Monge-Ampère operators and valuations

Jonas Knoerr

TU Wien

September 25, 2023

Conference on Convex Geometry and Geometric Probability
Salzburg 2023

Valuations on sets and functions

$\mathcal{S}$: family of sets, $(\mathcal{A}, +)$: Abelian semi-group

Definition

$\mu : \mathcal{S} \to \mathcal{A}$ is called a valuation if

$$\mu(K \cup L) + \mu(K \cap L) = \mu(K) + \mu(L)$$

for all $K, L \in \mathcal{S}$ s.t. $K \cup L, K \cap L \in \mathcal{S}$.

Valuations on sets and functions

$\mathcal{S}$: family of sets, $(\mathcal{A}, +)$: Abelian semi-group

Definition

$\mu : \mathcal{S} \to \mathcal{A}$ is called a valuation if

$$\mu(K \cup L) + \mu(K \cap L) = \mu(K) + \mu(L)$$

for all $K, L \in \mathcal{S}$ s.t. $K \cup L, K \cap L \in \mathcal{S}$.

X: some family of real valued functions

Definition

$\mu : X \to \mathcal{A}$ is called a valuation if

$$\mu(f \vee h) + \mu(f \wedge h) = \mu(f) + \mu(h)$$

for all $f, h \in X$ such that $f \vee h, f \wedge h \in X$.

$f \vee h$: pointwise maximum, $f \wedge h$: pointwise minimum

Valuations functions

- Valuations on L_p and Orlicz functions
  Tsang 2010, 2012; Ludwig 2013; Ober 2014; Kone 2014; Li & Ma 2017
- Valuations on Sobolev functions and BV functions
  Ludwig 2011; Wang 2014; Ma 2016
- Valuations on continuous and Lipschitz functions
  Villanueva 2016; Tradacete & Villanueva 2017, 2018, 2020; Colesanti, Pagnini, Tradacete & Villanueva 2020, 2021

Valuations functions

- Valuations on L_p and Orlicz functions
  Tsang 2010, 2012; Ludwig 2013; Ober 2014; Kone 2014; Li & Ma 2017
- Valuations on Sobolev functions and BV functions
  Ludwig 2011; Wang 2014; Ma 2016
- Valuations on continuous and Lipschitz functions
  Villanueva 2016; Tradacete & Villanueva 2017, 2018, 2020; Colesanti, Pagnini, Tradacete & Villanueva 2020, 2021
- **Valuations on convex functions**
  Cavallina & Colesanti 2015, Colesanti, Ludwig & Mussnig 2017, . . . , 2023; Alesker 2019; Mussnig 2019, 2020; K. 2020, 2021; Hofstätter & Schuster 2023; Hofstätter & K. 2023,...

Examples

- $\mathrm{Conv}(\mathbb{R}^n, \mathbb{R})$: convex functions $f : \mathbb{R}^n \to \mathbb{R}$ (loc. uniform conv.)
- $\mathcal{M}(\mathbb{R}^n) := (C_c(\mathbb{R}^n))'$: signed Radon measures on $\mathbb{R}^n$ (weak-$*$ conv.)

Example (real Monge-Ampère operator)

$$\begin{aligned} \mathrm{MA} : \mathrm{Conv}(\mathbb{R}^n, \mathbb{R}) \cap C^2(\mathbb{R}^n) &\to \mathcal{M}(\mathbb{R}^n) \\ f &\mapsto \left[B \mapsto \int_B \det(D^2 f(x)) dx, \quad B \text{ Borel set} \right] \end{aligned}$$

extends uniquely by continuity to a continuous valuation.

Examples

- $\mathrm{Conv}(\mathbb{R}^n, \mathbb{R})$: convex functions $f : \mathbb{R}^n \to \mathbb{R}$ (loc. uniform conv.)
- $\mathcal{M}(\mathbb{R}^n) := (C_c(\mathbb{R}^n))'$: signed Radon measures on $\mathbb{R}^n$ (weak-$*$ conv.)

Example (real Monge-Ampère operator)

$$\begin{aligned}\mathrm{MA} : \mathrm{Conv}(\mathbb{R}^n, \mathbb{R}) \cap C^2(\mathbb{R}^n) &\to \mathcal{M}(\mathbb{R}^n) \\ f &\mapsto \left[B \mapsto \int_B \det(D^2 f(x)) dx, \quad B \text{ Borel set} \right]\end{aligned}$$

extends uniquely by continuity to a continuous valuation.

mixed Monge-Ampère operators:

$$\mathrm{MA}(f_1, \dots, f_n) := \frac{1}{n!} \frac{\partial^n}{\partial \lambda_1 \dots \partial \lambda_n} \Big|_0 \mathrm{MA}\left(\sum_{i=1}^n \lambda_i f_i \right)$$

Further examples

Theorem (Alesker 2019)

For every $B \in C_c(\mathbb{R}^n)$, $A_1, \dots, A_{n-k} \in C_c(\mathbb{R}^n, \mathrm{Sym}^2\mathbb{R}^n)$ *there exists a unique continuous valuation* $\mu : \mathrm{Conv}(\mathbb{R}^n, \mathbb{R}) \to \mathbb{R}$ *s.t.*

$$\mu(f) = \int_{\mathbb{R}^n} B(x) \operatorname{det}(D^2 f(x)[k], A_1(x), \dots, A_{n-k}(x)) dx$$

for all $f \in \mathrm{Conv}(\mathbb{R}^n, \mathbb{R}) \cap C^2(\mathbb{R}^n)$. det*: mixed discriminant*

Further examples

Theorem (Alesker 2019)

For every $B \in C_c(\mathbb{R}^n)$*,* $A_1, \dots, A_{n-k} \in C_c(\mathbb{R}^n, \mathrm{Sym}^2\mathbb{R}^n)$ *there exists a unique continuous valuation* $\mu : \mathrm{Conv}(\mathbb{R}^n, \mathbb{R}) \to \mathbb{R}$ *s.t.*

$$\mu(f) = \int_{\mathbb{R}^n} B(x) \det(D^2 f(x)[k], A_1(x), \dots, A_{n-k}(x)) dx$$

for all $f \in \mathrm{Conv}(\mathbb{R}^n, \mathbb{R}) \cap C^2(\mathbb{R}^n)$*.* $\det$*: mixed discriminant*

Remark:

- μ is dually epi-translation invariant:

$$\mu(f + \ell) = \mu(f) \quad \text{for } f \in \mathrm{Conv}(\mathbb{R}^n, \mathbb{R}), \ell : \mathbb{R}^n \to \mathbb{R} \text{ affine}$$

- μ is k-homogeneous: $\mu(tf) = t^k \mu(f)$, $t \geq 0$

Dually epi-translation invariant valuations

F: locally convex vector space

Definition

$\mathrm{VConv}(\mathbb{R}^n, F)$: continuous, dually epi-translation invariant valuations $\mu : \mathrm{Conv}(\mathbb{R}^n, \mathbb{R}) \to F$.

Dually epi-translation invariant valuations

F: locally convex vector space

Definition

$\mathrm{VConv}(\mathbb{R}^n, F)$: continuous, dually epi-translation invariant valuations $\mu : \mathrm{Conv}(\mathbb{R}^n, \mathbb{R}) \to F$.

Theorem (Colesanti-Ludwig-Mussnig 2020, K. 2021)

$$\mathrm{VConv}(\mathbb{R}^n, F) = \bigoplus_{k=0}^{n} \mathrm{VConv}_k(\mathbb{R}^n, F)$$

$\mathrm{VConv}_k(\mathbb{R}^n, F)$: k-homogeneous valuations, $\mu(tf) = t^k \mu(f)$

Valuations of degree n

Theorem (Colesanti-Ludwig-Mussnig 2020)

$\mu \in \mathrm{VConv}_n(\mathbb{R}^n, \mathbb{R})$ *iff there exists* $\zeta \in C_c(\mathbb{R}^n)$ *s.t.*

$$\mu(f) = \int_{\mathbb{R}^n} \zeta(x) d\mathrm{MA}(f)[x].$$

Valuations of degree n

Theorem (Colesanti-Ludwig-Mussnig 2020)

$\mu \in \mathrm{VConv}_n(\mathbb{R}^n, \mathbb{R})$ *iff there exists* $\zeta \in C_c(\mathbb{R}^n)$ *s.t.*

$$\mu(f) = \int_{\mathbb{R}^n} \zeta(x) d\mathrm{MA}(f)[x].$$

Theorem (K. 2023+)

Let $\Psi \in \mathrm{VConv}_n(\mathbb{R}^n, \mathcal{M}(\mathbb{R}^n))$

Valuations of degree n

Theorem (Colesanti-Ludwig-Mussnig 2020)

$\mu \in \mathrm{VConv}_n(\mathbb{R}^n, \mathbb{R})$ *iff there exists* $\zeta \in C_c(\mathbb{R}^n)$ *s.t.*

$$\mu(f) = \int_{\mathbb{R}^n} \zeta(x) d\mathrm{MA}(f)[x].$$

Theorem (K. 2023+)

Let $\Psi \in \mathrm{VConv}_n(\mathbb{R}^n, \mathcal{M}(\mathbb{R}^n))$ *be* **locally determined**, *that is, for* $U \subset \mathbb{R}^n$ *open,* $f, h \in \mathrm{Conv}(\mathbb{R}^n, \mathbb{R})$*:*

$$f|_U = h|_U \quad \Rightarrow \quad \Psi(f)|_U = \Psi(h)|_U.$$

Valuations of degree n

Theorem (Colesanti-Ludwig-Mussnig 2020)

$\mu \in \mathrm{VConv}_n(\mathbb{R}^n, \mathbb{R})$ *iff there exists* $\zeta \in C_c(\mathbb{R}^n)$ *s.t.*

$$\mu(f) = \int_{\mathbb{R}^n} \zeta(x) d\mathrm{MA}(f)[x].$$

Theorem (K. 2023+)

Let $\Psi \in \mathrm{VConv}_n(\mathbb{R}^n, \mathcal{M}(\mathbb{R}^n))$ *be* ***locally determined****, that is, for* $U \subset \mathbb{R}^n$ *open,* $f, h \in \mathrm{Conv}(\mathbb{R}^n, \mathbb{R})$*:*

$$f|_U = h|_U \quad \Rightarrow \quad \Psi(f)|_U = \Psi(h)|_U.$$

Then there exists $\psi \in C(\mathbb{R}^n)$ *s.t. for* $B \subset \mathbb{R}^n$ *bounded Borel set*

$$\Psi(f)[B] = \int_B \psi(x) d\mathrm{MA}(f)[x].$$

Characterization of the real Monge-Ampère operator

$\Psi \in \mathrm{VConv}(\mathbb{R}^n, \mathcal{M}(\mathbb{R}^n))$ is **translation equivariant** iff for $x \in \mathbb{R}^n$, $f \in \mathrm{Conv}(\mathbb{R}^n, \mathbb{R})$:

$$\Psi(f(\cdot + x))[B] = \Psi(f)[B + x] \quad \text{for all bounded Borel sets } B \subset \mathbb{R}^n.$$

Characterization of the real Monge-Ampère operator

$\Psi \in \mathrm{VConv}(\mathbb{R}^n, \mathcal{M}(\mathbb{R}^n))$ is **translation equivariant** iff for $x \in \mathbb{R}^n$, $f \in \mathrm{Conv}(\mathbb{R}^n, \mathbb{R})$:

$$\Psi(f(\cdot + x))[B] = \Psi(f)[B + x] \quad \text{for all bounded Borel sets } B \subset \mathbb{R}^n.$$

Corollary (K. 2023+)

$\Psi \in \mathrm{VConv}_n(\mathbb{R}^n, \mathcal{M}(\mathbb{R}^n))$ *is locally determined and translation equivariant iff* $\Psi = c \cdot \mathrm{MA}$ *for some* $c \in \mathbb{R}$.

Further characterization results: Li-Ludwig 2023+

Translation equivariant Monge-Ampère operators

Definition

$$\mathrm{MAVal}(\mathbb{R}^n) := \{\Psi \in \mathrm{VConv}(\mathbb{R}^n, \mathcal{M}(\mathbb{R}^n)) : \ \Psi \text{ locally determined and translation equivariant}\}$$

Translation equivariant Monge-Ampère operators

Definition

$$\mathrm{MAVal}(\mathbb{R}^n) := \{\Psi \in \mathrm{VConv}(\mathbb{R}^n, \mathcal{M}(\mathbb{R}^n)) : \ \Psi \text{ locally determined and translation equivariant}\}$$

$\mu \in \mathrm{MAVal}_k(\mathbb{R}^n) :\Leftrightarrow \mu(tf) = t^k \mu(f)$ for $t \geq 0$

Translation equivariant Monge-Ampère operators

Definition

$$\mathrm{MAVal}(\mathbb{R}^n) := \{\Psi \in \mathrm{VConv}(\mathbb{R}^n, \mathcal{M}(\mathbb{R}^n)) : \ \Psi \text{ locally determined and translation equivariant}\}$$

$\mu \in \mathrm{MAVal}_k(\mathbb{R}^n) :\Leftrightarrow \mu(tf) = t^k \mu(f)$ for $t \geq 0$

Corollary

$$\mathrm{MAVal}(\mathbb{R}^n) = \bigoplus_{k=0}^{n} \mathrm{MAVal}_k(\mathbb{R}^n)$$

- $\mathrm{MAVal}_0(\mathbb{R}^n)$ is spanned by the Lebesgue measure.
- $\mathrm{MAVal}_n(\mathbb{R}^n)$ is spanned by MA.

Construction of valuations with the differential cycle

- For $f \in C^2(\mathbb{R}^n)$

$$\mathrm{graph}(df) := \{(x, df(x)) \in \mathbb{R}^n \times (\mathbb{R}^n)^* : x \in \mathbb{R}^n\} \subset T^*\mathbb{R}^n$$

is an oriented C^1-submanifold of the cotangent bundle $\pi : T^*\mathbb{R}^n \to \mathbb{R}^n$.

Construction of valuations with the differential cycle

- For $f \in C^2(\mathbb{R}^n)$

  $$\mathrm{graph}(df) := \{(x, df(x)) \in \mathbb{R}^n \times (\mathbb{R}^n)^* : x \in \mathbb{R}^n\} \subset T^*\mathbb{R}^n$$

  is an oriented C^1-submanifold of the cotangent bundle $\pi : T^*\mathbb{R}^n \to \mathbb{R}^n$.
- Fu 1989: Considered as an n-dimensional integral current, this extends to all convex functions. $\Rightarrow$ **differential cycle** $D(f)$ of f

Construction of valuations with the differential cycle

- For $f \in C^2(\mathbb{R}^n)$

$$\mathrm{graph}(df) := \{(x, df(x)) \in \mathbb{R}^n \times (\mathbb{R}^n)^* : x \in \mathbb{R}^n\} \subset T^*\mathbb{R}^n$$

  is an oriented C^1-submanifold of the cotangent bundle $\pi : T^*\mathbb{R}^n \to \mathbb{R}^n$.
- Fu 1989: Considered as an n-dimensional integral current, this extends to all convex functions. $\Rightarrow$ **differential cycle** $D(f)$ of f

Theorem (K. 2023+)

For any differential form $\tau \in \Lambda^{n-k,k} := \Lambda^{n-k}\mathbb{R}^n \otimes \Lambda^k(\mathbb{R}^n)^*$,

$$\Psi_\tau(f)[B] := D(f)\left[1_{\pi^{-1}(B)}\tau\right] = \int_{\mathrm{graph}(df)\cap\pi^{-1}(B)} \tau$$

defines an element of $\mathrm{MAVal}_k(\mathbb{R}^n)$.

Construction of valuations with the differential cycle

Example

For standard coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ on $T^*\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n$

$$\int_B \det(D^2 f(x))dx = \int_{\mathrm{graph}(df)\cap\pi^{-1}(B)} dy_1 \wedge \cdots \wedge dy_n$$

Construction of valuations with the differential cycle

Example

For standard coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ on $T^*\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n$

$$\int_B \det(D^2 f(x))dx = \int_{\mathrm{graph}(df)\cap\pi^{-1}(B)} dy_1 \wedge \cdots \wedge dy_n$$

and

$$\int_B \Delta f(x)dx = \int_{\mathrm{graph}(df)\cap\pi^{-1}(B)} \sum_{i=1}^n dx_1 \ldots dx_{i-1} \wedge dy_i \wedge dx_{i+1} \ldots dx_n$$

for $f \in C^2(\mathbb{R}^n)$.

Translation equivariant Monge-Ampère operators

Theorem (K. 2023+)

For a continuous map $\Psi : \mathrm{Conv}(\mathbb{R}^n, \mathbb{R}) \to \mathcal{M}(\mathbb{R}^n)$ *the following are equivalent:*

1. $\Psi \in \mathrm{MAVal}_k(\mathbb{R}^n)$.

Translation equivariant Monge-Ampère operators

Theorem (K. 2023+)

For a continuous map $\Psi : \mathrm{Conv}(\mathbb{R}^n, \mathbb{R}) \to \mathcal{M}(\mathbb{R}^n)$ *the following are equivalent:*

1. $\Psi \in \mathrm{MAVal}_k(\mathbb{R}^n)$.
2. $\Psi(f)[B] = D(f)[1_{\pi^{-1}(B)}\tau]$ *for some* $\tau \in \Lambda^{n-k,k}$.

Translation equivariant Monge-Ampère operators

Theorem (K. 2023+)

For a continuous map $\Psi : \mathrm{Conv}(\mathbb{R}^n, \mathbb{R}) \to \mathcal{M}(\mathbb{R}^n)$ *the following are equivalent:*

1. $\Psi \in \mathrm{MAVal}_k(\mathbb{R}^n)$.
2. $\Psi(f)[B] = D(f)[1_{\pi^{-1}(B)}\tau]$ *for some* $\tau \in \Lambda^{n-k,k}$.
3. Ψ *is a linear combination of the mixed Monge-Ampère operators*

$$f \mapsto \mathrm{MA}(f[k], A_1, \ldots, A_{n-k}), \quad A_j \text{ quadratic polynomial.}$$

Translation equivariant Monge-Ampère operators

Theorem (K. 2023+)

For a continuous map $\Psi : \mathrm{Conv}(\mathbb{R}^n, \mathbb{R}) \to \mathcal{M}(\mathbb{R}^n)$ *the following are equivalent:*

1. $\Psi \in \mathrm{MAVal}_k(\mathbb{R}^n)$.
2. $\Psi(f)[B] = D(f)[1_{\pi^{-1}(B)}\tau]$ *for some* $\tau \in \Lambda^{n-k,k}$.
3. Ψ *is a linear combination of the mixed Monge-Ampère operators*

$$f \mapsto \mathrm{MA}(f[k], A_1, \dots, A_{n-k}), \quad A_j \text{ quadratic polynomial.}$$

In particular, $\dim \mathrm{MAVal}_k(\mathbb{R}^n) = \binom{n}{k}^2 - \binom{n}{k-1}\binom{n}{n-k-1}$.

Restrictions to subspaces

Restrictions to subspaces

For $E \in \mathrm{Gr}_k(\mathbb{R}^n)$, $\Psi \in \mathrm{MAVal}_k(\mathbb{R}^n)$, consider

$$f_E \mapsto \Psi(\pi_E^* f) \in \mathcal{M}(\mathbb{R}^n) = \mathcal{M}(E \oplus E^\perp),$$

$\pi_E : \mathbb{R}^n \to E$ orthogonal projection, $f_E \in \mathrm{Conv}(E, \mathbb{R})$.

Restrictions to subspaces

For $E \in \mathrm{Gr}_k(\mathbb{R}^n)$, $\Psi \in \mathrm{MAVal}_k(\mathbb{R}^n)$, consider

$$f_E \mapsto \Psi(\pi_E^* f) \in \mathcal{M}(\mathbb{R}^n) = \mathcal{M}(E \oplus E^\perp),$$

$\pi_E : \mathbb{R}^n \to E$ orthogonal projection, $f_E \in \mathrm{Conv}(E, \mathbb{R})$.

- $\pi_E^* f_E$ is invariant w.r.t. translations in $E^\perp$

Restrictions to subspaces

For $E \in \mathrm{Gr}_k(\mathbb{R}^n)$, $\Psi \in \mathrm{MAVal}_k(\mathbb{R}^n)$, consider

$$f_E \mapsto \Psi(\pi_E^* f) \in \mathcal{M}(\mathbb{R}^n) = \mathcal{M}(E \oplus E^\perp),$$

$\pi_E : \mathbb{R}^n \to E$ orthogonal projection, $f_E \in \mathrm{Conv}(E, \mathbb{R})$.

- $\pi_E^* f_E$ is invariant w.r.t. translations in $E^\perp$

$\Rightarrow$ $\Psi(\pi_E^* f)$ is a translation invariant measure in the second component.

Restrictions to subspaces

For $E \in \mathrm{Gr}_k(\mathbb{R}^n)$, $\Psi \in \mathrm{MAVal}_k(\mathbb{R}^n)$, consider

$$f_E \mapsto \Psi(\pi_E^* f) \in \mathcal{M}(\mathbb{R}^n) = \mathcal{M}(E \oplus E^\perp),$$

$\pi_E : \mathbb{R}^n \to E$ orthogonal projection, $f_E \in \mathrm{Conv}(E, \mathbb{R})$.

- $\pi_E^* f_E$ is invariant w.r.t. translations in $E^\perp$

$\Rightarrow$ $\Psi(\pi_E^* f)$ is a translation invariant measure in the second component.

$\Rightarrow$ $\Psi(\pi_E^* f)[A \times B] = \tilde{\Psi}(f)[A] \cdot \mathrm{vol}_{E^\perp}(B)$, $A \subset E$, $B \subset E^\perp$

Restrictions to subspaces

For $E \in \mathrm{Gr}_k(\mathbb{R}^n)$, $\Psi \in \mathrm{MAVal}_k(\mathbb{R}^n)$, consider

$$f_E \mapsto \Psi(\pi_E^* f) \in \mathcal{M}(\mathbb{R}^n) = \mathcal{M}(E \oplus E^\perp),$$

$\pi_E : \mathbb{R}^n \to E$ orthogonal projection, $f_E \in \mathrm{Conv}(E, \mathbb{R})$.

- $\pi_E^* f_E$ is invariant w.r.t. translations in $E^\perp$

$\Rightarrow$ $\Psi(\pi_E^* f)$ is a translation invariant measure in the second component.

$\Rightarrow$ $\Psi(\pi_E^* f)[A \times B] = \tilde{\Psi}(f)[A] \cdot \mathrm{vol}_{E^\perp}(B)$, $A \subset E$, $B \subset E^\perp$

- $\tilde{\Psi} \in \mathrm{MAVal}_k(E) \cong \mathbb{R} \cdot \mathrm{MA}_E$

Restrictions to subspaces

For $E \in \mathrm{Gr}_k(\mathbb{R}^n)$, $\Psi \in \mathrm{MAVal}_k(\mathbb{R}^n)$, consider

$$f_E \mapsto \Psi(\pi_E^* f) \in \mathcal{M}(\mathbb{R}^n) = \mathcal{M}(E \oplus E^\perp),$$

$\pi_E : \mathbb{R}^n \to E$ orthogonal projection, $f_E \in \mathrm{Conv}(E, \mathbb{R})$.

- $\pi_E^* f_E$ is invariant w.r.t. translations in $E^\perp$

$\Rightarrow$ $\Psi(\pi_E^* f)$ is a translation invariant measure in the second component.

$\Rightarrow$ $\Psi(\pi_E^* f)[A \times B] = \tilde{\Psi}(f)[A] \cdot \mathrm{vol}_{E^\perp}(B)$, $A \subset E$, $B \subset E^\perp$

- $\tilde{\Psi} \in \mathrm{MAVal}_k(E) \cong \mathbb{R} \cdot \mathrm{MA}_E$

Lemma

Let $\Psi \in \mathrm{MAVal}_k(\mathbb{R}^n)$. For $E \in \mathrm{Gr}_k(\mathbb{R}^n)$ there exists $\mathrm{Kl}_\Psi(E) \in \mathbb{R}$ s.t.

$$\Psi(\pi_E^* f) = \mathrm{Kl}_\Psi(E) \cdot \mathrm{MA}_E(f) \otimes \mathrm{vol}_{E^\perp} \quad \textit{for all } f \in \mathrm{Conv}(E, \mathbb{R}).$$

The Goodey-Weil distributions

For $\mu \in \mathrm{VConv}_k(\mathbb{R}^n, \mathbb{R})$: $\bar{\mu}(f_1, \ldots, f_k) := \frac{1}{k!} \frac{\partial^k}{\partial\lambda_1 \ldots \partial\lambda_k}\Big|_0 \mu\left(\sum_{i=1}^k \lambda_i f_i\right)$

The Goodey-Weil distributions

For $\mu \in \mathrm{VConv}_k(\mathbb{R}^n, \mathbb{R})$: $\bar{\mu}(f_1, \dots, f_k) := \frac{1}{k!} \frac{\partial^k}{\partial\lambda_1 \dots \partial\lambda_k}\Big|_0 \mu\left(\sum_{i=1}^k \lambda_i f_i\right)$ satisfies

- $\bar{\mu}$ is symmetric,
- $\bar{\mu}(f, \dots, f) = \mu(f)$,

The Goodey-Weil distributions

For $\mu \in \mathrm{VConv}_k(\mathbb{R}^n, \mathbb{R})$: $\bar{\mu}(f_1, \dots, f_k) := \frac{1}{k!} \frac{\partial^k}{\partial\lambda_1 \dots \partial\lambda_k}\Big|_0 \mu\left(\sum_{i=1}^k \lambda_i f_i\right)$ satisfies

- $\bar{\mu}$ is symmetric,
- $\bar{\mu}(f, \dots, f) = \mu(f)$,
- $\bar{\mu}(f + \lambda h, f_2, \dots, f_k) = \bar{\mu}(f, f_2, \dots, f_k) + \lambda\bar{\mu}(h, f_2, \dots, f_k)$, $\lambda \geq 0$.

The Goodey-Weil distributions

For $\mu \in \mathrm{VConv}_k(\mathbb{R}^n, \mathbb{R})$: $\bar{\mu}(f_1, \ldots, f_k) := \frac{1}{k!} \frac{\partial^k}{\partial\lambda_1 \ldots \partial\lambda_k}\Big|_0 \mu\left(\sum_{i=1}^k \lambda_i f_i\right)$ satisfies

- $\bar{\mu}$ is symmetric,
- $\bar{\mu}(f, \ldots, f) = \mu(f)$,
- $\bar{\mu}(f + \lambda h, f_2, \ldots, f_k) = \bar{\mu}(f, f_2, \ldots, f_k) + \lambda \bar{\mu}(h, f_2, \ldots, f_k)$, $\lambda \geq 0$.

Theorem (K. 2021)

For every $\mu \in \mathrm{VConv}_k(\mathbb{R}^n, \mathbb{R})$ there exists a unique symmetric distribution $\mathrm{GW}(\mu) \in \mathcal{D}((\mathbb{R}^n)^k)$ **with compact support** *such that*

$$\mathrm{GW}(\mu)[f_1 \otimes \cdots \otimes f_k] = \bar{\mu}(f_1, \ldots, f_k)$$

for all $f_1, \ldots, f_k \in \mathrm{Conv}(\mathbb{R}^n, \mathbb{R}) \cap C^\infty(\mathbb{R}^n)$.

The Goodey-Weil distributions

For $\mu \in \mathrm{VConv}_k(\mathbb{R}^n,\mathbb{R})$: $\bar{\mu}(f_1,\dots,f_k) := \frac{1}{k!}\frac{\partial^k}{\partial\lambda_1\dots\partial\lambda_k}\Big|_0 \mu\left(\sum_{i=1}^k \lambda_i f_i\right)$ satisfies

- $\bar{\mu}$ is symmetric,
- $\bar{\mu}(f,\dots,f) = \mu(f)$,
- $\bar{\mu}(f+\lambda h, f_2,\dots,f_k) = \bar{\mu}(f,f_2,\dots,f_k) + \lambda\bar{\mu}(h,f_2,\dots,f_k)$, $\lambda \geq 0$.

Theorem (K. 2021)

For every $\mu \in \mathrm{VConv}_k(\mathbb{R}^n,\mathbb{R})$ there exists a unique symmetric distribution $\mathrm{GW}(\mu) \in \mathcal{D}((\mathbb{R}^n)^k)$ **with compact support** *such that*

$$\mathrm{GW}(\mu)[f_1 \otimes \cdots \otimes f_k] = \bar{\mu}(f_1,\dots,f_k)$$

for all $f_1,\dots,f_k \in \mathrm{Conv}(\mathbb{R}^n,\mathbb{R}) \cap C^\infty(\mathbb{R}^n)$.

- $\mathrm{GW}(\mu)[f \otimes \cdots \otimes f] = \mu(f)$ for $f \in \mathrm{Conv}(\mathbb{R}^n,\mathbb{R}) \cap C^\infty(\mathbb{R}^n)$
- original construction due to Goodey and Weil for $\mathrm{Val}_k(\mathbb{R}^n)$

The Fourier transform of Goodey-Weil distributions

Recall: $\mathrm{GW}(\mu)$ compactly supported distribution on $(\mathbb{R}^n)^k$,
$\mathrm{GW}(\mu)[f_1 \otimes \cdots \otimes f_k] = \bar{\mu}(f_1, \ldots, f_k)$, $\mu \in \mathrm{VConv}_k(\mathbb{R}^n, \mathbb{R})$

The Fourier transform of Goodey-Weil distributions

Recall: $\mathrm{GW}(\mu)$ compactly supported distribution on $(\mathbb{R}^n)^k$,
$\mathrm{GW}(\mu)[f_1 \otimes \cdots \otimes f_k] = \bar{\mu}(f_1, \ldots, f_k)$, $\mu \in \mathrm{VConv}_k(\mathbb{R}^n, \mathbb{R})$

- The Fourier transform $\mathcal{F}(\mathrm{GW}(\mu))$ is an **entire function** on $(\mathbb{C}^n)^k$:

$$\mathcal{F}(\mathrm{GW}(\mu))[z_1, \ldots, z_k] = \mathrm{GW}(\mu)[\exp(i\langle z_1, \cdot\rangle) \otimes \cdots \otimes \exp(i\langle z_k, \cdot\rangle)].$$

The Fourier transform of Goodey-Weil distributions

Recall: $\mathrm{GW}(\mu)$ compactly supported distribution on $(\mathbb{R}^n)^k$,
$\mathrm{GW}(\mu)[f_1 \otimes \cdots \otimes f_k] = \bar{\mu}(f_1, \ldots, f_k)$, $\mu \in \mathrm{VConv}_k(\mathbb{R}^n, \mathbb{R})$

- The Fourier transform $\mathcal{F}(\mathrm{GW}(\mu))$ is an **entire function** on $(\mathbb{C}^n)^k$:

$$\mathcal{F}(\mathrm{GW}(\mu))[z_1, \ldots, z_k] = \mathrm{GW}(\mu)[\exp(i\langle z_1, \cdot\rangle) \otimes \cdots \otimes \exp(i\langle z_k, \cdot\rangle)].$$

- It is determined by the restriction to $(i\mathbb{R}^n)^k$.

The Fourier transform of Goodey-Weil distributions

Recall: $\mathrm{GW}(\mu)$ compactly supported distribution on $(\mathbb{R}^n)^k$,
$\mathrm{GW}(\mu)[f_1 \otimes \cdots \otimes f_k] = \bar{\mu}(f_1, \dots, f_k)$, $\mu \in \mathrm{VConv}_k(\mathbb{R}^n, \mathbb{R})$

- The Fourier transform $\mathcal{F}(\mathrm{GW}(\mu))$ is an **entire function** on $(\mathbb{C}^n)^k$:

$$\mathcal{F}(\mathrm{GW}(\mu))[z_1, \dots, z_k] = \mathrm{GW}(\mu)[\exp(i\langle z_1, \cdot\rangle) \otimes \cdots \otimes \exp(i\langle z_k, \cdot\rangle)].$$

- It is determined by the restriction to $(i\mathbb{R}^n)^k$. For $y_1, \dots, y_k \in \mathbb{R}^n$:

$$\mathcal{F}(\mathrm{GW}(\mu))[iy_1, \dots, iy_k] = \frac{1}{k!} \frac{\partial^k}{\partial \lambda_1 \cdots \partial \lambda_k}\Big|_0 \mu\left(\sum_{j=1}^k \lambda_j \exp(-\langle y_j, \cdot\rangle)\right).$$

The Fourier transform of Goodey-Weil distributions

Recall: $\mathrm{GW}(\mu)$ compactly supported distribution on $(\mathbb{R}^n)^k$,

$$\mathrm{GW}(\mu)[f_1 \otimes \cdots \otimes f_k] = \bar{\mu}(f_1, \dots, f_k),\ \mu \in \mathrm{VConv}_k(\mathbb{R}^n, \mathbb{R})$$

- The Fourier transform $\mathcal{F}(\mathrm{GW}(\mu))$ is an **entire function** on $(\mathbb{C}^n)^k$:

$$\mathcal{F}(\mathrm{GW}(\mu))[z_1, \dots, z_k] = \mathrm{GW}(\mu)[\exp(i\langle z_1, \cdot\rangle) \otimes \cdots \otimes \exp(i\langle z_k, \cdot\rangle)].$$

- It is determined by the restriction to $(i\mathbb{R}^n)^k$. For $y_1, \dots, y_k \in \mathbb{R}^n$:

$$\mathcal{F}(\mathrm{GW}(\mu))[iy_1, \dots, iy_k] = \frac{1}{k!} \frac{\partial^k}{\partial\lambda_1 \cdots \partial\lambda_k}\Big|_0 \mu\left(\sum_{j=1}^k \lambda_j \exp(-\langle y_j, \cdot\rangle)\right).$$

- Right hand side: evaluation in functions defined on $\mathrm{span}(y_1, \dots, y_k)$.

Restriction to subspaces

1. Take the orthogonal projection $\pi_E : \mathbb{R}^n \to E$.
2. Consider the **restriction** $\pi_{E*}\mu \in \mathrm{VConv}(E, \mathbb{R})$,

$$[\pi_{E*}\mu](f) := \mu(\pi_E^* f), \quad f \in \mathrm{Conv}(E, \mathbb{R}),$$

of $\mu \in \mathrm{VConv}(\mathbb{R}^n, \mathbb{R})$ to $E \in \mathrm{Gr}_k(\mathbb{R}^n)$.

Restriction to subspaces

1. Take the orthogonal projection $\pi_E : \mathbb{R}^n \to E$.
2. Consider the **restriction** $\pi_{E*}\mu \in \mathrm{VConv}(E, \mathbb{R})$,

$$[\pi_{E*}\mu](f) := \mu(\pi_E^* f), \quad f \in \mathrm{Conv}(E, \mathbb{R}),$$

of $\mu \in \mathrm{VConv}(\mathbb{R}^n, \mathbb{R})$ to $E \in \mathrm{Gr}_k(\mathbb{R}^n)$.

Corollary

For $\mu \in \mathrm{VConv}_k(\mathbb{R}^n, \mathbb{R})$ the following are equivalent:

1. $\mu = 0$,
2. $\pi_{E*}\mu = 0$ *for all* $E \in \mathrm{Gr}_k(\mathbb{R}^n)$.

In other words, μ is uniquely determined by its restrictions.

Application to measure-valued valuations

For $\phi \in C_c(\mathbb{R}^n)$, $\Psi \in \mathrm{MAVal}_k(\mathbb{R}^n)$, consider $\Psi[\phi] \in \mathrm{VConv}_k(\mathbb{R}^n, \mathbb{R})$:

$$\Psi[\phi](f) := \int_{\mathbb{R}^n} \phi d\Psi(f).$$

Application to measure-valued valuations

For $\phi \in C_c(\mathbb{R}^n)$, $\Psi \in \mathrm{MAVal}_k(\mathbb{R}^n)$, consider $\Psi[\phi] \in \mathrm{VConv}_k(\mathbb{R}^n, \mathbb{R})$:

$$\Psi[\phi](f) := \int_{\mathbb{R}^n} \phi \, d\Psi(f).$$

For $y_1, \dots, y_k \in E$:

$$\mathcal{F}(\mathrm{GW}(\Psi[\phi]))[iy_1, \dots, iy_k] = \mathrm{GW}(\Psi[\phi])[\exp(-\langle y_1, \cdot \rangle) \otimes \cdots \otimes \exp(-\langle y_k, \cdot \rangle)]$$

Application to measure-valued valuations

For $\phi \in C_c(\mathbb{R}^n)$, $\Psi \in \mathrm{MAVal}_k(\mathbb{R}^n)$, consider $\Psi[\phi] \in \mathrm{VConv}_k(\mathbb{R}^n, \mathbb{R})$:

$$\Psi[\phi](f) := \int_{\mathbb{R}^n} \phi d\Psi(f).$$

For $y_1, \dots, y_k \in E$:

$$\begin{aligned} \mathcal{F}(\mathrm{GW}(\Psi[\phi]))[iy_1, \dots, iy_k] &= \mathrm{GW}(\Psi[\phi])[\exp(-\langle y_1, \cdot \rangle) \otimes \cdots \otimes \exp(-\langle y_k, \cdot \rangle)] \\ &= \mathcal{F}(\mathrm{GW}(\pi_{E*}\Psi[\phi]))[iy_1, \dots, iy_k] \end{aligned}$$

Application to measure-valued valuations

For $\phi \in C_c(\mathbb{R}^n)$, $\Psi \in \mathrm{MAVal}_k(\mathbb{R}^n)$, consider $\Psi[\phi] \in \mathrm{VConv}_k(\mathbb{R}^n, \mathbb{R})$:

$$\Psi[\phi](f) := \int_{\mathbb{R}^n} \phi d\Psi(f).$$

For $y_1, \dots, y_k \in E$:

$$\begin{aligned}
\mathcal{F}(\mathrm{GW}(\Psi[\phi]))[iy_1, \dots, iy_k] &= \mathrm{GW}(\Psi[\phi])[\exp(-\langle y_1, \cdot\rangle) \otimes \cdots \otimes \exp(-\langle y_k, \cdot\rangle)] \\
&= \mathcal{F}(\mathrm{GW}(\pi_{E*}\Psi[\phi]))[iy_1, \dots, iy_k] \\
&= \mathcal{F}(\mathrm{GW}(\mathrm{Kl}_\Psi(E)\mathrm{MA}_E \otimes \mathrm{vol}_{E^\perp}[\phi]))[iy_1, \dots, iy_k]
\end{aligned}$$

Application to measure-valued valuations

For $\phi \in C_c(\mathbb{R}^n)$, $\Psi \in \mathrm{MAVal}_k(\mathbb{R}^n)$, consider $\Psi[\phi] \in \mathrm{VConv}_k(\mathbb{R}^n, \mathbb{R})$:

$$\Psi[\phi](f) := \int_{\mathbb{R}^n} \phi d\Psi(f).$$

For $y_1, \dots, y_k \in E$:

$$\begin{aligned}
\mathcal{F}(\mathrm{GW}(\Psi[\phi]))[iy_1, \dots, iy_k] &= \mathrm{GW}(\Psi[\phi])[\exp(-\langle y_1, \cdot \rangle) \otimes \cdots \otimes \exp(-\langle y_k, \cdot \rangle)] \\
&= \mathcal{F}(\mathrm{GW}(\pi_{E*}\Psi[\phi]))[iy_1, \dots, iy_k] \\
&= \mathcal{F}(\mathrm{GW}(\mathrm{Kl}_\Psi(E)\mathrm{MA}_E \otimes \mathrm{vol}_{E^\perp}[\phi]))[iy_1, \dots, iy_k]
\end{aligned}$$

Corollary

For $y_1, \dots, y_k \in E \in \mathrm{Gr}_k(\mathbb{R}^n)$:

$$\mathcal{F}(\mathrm{GW}(\Psi[\phi]))[iy_1, \dots, iy_k] = \mathrm{Kl}_\Psi(E) \frac{\det_k \left(\langle y_j, y_l \rangle\right)_{j,l=1}^k}{(-1)^k k!} \mathcal{F}(\phi) \left(\sum_{j=1}^k iy_j \right).$$

From valuations to polynomials

Theorem (K. 2023+)

For every $\Psi \in \mathrm{MAVal}_k(\mathbb{R}^n)$ *there exists a unique polynomial* $Q[\Psi]$ *on* $(\mathbb{C}^n)^k$ *of degree at most* $2k$ *such that for all* $\phi \in C_c(\mathbb{R}^n)$

$$\mathcal{F}(\mathrm{GW}(\Psi[\phi]))[z_1, \ldots, z_k] = Q[\Psi](z_1, \ldots, z_k)\mathcal{F}(\phi)\left(\sum_{j=1}^{k} z_j\right).$$

From valuations to polynomials

Theorem (K. 2023+)

For every $\Psi \in \mathrm{MAVal}_k(\mathbb{R}^n)$ *there exists a unique polynomial* $Q[\Psi]$ *on* $(\mathbb{C}^n)^k$ *of degree at most* $2k$ *such that for all* $\phi \in C_c(\mathbb{R}^n)$

$$\mathcal{F}(\mathrm{GW}(\Psi[\phi]))[z_1, \dots, z_k] = Q[\Psi](z_1, \dots, z_k)\mathcal{F}(\phi)\left(\sum_{j=1}^k z_j\right).$$

In fact, $Q[\Psi]$ is a symmetric homogeneous polynomial of degree $(2, \dots, 2) \in \mathbb{N}^k$ that vanishes if its arguments are linearly dependent.

From valuations to polynomials

Theorem (K. 2023+)

For every $\Psi \in \mathrm{MAVal}_k(\mathbb{R}^n)$ *there exists a unique polynomial* $Q[\Psi]$ *on* $(\mathbb{C}^n)^k$ *of degree at most* $2k$ *such that for all* $\phi \in C_c(\mathbb{R}^n)$

$$\mathcal{F}(\mathrm{GW}(\Psi[\phi]))[z_1, \dots, z_k] = Q[\Psi](z_1, \dots, z_k)\mathcal{F}(\phi)\left(\sum_{j=1}^k z_j\right).$$

In fact, $Q[\Psi]$ is a symmetric homogeneous polynomial of degree $(2, \dots, 2) \in \mathbb{N}^k$ that vanishes if its arguments are linearly dependent.

Corollary

$Q[\Psi](z_1, \dots, z_k) = P(\sum_{i=1}^k z_i \cdot z_i^T)$, *where* P *is a linear combination of* $(k \times k)$*-minors.*

From valuations to polynomials

Theorem (K. 2023+)

For every $\Psi \in \mathrm{MAVal}_k(\mathbb{R}^n)$ *there exists a unique polynomial* $Q[\Psi]$ *on* $(\mathbb{C}^n)^k$ *of degree at most* $2k$ *such that for all* $\phi \in C_c(\mathbb{R}^n)$

$$\mathcal{F}(\mathrm{GW}(\Psi[\phi]))[z_1, \dots, z_k] = Q[\Psi](z_1, \dots, z_k)\mathcal{F}(\phi)\left(\sum_{j=1}^{k} z_j\right).$$

In fact, $Q[\Psi]$ is a symmetric homogeneous polynomial of degree $(2, \dots, 2) \in \mathbb{N}^k$ that vanishes if its arguments are linearly dependent.

Corollary

$Q[\Psi](z_1, \dots, z_k) = P(\sum_{i=1}^{k} z_i \cdot z_i^T)$, *where* P *is a linear combination of* $(k \times k)$*-minors.*

$(k \times k)$-minors: irreducible representation of $\mathrm{GL}(n, \mathbb{R})$.

Irreducibility under $\mathrm{GL}(n, \mathbb{R})$

Corollary

The map $\mathrm{MAVal}_k(\mathbb{R}^n) \ni \Psi \mapsto Q[\Psi]$ *defines a* $\mathrm{GL}(n, \mathbb{R})$*-equivariant map into an irreducible representation of* $\mathrm{GL}(n, \mathbb{R})$.

Irreducibility under $\mathrm{GL}(n, \mathbb{R})$

Corollary

The map $\mathrm{MAVal}_k(\mathbb{R}^n) \ni \Psi \mapsto Q[\Psi]$ *defines a* $\mathrm{GL}(n, \mathbb{R})$*-equivariant map into an irreducible representation of* $\mathrm{GL}(n, \mathbb{R})$*. Consequently,* $\mathrm{MAVal}_k(\mathbb{R}^n)$ *is an irreducible representation of* $\mathrm{GL}(n, \mathbb{R})$*.*

Irreducibility under $\mathrm{GL}(n, \mathbb{R})$

Corollary

The map $\mathrm{MAVal}_k(\mathbb{R}^n) \ni \Psi \mapsto Q[\Psi]$ *defines a* $\mathrm{GL}(n, \mathbb{R})$*-equivariant map into an irreducible representation of* $\mathrm{GL}(n, \mathbb{R})$*. Consequently,* $\mathrm{MAVal}_k(\mathbb{R}^n)$ *is an irreducible representation of* $\mathrm{GL}(n, \mathbb{R})$*.*

Lemma

The following valuations span $\mathrm{GL}(n, \mathbb{R})$*-invariant subspaces of* $\mathrm{MAVal}_k(\mathbb{R}^n)$*:*

1. $f \mapsto D(f)[1_{\pi^{-1}(\cdot)}\tau]$, $\tau \in \Lambda^{n-k,k}$.
2. $f \mapsto \mathrm{MA}(f[k], A_1, \dots, A_{n-k})$, A_j *quadratic polynomial.*

Thus, these spaces coincide with $\mathrm{MAVal}_k(\mathbb{R}^n)$.

Translation equivariant Monge-Ampère operators

Theorem (K. 2023+)

For a continuous map $\Psi : \mathrm{Conv}(\mathbb{R}^n, \mathbb{R}) \to \mathcal{M}(\mathbb{R}^n)$ *the following are equivalent:*

1. $\Psi \in \mathrm{MAVal}_k(\mathbb{R}^n)$.
2. $\Psi(f)[B] = D(f)[1_{\pi^{-1}(B)}\tau]$ *for some* $\tau \in \Lambda^{n-k,k}$.
3. Ψ *is a linear combination of the mixed Monge-Ampère operators*

$$f \mapsto \mathrm{MA}(f[k], A_1, \dots, A_{n-k}), \quad A_j \text{ quadratic polynomial.}$$

Thank you for your attention!