X-Rays in Convex Geometry

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Motivation

Point X-rays and discrete X-rays will not be discussed!

Lemma 8.1.16. *Let K be a centered convex body in $\mathbb{E}^n$ of class C^2, and suppose that there is an even Borel function g on S^{n-1} such that $\rho_K = Rg$. For λ_{n-1}-almost all $u \in S^{n-1}$, we have*

$$g(u) = -\frac{1}{4\pi^2}\int_0^\infty \frac{A_K(t,u) - A_K(0,u)}{t^2}\,dt, \qquad \text{if } n = 3 \tag{8.9}$$

and

(*n*-1)-dimensional X-ray of *K* in the direction *u*

$$g(u) = -\frac{1}{16\pi^2} A_K''(0,u), \qquad \text{if } n = 4. \tag{8.10}$$

(parallel) X-ray of *K* in the direction *u*

R.J.G., A positive answer to the Busemann-Petty problem in three dimensions, *Ann. of Math. Soc.* (2) **140** (1994), 435-447.

G. Zhang, A positive answer to the Busemann-Petty problem in four dimensions, *Ann. of Math. Soc.* (2) **149** (1999), 535-543.

R.J.G., A. Koldobsky, and T. Schlumprecht, An analytic solution to the Busemann-Petty problem on sections of convex bodies, *Ann. of Math. Soc.* (2) **149** (1999), 691-703.

F. Barthe, M. Fradelizi, and B. Maurey, A short solution to the Busemann-Petty problem, *Positivity* **3** (1999), 95-100.

X-Ray Transform

Suppose that f is a bounded measurable function on $\mathbb{R}^n$ that vanishes outside a bounded measurable set. If $u \in S^{n-1}$, then

$$(X_u f)(x) = \int_{\mathbb{R}} f(x+tu)\, dt, \quad x \in u^{\perp}$$

is the X-ray transform of f in the direction u.

Johann Radon
(1887 - 1956)

Radon transform: $\tilde{f}(H) = \int_H f(y)\, dy,$

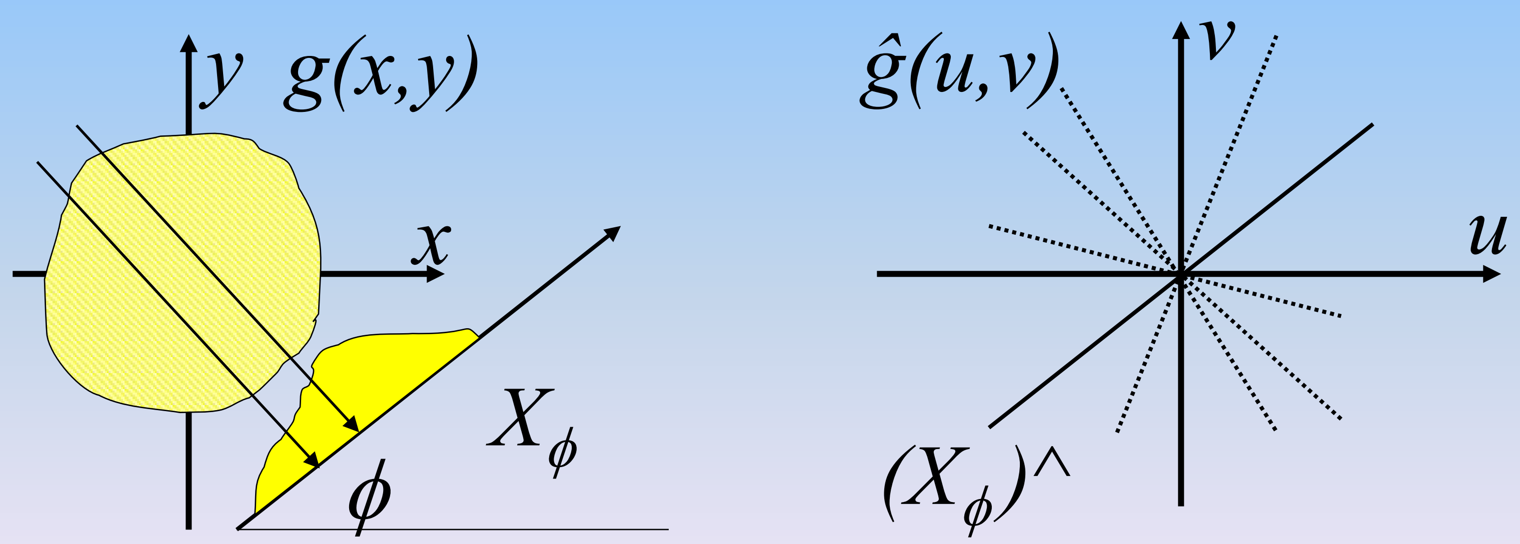
where H is a hyperplane. Case $n = 2$ is similar.

Uniqueness Theorem. Let f be as above and let D be an infinite subset of S^{n-1}. If $X_u f = 0$ for all u in D, then $f = 0$ a.e.

J. Radon, Über die Bestimmung von Funktionen durch ihre Integralwerte längs gewisser Mannigfaltigkeiten, *Ber. Verh. Sächs. Akad. Wiss. Leipzig Math.-Phys. Kl.* **69** (1917), 262-267.

Projection-Slice Theorem

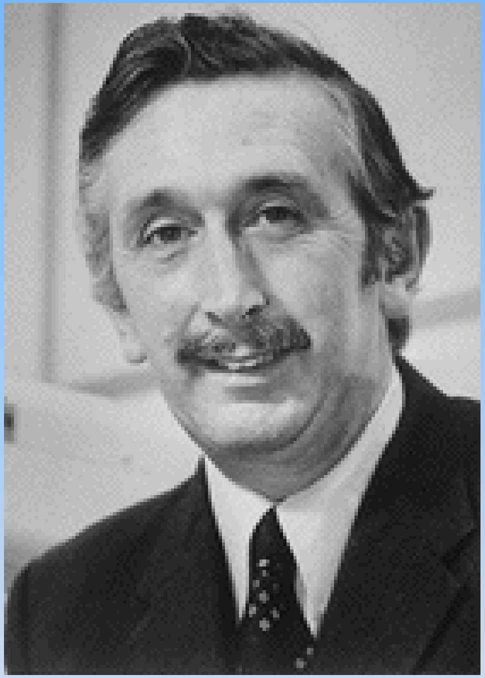
The (one-dimensional) Fourier transform of the X-ray transform of a function $g(x,y)$ at a given angle equals the slice of the (two-dimensional) Fourier transform of g at the same angle.



1979 Nobel Prize in Medicine

(Work published in 1963 to 1973)

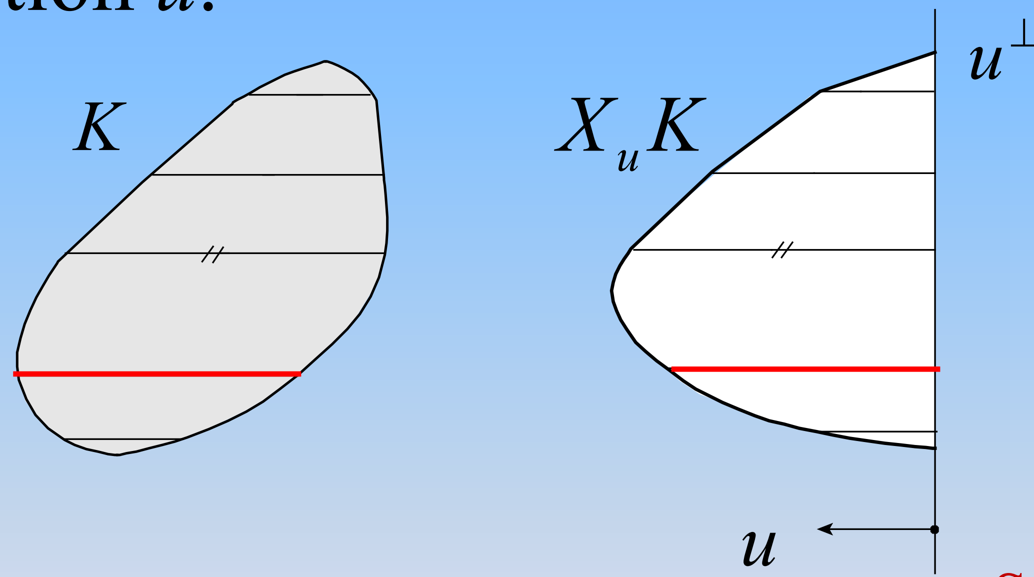
Allan MacLeod Cormack
physicist
(1924 - 1998)

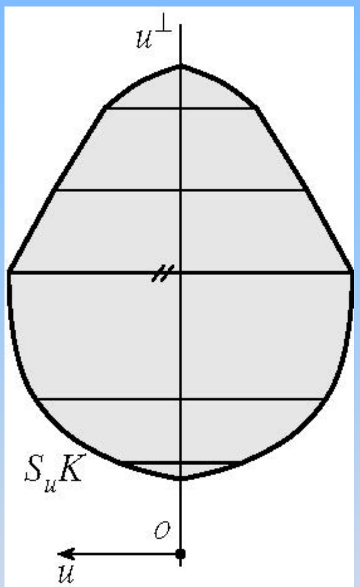


Godfrey Newbold Hounsfield
engineer
(1919 - 2004)

Parallel X-rays

Let K be a bounded Borel subset of $\mathbb{R}^n$. For u in S^{n-1}, $X_uK := X_u 1_K$ is the (parallel) X-ray of K in the direction u.

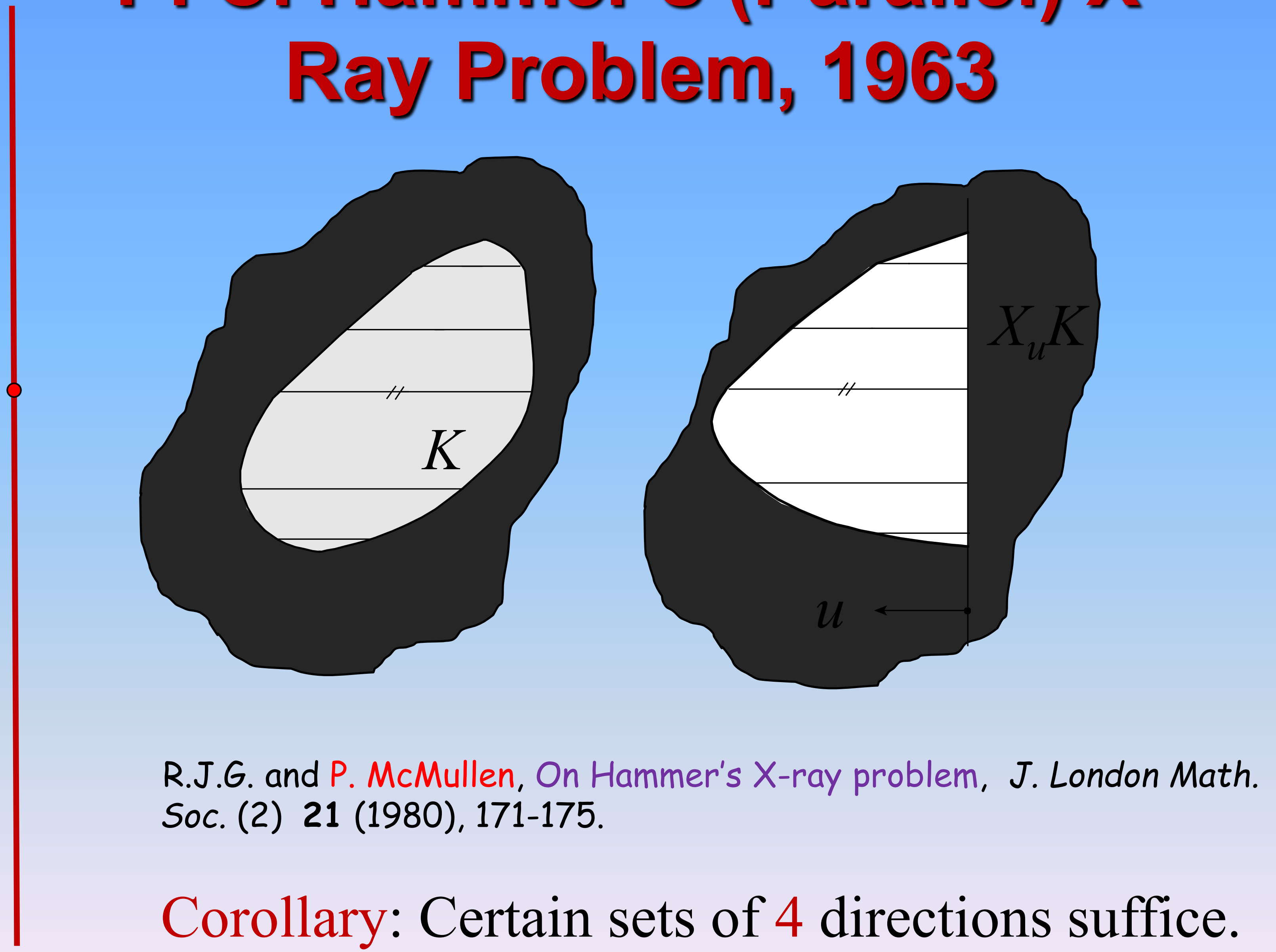




Steiner symmetral

The Uniqueness Theorem for the X-ray transform fails when the set D is finite, even for quite special classes of characteristic functions.

P. C. Hammer's (Parallel) X-Ray Problem, 1963



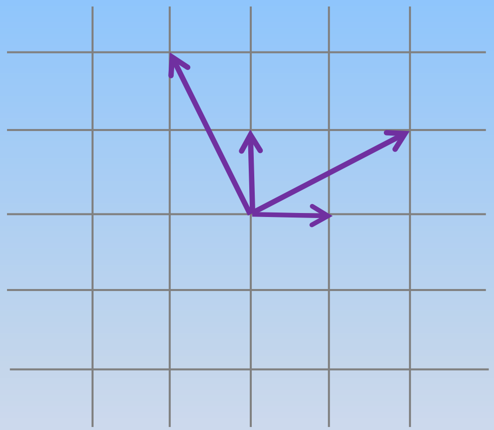
R.J.G. and P. McMullen, On Hammer's X-ray problem, *J. London Math. Soc.* (2) **21** (1980), 171-175.

Corollary: Certain sets of 4 directions suffice.

Good Sets of 4 Directions

- A set of 4 directions such that the cross ratio of the slopes is a transcendental number.
- A set of 4 directions whose rational slopes do not have a cross ratio equal to 2, 3, or 4 (in any ordering).

$$\langle x_1, x_2, x_3, x_4 \rangle = \frac{(x_3 - x_1)(x_4 - x_2)}{(x_4 - x_1)(x_3 - x_2)}$$

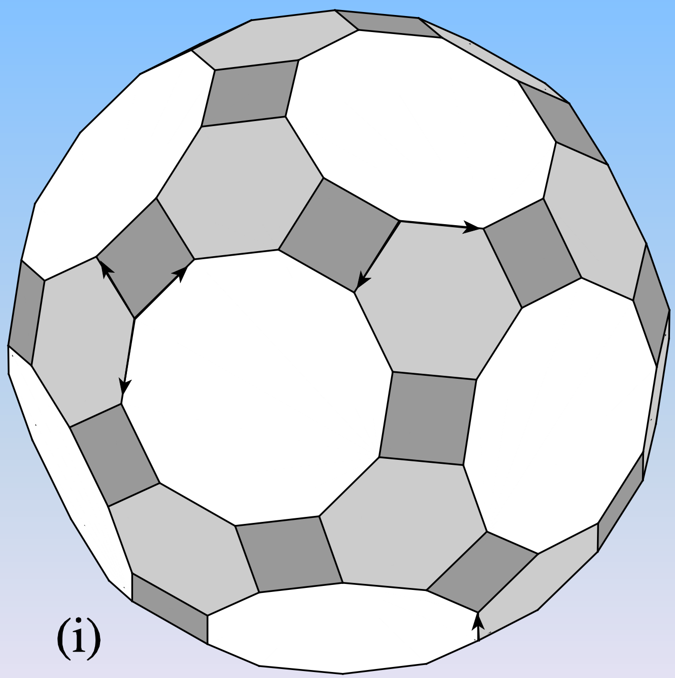


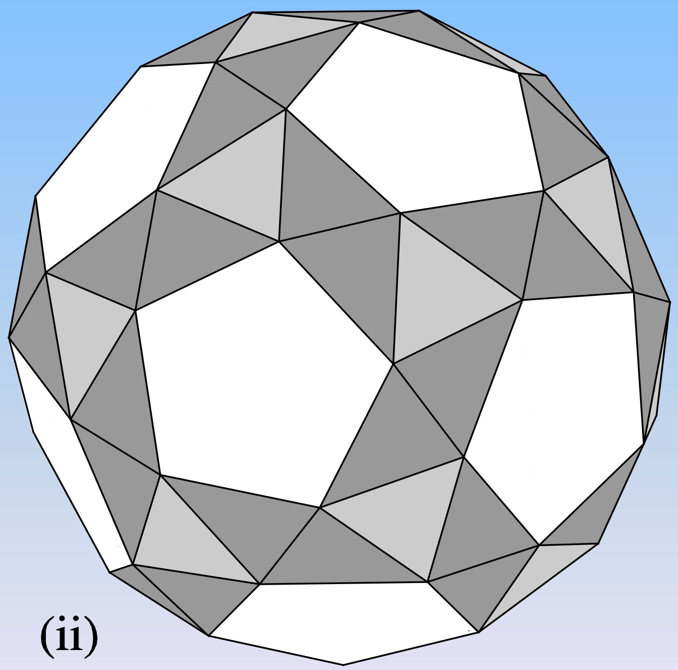
R.J.G. and P. Gritzmann, Discrete tomography: Determination of finite sets by X-rays, *Trans. Amer. Math. Soc.* **349** (1997), 2271-2296.

R.J.G. and M. Kiderlen, A solution to Hammer's X-ray reconstruction problem, *Adv. Math.* **214** (2007), 323-343.

Open Problem 1

([G, Problem 2.1], X-rays in higher dimensions.) Are convex bodies in $\mathbb{R}^3$ determined by any set of 7 X-rays in directions in general position (no 3 in a plane)?





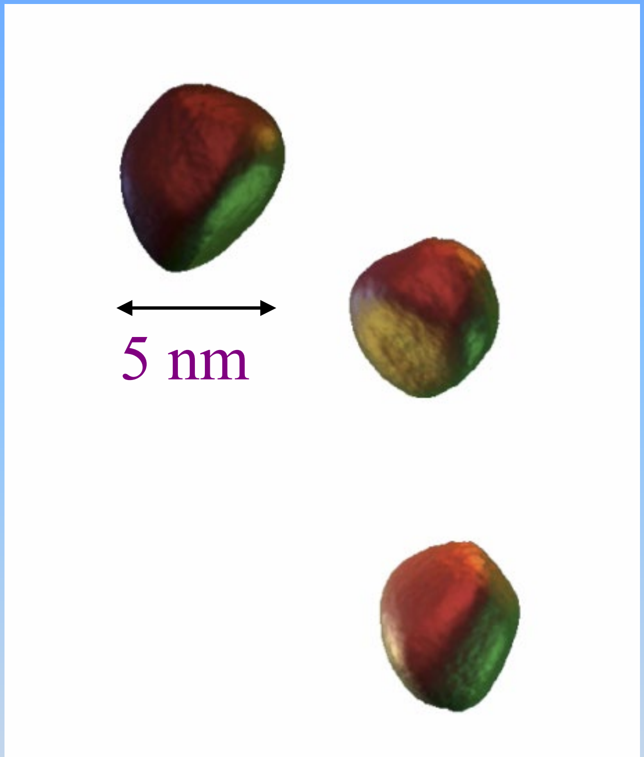
(with Jörg Wills: Oberwolfach 1982 or 1984)

Outside Mathematics, Who Cares?

Rafal Dunin-Borkowski, physicist at the Ernst-Ruska-Centre for Microscopy and Spectroscopy with Electrons Institute for Microstructure Research, Jülich, Germany.

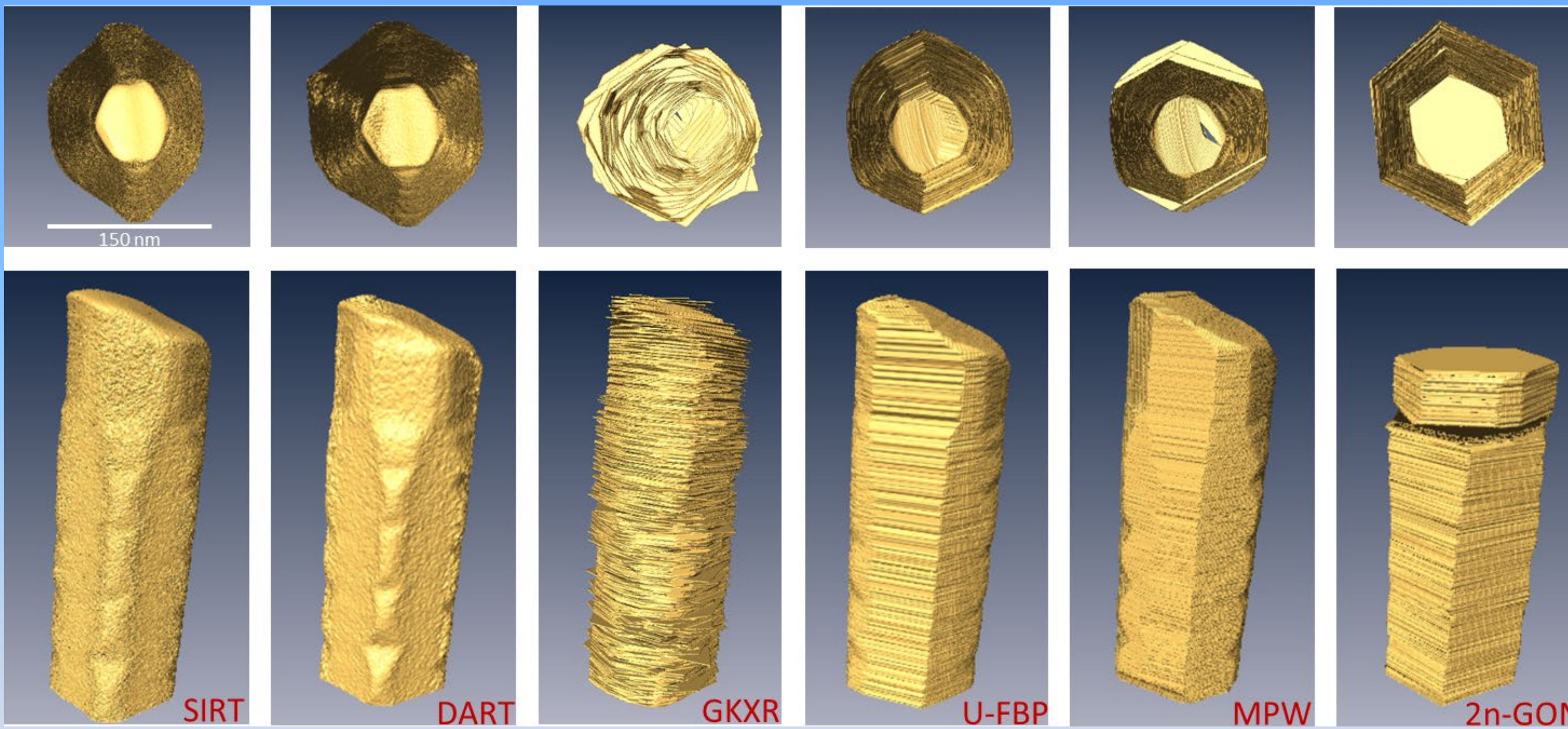
5 nanometers = 5 x 10^{-9} meters

HAADF (High-Angle Annular Dark-Field) electron tomography of platinum catalyst nanoparticles.



A. Alpers, R.J.G, S. König, R. S. Pennington, C. B. Boothroyd, L. Houben, R. Dunin-Borkowski, and K. J. Batenburg, Geometric reconstruction methods in electron tomography, *Ultramicroscopy* **128** (2013), 42-54.

Nanowire Reconstructions



change at each step.

k-Plane Transform

Suppose that f is a bounded measurable function on $\mathbb{R}^n$ that vanishes outside a bounded measurable set. If $1 \le k \le n$-1 and $S \in \mathcal{G}(n,k)$, then

S. Helgason, 1959 (?)

$$(X_S f)(x) = \int_S f(x+y)\,dy, \quad x \in S^\perp$$

$k = 1 \rightarrow$ X-ray transform

$k = n$-1 ~ Radon transform

is the k-plane transform of f parallel to S.

Central Slice Theorem. $(X_S f)^\wedge(x) = f^\wedge(x), \;\; x \in S^\perp.$

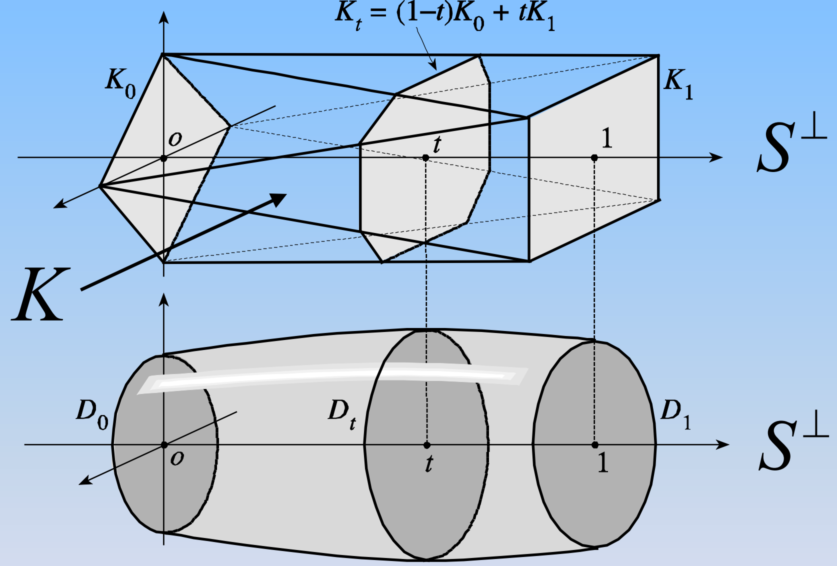
Uniqueness Theorem. If $X_{S_m} f = 0, \; m \in \mathbb{N},$ where $S_m{}^\perp, \; m \in \mathbb{N},$ are not contained in a proper algebraic variety, then $f = 0$ a.e.

There are infinite families of subspaces providing uniqueness.

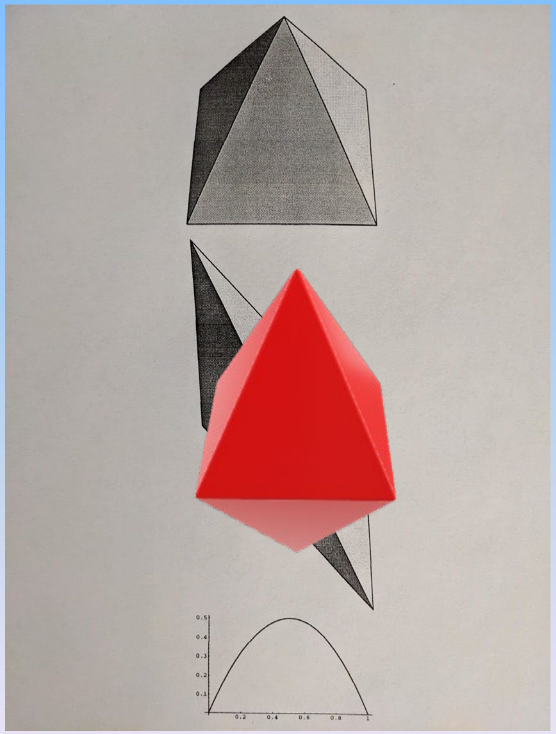
F. Keinert, Inversion of k-plane transforms and applications in computer tomography, *SIAM Rev.* **31** (1989), 273-298.

Higher-Dimensional X-rays

Let K be a bounded Borel subset of $\mathbb{R}^n$. If $1 \leq k \leq n$-1 and $S \in \mathcal{G}(n,k)$, then $X_S K = X_S 1_K$ is the k-dimensional X-ray of K parallel to S.



Schwarz symmetral



Pyramid and tetrahedron with equal 2-dimensional X-rays

Unsolved Problems

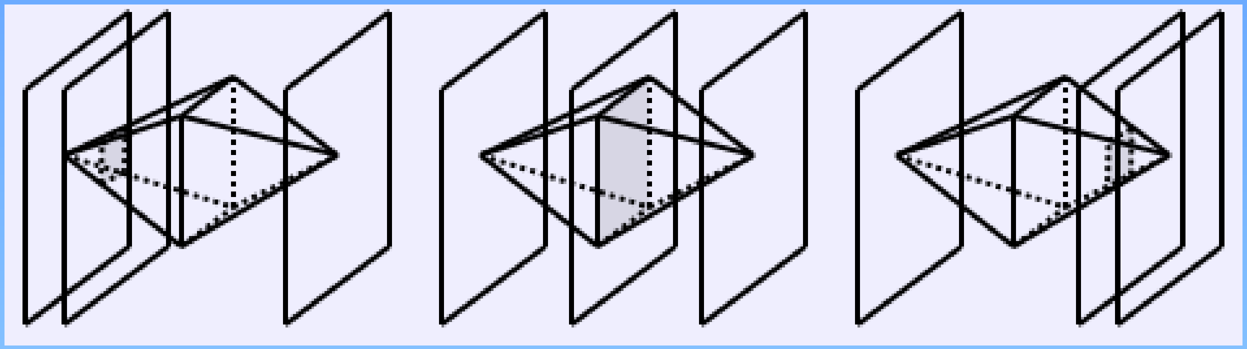
Some examples of problems on X-rays still open:

- [G, Problem 2.5] Are there finite sets of directions such that a convex body in $\mathbb{R}^3$ is determined by its 2-dimensional X-rays orthogonal to these directions?
- [G, Problem 2.5] Are there finite sets of directions such that a convex polytope in $\mathbb{R}^n$ is successively determined by its 2-dimensional X-rays orthogonal to these directions? (Three 1-dimensional X-rays suffice in $\mathbb{R}^2$ and two suffice in $\mathbb{R}^3$.)
- [G, Problem 2.4] Can planar convex bodies be successively determined by 3 X-rays?

R.J.G. and P. Gritzmann, Successive determination and verification of polytopes by their X-rays, *J. London Math. Soc.* (2) **50** (1994), 375-391.

Maximal Section Function

Suppose that $n \geq 3$.



u

$$m_K(u) = \max_{-\infty<t<\infty} V_{n-1}(K \cap (u^\perp + tu))$$

Inner Quermass, HA-measurement (de Haas-van Alpen effect, Fermi surfaces)

Open Problem (Bonnesen, 1926). Do the brightness function and maximal section function together determine every convex body, up to translation and reflection in the origin? Even open for balls.

(Problem of Klee, 1969.) Maximal section function alone does not suffice, even for balls:

F. Nazarov, D. Ryabogin, and A. Zvavitch, An asymmetrical convex body with maximal sections of constant volume, *J. Amer. Math. Soc.* **27** (2014), 43-68.

(n-1)-dim.-X-Ray pth Mean Bodies

If K is a convex body in $\mathbb{R}^n$ and $p \neq 0$, define

$$\rho_{E_pK}(u) = \left(\frac{1}{w_K(u)} \int_{\mathbb{R}} V(K \cap (u^\perp + tu))^p \, dt \right)^{1/p}, \qquad u \in S^{n-1},$$

$$\rho_{E_0K}(u) = \exp\left(\frac{1}{w_K(u)} \int_{\mathbb{R}} \log X_{u^\perp} K(t) \, dt \right), \qquad u \in S^{n-1},$$

and

$$\rho_{E_\infty K}(u) = \max\left\{ X_{u^\perp} K(t) : t \in \mathbb{R} \right\} = m_K(u).$$

By Jensen's inequality, H. Martini, 1992; C. M. Petty, 1952

$$V(K) D^o K = E_1 K \subset E_p K \subset E_q K \subset E_\infty K = CK, \quad 1 \le p \le q,$$

where CK is the cross-section body of K, since

$$V(K)^{-1} \rho_{E_1K}(u) = w_K(u)^{-1} = h_{DK}(u)^{-1} = \rho_{D^oK}(u).$$

Spectral *p*th Mean Bodies

If K is a convex body in $\mathbb{R}^n$ and $0 \neq p \geq -1$, define

$$\rho_{S_pK}(u) = \left(\boxed{\frac{1}{V(K)}}\int_{K|u^\perp} X_uK(y)^{p+1}\, dy\right)^{1/p}, \qquad u \in S^{n-1},$$

$$\rho_{S_0K}(u) = \exp\left(\frac{1}{V(K)}\int_{K|u^\perp} X_uK(y)\log X_uK(y)\, dy\right),$$

$$\rho_{S_\infty K}(u) = \max\left\{X_uK(y) : y \in K \,|\, u^\perp\right\}.$$

Then

$$V(K)\Pi^o K = S_{-1}K \subset S_pK \subset S_qK \subset S_\infty K = DK, \qquad p < q,$$

since

$$V(K)^{-1}\rho_{S_{-1}K}(u) = V(K\,|\,u^\perp)^{-1} = h_{\Pi K}(u)^{-1} = \rho_{\Pi^o K}(u).$$

Could prove that S_pK is an o-symmetric convex body when $p \geq 0$ and retrieve the Rogers-Shephard and Zhang projection inequalities.

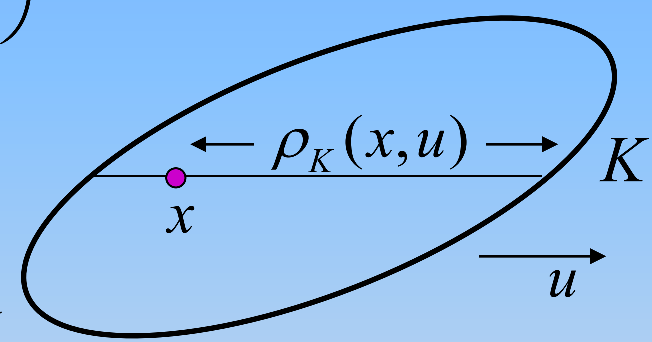
*Following D. Langharst

Radial *p*th Mean Bodies

If K is a convex body in $\mathbb{R}^n$ and $p \neq 0$, define

$$\rho_{R_pK}(u) = \left(\frac{1}{V(K)} \int_K \rho_K(x,u)^p \, dx \right)^{1/p}, \qquad u \in S^{n-1},$$

etc. Then



$$R_pK = (p+1)^{-1/p} S_pK, \quad 0 \neq p > \text{-}1,$$

shape of $R_pK \longrightarrow S_{-1}K$ as $p \longrightarrow$ -1, and

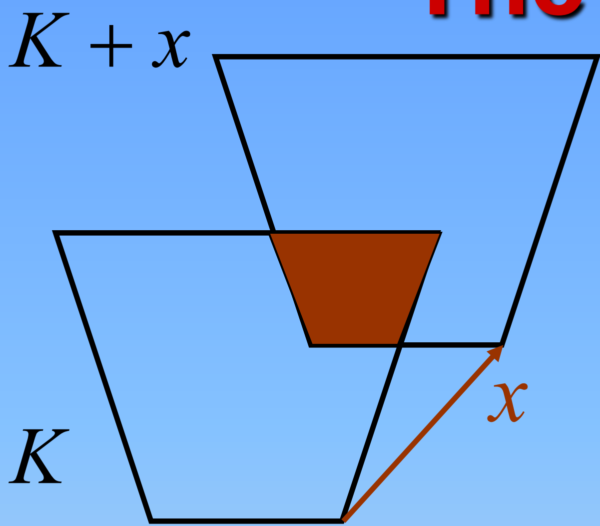
$$R_pK \subset R_qK \subset R_\infty K = DK, \ \ \text{-}1 < p \leq q.$$

Moreover,

$$\rho_{R_pK}(u) = \left(pV(K) \int_0^{\rho_{DK}(u)} g_K(ru)\, r^{p-1} dr \right)^{1/p}, \qquad u \in S^{n-1}.$$

R.J.G. and G. Zhang, Affine inequalities and radial mean bodies, *Amer. J. Math.* **120** (1998), 505-528.

The Covariogram



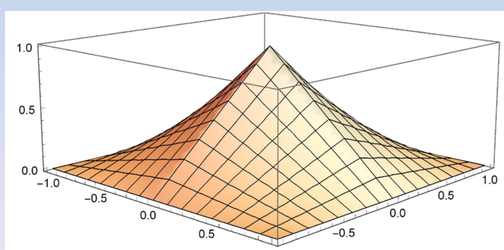
$$\boxed{g_K(x)} = V(K \cap (K+x))$$

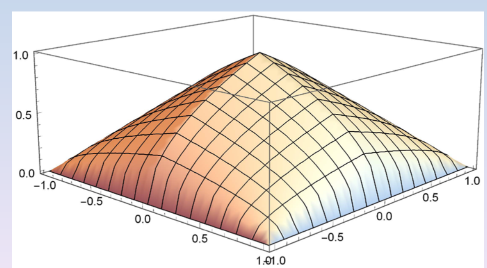
$$= \int_{\mathbb{R}^n} 1_K(z) 1_K(z-x)\, dz = \boxed{1_K * 1_{-K}(x)}$$

G. Matheron, *Random Sets and Integral Geometry,* Wiley, New York, 1975.

- g_K is invariant under translations of K or reflections of K in o.
- (Brunn-Minkowski) $g_K^{1/n}$ is concave on its support DK.
- g_K is log concave.

$K=[0,1]^2$: $\quad g_K(x) = (1-|x|)(1-|y|), \quad -1 \leq x, y \leq 1.$





G. Bianchi, The covariogram problem, *to appear.*

p-Cross-Section Bodies

If K is a convex body in $\mathbb{R}^n$ and $0 \neq p \geq -1$, define

$$\rho_{C_pK}(u) = \left(\frac{1}{V(K)} \int_{\mathbb{R}} X_{u^\perp} K(tu)^{p+1} \, dt \right)^{1/p}, \qquad u \in S^{n-1},$$

etc. Then

$$V(K) D^o K = C_{-1}K \subset C_pK \subset C_qK \subset C_\infty K = CK, \quad 1 \leq p \leq q.$$

It is known that CK is convex when $n = 3$ (M. Meyer, 1999; K regular tetrahedron $\rightarrow$ CK cube!) but not when $n \geq 4$ (U. Brehm, 1999). Also, C_1K is convex due to the strange fact that

$$C_1K = I(R_{n-1}K)!$$

R.J.G. and A. Giannopoulos, *p*-Cross-section bodies, *Indiana Univ. Math. J.* **48** (1999), 593-613.

More on Convexity

Ball, 1988: Let f be a nonnegative, integrable, log concave function on $\mathbb{R}^n$ and let $p > 0$. Then

$$\rho(x) = \left(\int_0^\infty f(tx)\, t^{p-1} dt\right)^{1/p}, \qquad o \neq x \in \mathbb{R}^n,$$

is the radial function of a convex body in $\mathbb{R}^n$.

So R_pK and S_pK are convex when $p \geq 0$.

Are R_pK and S_pK convex when $-1 < p < 0$?

Is C_pK convex when $n = 3$ or when K is o-symmetric?

K. M. Ball, Logarithmically concave functions and sections of convex sets in $\mathbf{R}^n$, *Studia Math.* **88** (1988), 64-94.

The Covariogram Problem

Matheron, 1986: Is a convex body determined, up to translation and reflection in the origin, by its covariogram?

R.J.G. & G. Zhang, 1998: Equivalently, is a convex body K so determined by (i) R_pK, for all $p > -1$ or (ii) S_pK, for all $p \geq -1$?

Yes, when $n = 2$:

G. Averkov and G. Bianchi, Confirmation of Matheron's conjecture on the covariogram of planar convex body, *J. Euro. Math. Soc.* (*JEMS*) **11** (2009), 1187-1202.

No, when $n \geq 4$:

G. Bianchi, Matheron's conjecture for the covariogram problem, *J. London Math. Soc.* (2) **71** (2005), 203-220.

Open, when $n = 3$!

We have $g_K = g_L$ if and only if for each u in S^{n-1}, the X-rays X_uK and X_uL are rearrangements of one another.

R.J.G., P. Gronchi, and C. Zong, Sums, projections, and sections of lattice sets, and the discrete covariogram, *Discrete Comput. Geom.* **34** (2005), 391-409.

Inequalities 1

$$DK \subset c_{n,q} R_q K \subset c_{n,p} R_p K \subset nV(K)\Pi^o K, \quad -1 < p < q,$$

$$c_{n,p} = (nB(p+1,n))^{1/p}, \qquad V(R_n K) = V(K),$$

with equality anywhere if and only if K is a simplex.

- Apply Berwald's inequality (Berwald, 1947; Borell, 1973) to the concave function $\rho_K(\cdot,u)$ on K.

$$(n,\infty): \quad V(K) \ge \binom{2n}{n}^{-1} V(DK)$$

Rogers-Shephard inequality

- Analogous inclusions hold for the p-cross-section bodies C_pK, though without global equality conditions.

$$(-1,n): \quad V(K) \le \binom{2n}{n}^{-1} nV(K)^n V(\Pi^o K)$$

Zhang projection inequality

G. D. Chakerian, Inequalities for the difference body of a convex body, *Proc. Amer. Math. Soc.* **18** (1967), 879-884.

D. Langharst, Generalizations of Berwald's inequality to measures, *arXiv:2210.04438v3.*

Ai-Jun Li, Cortona meeting talk, *June 2023.* (k-dim. X-rays, $1 < k < n-1$.)

Inequalities 2

$$V(R_q K)/V(K) \ge V(R_p B^n)/V(B^n), \quad -1 < p < 0,\ p > n,$$

$$V(R_q K)/V(K) \le V(R_p B^n)/V(B^n), \quad 0 < p < n.$$

$$\tilde{V}(L, R_p K) = \frac{p}{nV(K)} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \|x - y\|_L^{p-n} \, dx \, dy, \quad p \ne 0$$

chord-power integral

+ dual Minkowski inequality + Riesz rearrangement inequality

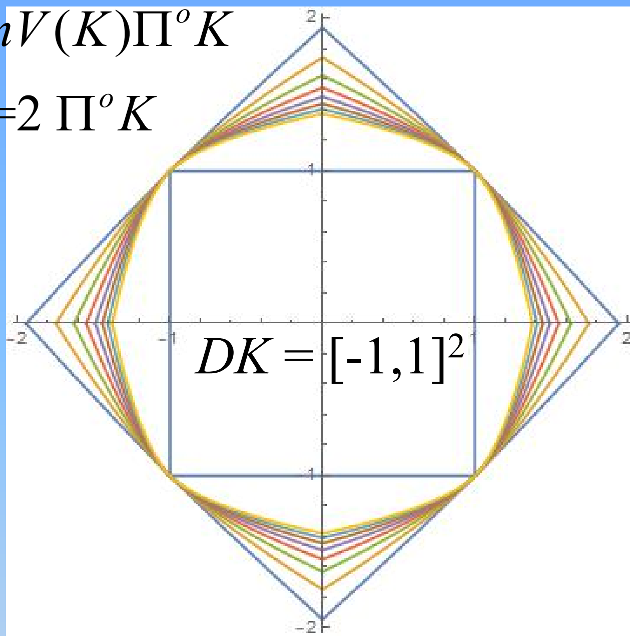
J. Haddad and M. Ludwig, Affine fractional Sobolev and isoperimetric inequalities, *arXiv*:2207.06375v1.

J. Haddad and M. Ludwig, Affine Hardy-Littlewood-Sobolev inequalities, *arXiv*:2212.12194v1.

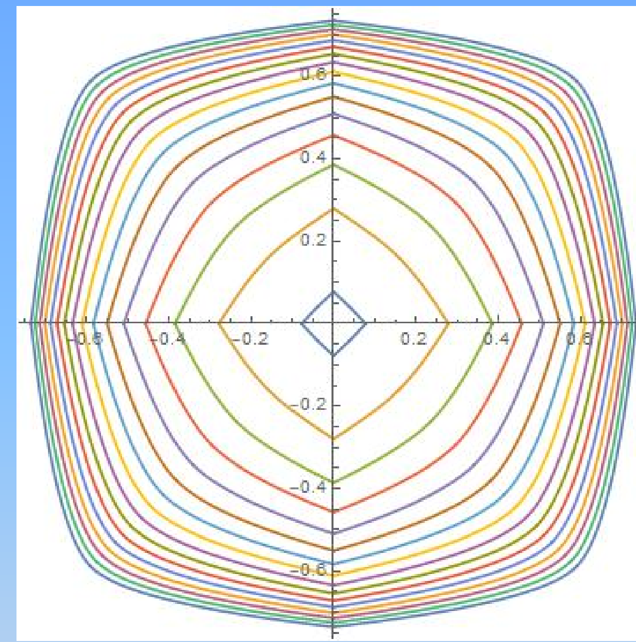
D. Langharst, M. Roysdon, and A. Zvavitch, General measure extensions of projection bodies, *J. London Math. Soc.* (3) **125** (2022), 1083-1129.

J. Haddad, D. Langharst, E. Putterman, M. Roysdon, and D. Ye, Affine isoperimetric inequalities for higher-order projection and centroid bodies, *preprint*.

Example 1



$K = [0,1]^2$

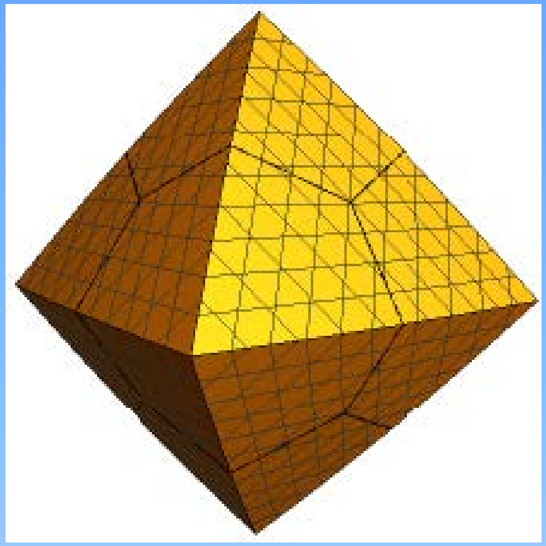


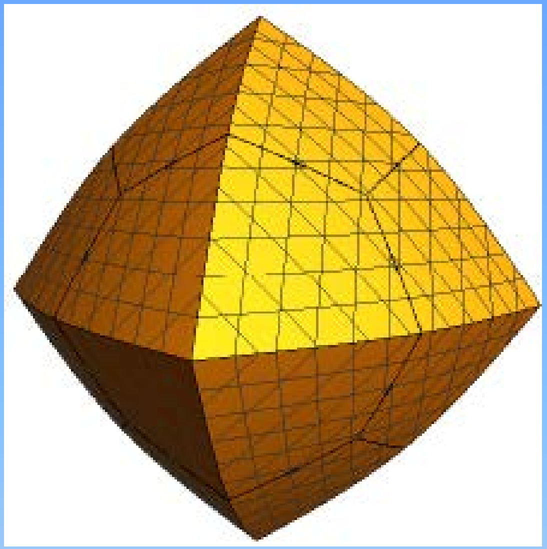
$c_{n,p} R_p K, \ -0.9 \le p \le 0.5$ $\qquad$ $R_p K, \ -0.9 \le p \le 0.5$

$R_p K$ is convex when $p \geq -1$ when $K = [0,1]$^2 or $K = [0,1]$^3.

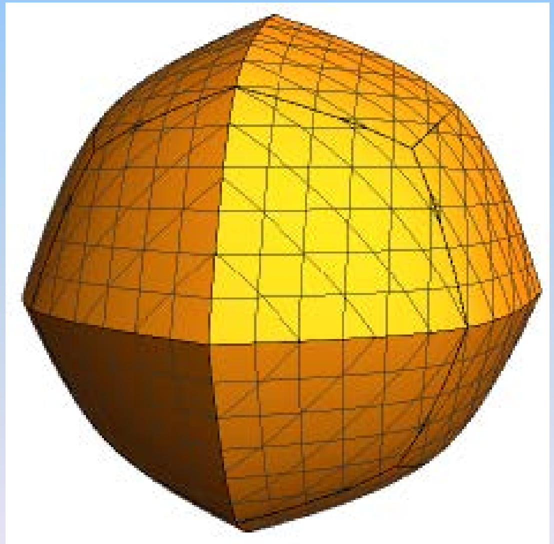
T. Koshida, Convexities of radial mean bodies, Masters thesis, Ibaraki University, Japan, 2023.

Example 2

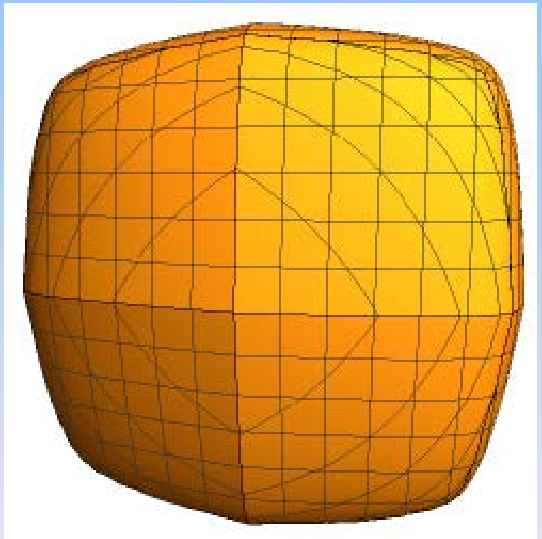




$K=[-1,1]^3$



Thanks to
Hiroshi Iriyeh



$R_pK,\ p=-0.9,\ \ -0.5,\ \ -0.5,\ 1,\ 10$