Higher rank antipodality

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The question

Antipodality — The classical definitions

Definition by Klee, 1960

A set $S \subset \mathbb{R}^d$ is *antipodal*, if for any two distinct points $q_1, q_2 \in S$, there exists two distinct parallel hyperplanes L_1, L_2 supporting S such that $q_1 \in L_1$ and $q_2 \in L_2$.

Definition by Erdős, 1957

S is *obtuse triangle-free*, if it does not contain the vertices of an obtuse triangle.

Danzer and Grünbaum, 1962 — Answering Klee's question

Maximum size of S is 2^d, and equality: vertices of a parallelotope.

Rank k antipodality

Definition by Klee, 1960 — Repeated

A set $S \subset \mathbb{R}^d$ is *antipodal*, if for any two distinct points $q_1, q_2 \in S$, there exists an affine transformation ϕ of $\mathbb{R}^d$ onto a line mapping S to the line segment conv $\{\phi(q_1), \phi(q_2)\}$.

Rank k antipodality

A set $S \subset \mathbb{R}^d$ is *rank k antipodal*, if for any $k+1$ distinct points $q_1, \ldots, q_{k+1} \in S$, there exists an affine map ϕ of $\mathbb{R}^d$ mapping S to the k-dimensional simplex conv $\{\phi(q_1), \ldots, \phi(q_{k+1})\}$.

Rank 1 antipodality = antipodality.

Variants, Upper bound

Erdős' notion generalized: for any $q_1, \ldots, q_{k+1}$, there is an orthogonal projection to $\mathrm{aff}(q_1, \ldots, q_{k+1})$ such that ...

Strict antipodality generalized: ...

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Question: How large can such a set be for a given d and k?

Upper bound [MN, Szilágyi, Weiner]

$$A(d,d) = A(d,d-1) = d+1$$

and

$$A(d,k) \leq k\left(\frac{k+1}{k}\right)^d.$$

General Probability Theory

Common framework for classical and quantum probability.

Set of states: S.
Possible outcomes of a measurement: say $[k+1]$.
Outcome-statistics: Probabilities, ie. a point in

$$\Delta_k = \{(p_1, \dots p_{k+1}) \in \mathbb{R}^{k+1} \ : \ \sum_j p_j = 1,\ p_j \geq 0 \text{ for all } j\}.$$

Measurement: a $\phi : S \longrightarrow \Delta_k$ map.

Convex structure on the set S of states

Mixed state: $\lambda s_1 + (1-\lambda)s_2$, where $s_1, s_2 \in S$, and $\lambda, 1-\lambda \in [0,1]$
Thus, S is a convex set.
Every measurement is an affine map $\phi : S \longrightarrow \Delta_k$.
No restriction principle: and vica versa.

Classical vs. quantum probability

Classical probability: $S = \Delta_n$ for some n.
Quantum probability: S is the set of density operators on $\mathbb{C}^n$.

Distinguishing states

Let $s_1, s_2, \ldots s_{k+1} \in S$. To use the system as memory, the system is put into state s_j according to some selected value $j \in [k+1]$.

To retrieve j, perform a measurement with $k+1$ possible outcomes: $\phi : S \to \Delta_k$.

$s_1, s_2, \ldots s_{k+1}$ are *jointly perfectly distinguishable*, if there is a ϕ such that $\phi_j(s_j) = 1$ for all $j \in [k+1]$.

Note: Both in quantum and in classical probability:
pairwise perfect distinguishability (ie. rank 1 antipodality)

$$\Longrightarrow$$

joint perfect distinguishability (ie. rank k antipodality).

Construction:

Hash $\longrightarrow$ Rank k antipodal set

Fix $b, k, m \in \mathbb{Z}_+$, $2 < k \leq b$.

A *perfect (b,k)-hash code of length m* is a set W of words of length m on the alphabet $[b] = \{1, 2, \ldots, b\}$ in which for every subset $\{w_1, \ldots, w_k\}$ of k elements of W, there is a $j \in [m]$ such that the j^{th} letters of the words $w_1, \ldots, w_k$ are all different.

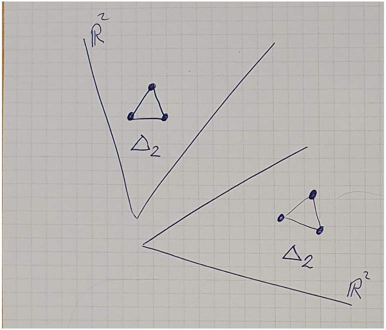
$N(b,k,m)$: size of the largest perfect (b,k)-hash code of length m.

[L. Lami, D. Goldwater, G. Adesso], [MN, Szilágyi, Weiner]

Assume that there is a rank k antipodal set in dimension d_0 of size b. Then for every $m = 1, 2, \ldots$, one can construct a rank k antipodal set in dimension $d = m \cdot d_0$ of size $N(b, k+1, m)$.

Goal: A rank k antipodal set in $d = m \cdot d_0$ of size $N(b, k+1, m)$

Given: a size b rank k antipodal set in $\mathbb{R}^{d_0}$.



Δ_2 is a $k = 2$ andtipodal set of size $b = 3$.

Here, $m = 2$ copies

$\mathbb{R}^4 = \mathbb{R}^{m \cdot d_0}$.

(In the hash: m — the length of the words; b — size of alphabet).

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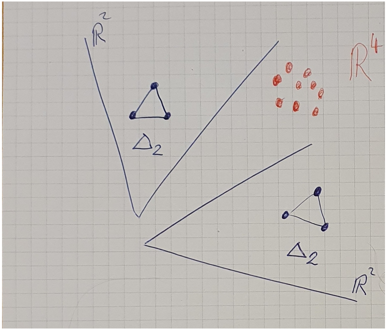

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Proof: $A(d,k) \le k\left(\frac{k+1}{k}\right)^d$

Lemma (Paraphrasing Danzer and Grünbaum)

$S \subset \mathbb{R}^d$ a convex body, $q_1, \dots q_{k+1} \in S$. Then the following are equivalent.

1. $q_1, \dots q_{k+1}$ are jointly antipodal with respect to S;
2. for any $\lambda_1, \dots \lambda_{k+1} \in (0,1)$ with $\lambda_1 + \dots + \lambda_{k+1} = k$, we have
$$\bigcap_{j=1}^{k+1} D_{q_j,\lambda_j}(\mathrm{int}\,(S)) = \emptyset;$$
3. for some $\lambda_1, \dots \lambda_{k+1} \in (0,1)$ with $\lambda_1 + \dots + \lambda_{k+1} = k$, we have
$$\bigcap_{j=1}^{k+1} D_{q_j,\lambda_j}(\mathrm{int}\,(S)) = \emptyset.$$

Proof: $A(d,k) \le k\left(\frac{k+1}{k}\right)^d$

$S := \operatorname{conv}(q_1, \ldots, q_n)$, and consider $S_j = D_{q_j, k/(k+1)}(\operatorname{int}(S))$.

$S_j \subseteq S$, and $\operatorname{vol}(S_j) = \left(\frac{k}{k+1}\right)^d \operatorname{vol}(S)$.

By the Lemma, no $k+1$ of these sets intersect, that is, every point of S is contained in at most k of the S_j. Thus,

$$\sum_{j=1}^{n} \operatorname{vol}(S_j) \le k \cdot \operatorname{vol}(S)\,.$$

$\square$

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Thank you!