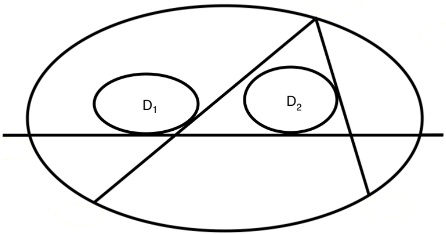
Barker and Larman also suggested a more general version of this problem by replacing the ball by a convex body.

Theorem (Yaskin and Zh., 17')

Let K and L be convex bodies in $\mathbb{R}^2$ that contain two admissible convex bodies D_1 and D_2 in their interiors. If $\mathrm{vol}_{n-1}(K \cap H) = \mathrm{vol}_{n-1}(L \cap H)$ for every hyperplane H that supports either D_1 or D_2, then $K = L$.

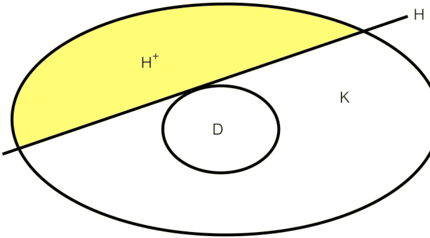


There is the following closely related problem by Groemer.

Problem (Groemer)

Let K an L be convex bodies in $\mathbb{R}^n$ and let D be a convex body in the interior of $K \cap L$. If $\mathrm{vol}_n(K \cap H^+) = \mathrm{vol}_n(L \cap H^+)$ for every hyperplane H supporting D, does it follow that $K = L$?

Here, H^+ is the half-space bounded by the hyperplane H that does not intersect the interior of D.



In $\mathbb{R}^n$ $(n \geq 3)$, we give a positive answer to a certain modification of this problem. In $\mathbb{R}^2$, we give some partial results. If D is a disk and $\mathrm{vol}_2(K \cap H^+) = \mathrm{const}$ for all H supporting D, then K is also a disk. We also solve a modification of this problem by adding another body inside $K \cap L$.

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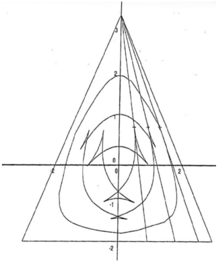
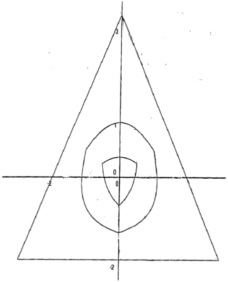
Floating bodies (Schütt and Werner, 90')

The convex floating body K_δ is the intersection of all halfspaces H^+ whose defining hyperplanes H cut off a set of volume δ from K, that is

$$K_\delta = \bigcap_{|K \cap H^-| \leq \delta} H^+$$

Floating bodies





These pictures are taken from Caglar's thesis

Application

Floating bodies give a geometric explaination of the long sought extensions of an important affine invariant, the affine surface area. This was carried out in form that affine surface area appears as a limit of the volume difference of the convex body and its floating body.

Floating bodies (Dupin, 1822)

Dupin's floating body $K_{[\delta]}$ is defined as the body whose boundary is the set of points that are the centroids of all the sections $K \cap H$, where H are the hyperplanes that cut a set of volume δ from K.

$$K_{[\delta]} = K_\delta \iff K_{[\delta]} \text{ is convex.}$$

Uniqueness of floating bodies

Problem (Werner)

Suppose that $\mathrm{K}, \mathrm{L} \subset \mathbb{R}^{\mathrm{n}}$,$\mathrm{n} \geq 3$ be convex bodies containing the origin in their interiors and $\delta \in (0,1)$. Assume that $\mathrm{K}_\delta = \mathrm{L}_\delta$. Do K and L coincide?

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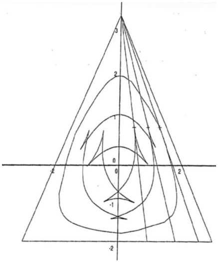

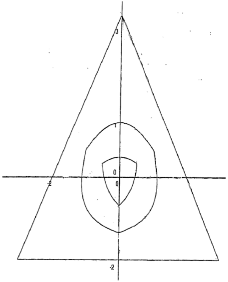

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Problem

Suppose that $\mathrm{K}, \mathrm{L} \subset \mathbb{R}^{\mathrm{n}}$, $\mathrm{n} \geq 3$ be convex bodies containing the origin in their interiors and $\delta \in (0,1)$. Assume that $\mathrm{K}_{[\delta]}$ and $\mathrm{L}_{[\delta]}$ are convex and $\mathrm{K}_\delta = \mathrm{L}_\delta$. Do K and L coincide?

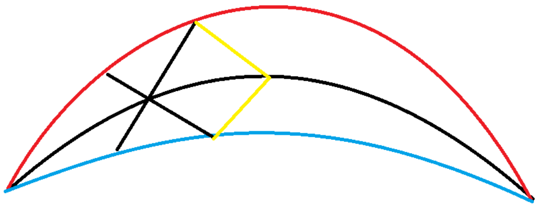
Symmetry of two-dimensional convex curves

Problem (Nazarov)

Are two convex curvex curves symmetric with respect to a circular arc at angle $\frac{p}{2}\pi$ with two end points on the arc, $p \in \mathbb{Q} \cap (0, 1)$?

Symmetry of two-dimensional convex curves

Here, we call the fact two convex curvex curves are symmetric with respect to a circular arc at angle $\frac{p}{2}\pi$ is that the $\frac{p}{2}\pi$ distance from the curves to the circular arc are equal.



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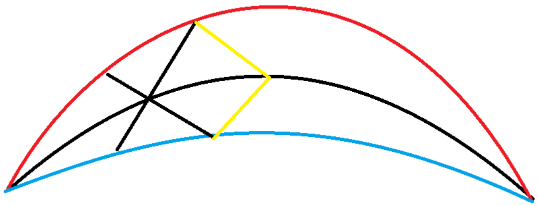

Recently, we give a negative to this problem.

Theorem (Uniqueness of noncentral sections)

Let K and L be convex bodies in $\mathbb{R}^2$ that contain a circle B in their interiors. If $\mathrm{vol}_1(K \cap H) = \mathrm{vol}_1(L \cap H)$ for every line H that supports B, then $K = L$.

Sketch of the proof

Start from a intersection point and take the tagent chord from this point. This will give you another intersection point and go on.

- This process do not have an end point.
- This process ends back to the starting point.

Sketch of the proof

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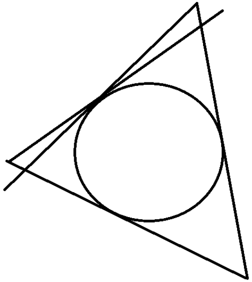


- There is not a limiting point.

This will give a dense set of intersection points.

Sketch of the proof

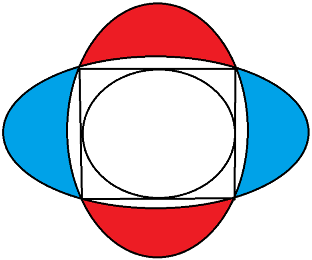
- There is a limiting point.

Do the same process from the limiting point. It will end at itself after finite steps. Now assume the two convex body do not coincide, then taking the X-rays methods gives a contradiction.

Sketch of the proof

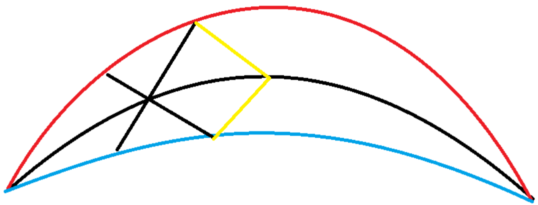
This process ends back to the starting point.

- Same structure as above.
- Similar structure by Nazarov.



Sketch of the proof

The idea is taking the bisection of tangent bundles.



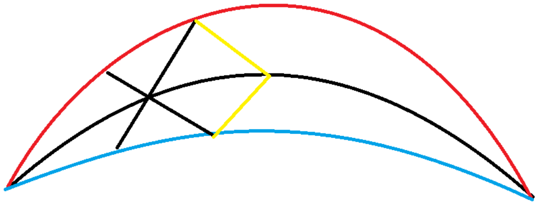
Theorem (Uniqueness of noncentral half sections)

Let K and L be convex bodies in $\mathbb{R}^2$ that contain a circle B in their interiors. If $\mathrm{vol}_1(K \cap H^+) = \mathrm{vol}_1(L \cap H^+)$ for every line H that supports B, then $K = L$.

Theorem (Uniqueness of noncentral sections)

Let K and L be convex bodies in $\mathbb{R}^2$ that contain a smooth convex body D in their interiors. If $\mathrm{vol}_1(K \cap H) = \mathrm{vol}_1(L \cap H)$ for every line H that supports D, then $K = L$.

The idea is slightly different from the above.



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Theorem (Uniqueness of floating bodies)

Let K and L be convex bodies in $\mathbb{R}^2$ that contain the origin. If K_δ and L_δ are convex and $K_\delta = L_\delta$, then $K = L$.

Ulam's 19 floating body

Problem (Ulam, 60')

If a (convex) body rests in equilibrium in every position on a flat horizontal surface is it sphere?

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Before this, in 1938, Auerbach find a counterexample in 2-dimension.

$$\begin{aligned} \mathrm{r}(\theta) = & (\mathrm{k}\cos(3\theta)\sin\theta + \cos\theta(3\mathrm{k}\sin(3\theta) - 1), \\ & - \mathrm{k}\cos(3\theta)\cos\theta + \sin\theta(3\mathrm{k}\sin(3\theta) - 1)) \end{aligned}$$

and $\delta = 1/2$.

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and $\delta = 1/2$.

Wegner gave counterexamples with $\delta \neq 1/2$ in $\mathbb{R}^2$ to Ulam's problem.

1. Let K be an origin-symmetric convex body in $\mathbb{R}^n$. If K is equilibrium for every direction with $\delta = 1/2$, then K is a ball.
2. There exists non-ball convex body being equilibrium for every direction with $\delta = 1/2$, [Ryabogin, 22'].

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What is the positive answer for Ulam's 19 problem in the plane.

Thank You!