Fluctuations of Dynamic Convex Hulls

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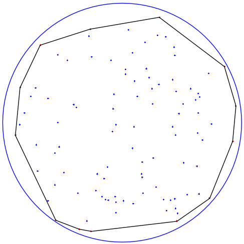
Salzburg Sept 25-29, 2023

Convex Geometry and Geometric Probability

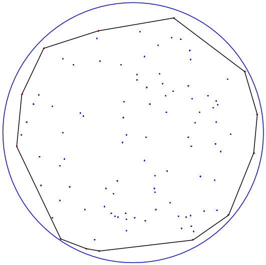
Introduction

- K: smooth convex body in $\mathbb{R}^d$.
- K_t: convex hull of t i.i.d. uniform points in K.
- $K := \mathbb{B}^2, t = 100$:



- Boundary: ∂K_t . As t increases, new points appear, creating new facets which may subsume existing facets.
- Dynamics: 'peaks' are smoothed, 'valleys' are filled in.

Introduction: The surface growth model $\partial(tK_t)$



We consider the surface growth model given by $\partial(tK_t)$. Goals:

- Find scaling limit for $\partial(tK_t)$ as $t \to \infty$.
- Distributional convergence of the process of 'heights' of convex hull boundary? Does process of heights satisfy PDEs reminiscent of those governing surface growth models?

Statistics of facets in $\partial(tK_t)$

- $\mathcal{F}_t$: Facet chosen at random from the facets of $\partial(tK_t)$.
- $\mathrm{dist}(\mathcal{F}_t, \partial(tK))$: distance between the hyperplane containing $\mathcal{F}_t$ and nearest supporting hyperplane on tK.
- **Prop.** (expected facet length and facet distance; $K \subset \mathbb{R}^2$ is C^3) $\mathbb{E}[\mathrm{length}(\mathcal{F}_t)] = Ct^{\frac{2}{3}}(1 + o(1))$; $\mathbb{E}[\mathrm{dist}(\mathcal{F}_t, \partial(tK))] = Ct^{\frac{1}{3}}(1 + o(1))$.
- **Theorem** (convergence in distribution of facet distance, $K = \mathbb{B}^2$)

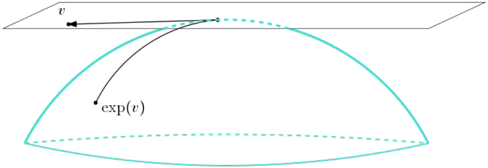
$$t^{\frac{-1}{3}} \mathrm{dist}(\mathcal{F}_t, \partial(t\mathbb{B}^2)) \xrightarrow{\mathcal{D}} \xi, \quad t \to \infty.$$

$$\mathbb{P}(\xi \geq s) \sim Cs^{\frac{3}{2}} \exp\left(-\frac{4\sqrt{2}}{3\pi} s^{\frac{3}{2}}\right), \quad s \to \infty.$$

- Random planar growth models exhibit $\frac{1}{3}, \frac{2}{3}$ scaling and follow Tracy-Widom distributions.

Exponential map

- To consider process convergence of the ‘height’ of tK_t it will be convenient to express the radius vector function of the convex hull as a process on $\mathbb{R}^{d-1}$.
- We will do this using the exponential map:



Convergence of one parameter height process for $t\mathbb{B}_t^d$

· **Theorem** Let $h(z,t)$ be the radius vector function (height) of $t\mathbb{B}_t^d$ in the direction $z \in t\mathbb{S}^{d-1}$. Then as $t \to \infty$

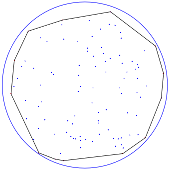
$$\left(\frac{h(t^{\frac{d}{d+1}} \exp_{t^{1/(d+1)}\mathbb{S}^{d-1}}(x), t) - t}{t^{\frac{d-1}{d+1}}} \right)_{|x| \le t^{1/(d+1)}} \xrightarrow{\mathbb{P}} \{H(x)\}_{x \in \mathbb{R}^{d-1}} \qquad (*)$$

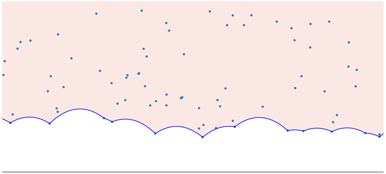
· The scaling limit $H(\cdot)$ is the 'Burgers' festoon', defined on next slide.

· $d = 2$: Fluctuations are on the order $t^{\frac{1}{3}}$, space scaled by $t^{\frac{2}{3}}$. The characteristic triplet 'Fluctuations:Space:Time' scales like $1:2:3$.

· (*) Convergence in space of cont. fcts. on compacts with sup norm.

Scaling limit: Burgers' festoon ($d = 2$)





- Down parabola with apex at $(x_0, h_0) \in \mathbb{R} \times \mathbb{R}^+$:

$$\Pi^{\downarrow}(x_0, h_0) := \{(x, h) \in \mathbb{R} \times \mathbb{R},\ h - h_0 \leq -\frac{|x - x_0|^2}{2}\}.$$

- $\mathcal{P}$: rate 1 Poisson pt process on $\mathbb{R} \times \mathbb{R}^+$. Burgers' festoon H:

$$H(x) := \sup_{(x_0,h_0) \in \mathbb{R}\times\mathbb{R}^+,\ \Pi^{\downarrow}(x_0,h_0) \cap \mathcal{P} = \emptyset} \left(h_0 - \frac{|x - x_0|^2}{2}\right).$$

- Parabolic faces in H are the re-scaled asymptotic images of the facets of K_t.

KPZ universality

- All models in the 1-dimensional Kardar–Parisi–Zhang (KPZ) universality class (random growth models, last passage percolation and deposition models, ...) have a 'height function' h(x, t) which is conjectured to converge at large time and length scales ($n \to \infty$), under the KPZ 1:2:3 scaling

$$n^{-\frac{1}{3}}(h(n^{\frac{2}{3}}x, n^{\frac{3}{3}}t) - C_n t)$$

  to a universal two parameter fluctuating field $H(x,t)$ which does not depend on the particular model, but does depend on the initial data.
- KPZ universality class should not be confused with KPZ equation

$$\frac{dh}{dt} = \nu \frac{d^2h}{dx^2} + c(\frac{dh}{dx})^2 + \text{Gaussian white noise},$$

  the canonical continuum equation for random growth.

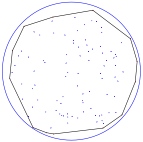
KPZ universality of dynamic convex hull

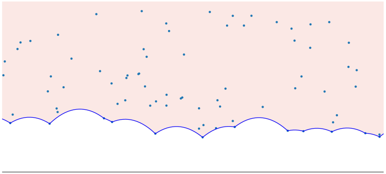
- Put $h(z,t)$ to be the radius vector function (height) of $t\mathbb{B}_t^2$ in the direction $z \in t\mathbb{S}$.
- **Theorem** (convergence of two parameter height process) At large time and length scales ($n \to \infty$), the re-scaled two-parameter height process satisfies for fixed $t > 0$

$$\left(\frac{h(n^{\frac{2}{3}} \exp_{n^{1/3}t\mathbb{S}}(x), n^{\frac{3}{3}}t) - nt}{n^{\frac{1}{3}}} \right)_{|x| \leq n^{1/3}t} \xrightarrow{\mathbb{P}} (H(x,t))_{x \in \mathbb{R}}.$$

- The dynamic convex hull tK_t is a growth model belonging to a KPZ sub-universality class. It would belong to KPZ universality class if the field $H(x,t)$ also contained an Airy process.

Two parameter scaling limit: Burgers' festoon





- Down parabola with apex at $(x_0, h_0) \in \mathbb{R} \times \mathbb{R}^+$:

$$\Pi_t^{\downarrow}(x_0, h_0) := \{(x, h) \in \mathbb{R} \times \mathbb{R},\ h - h_0 \leq -\frac{|x - x_0|^2}{2t}\}.$$

- $\mathcal{P}$: rate 1 Poisson pt process on $\mathbb{R} \times \mathbb{R}^+$. Burgers' festoon H:

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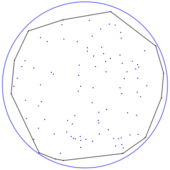
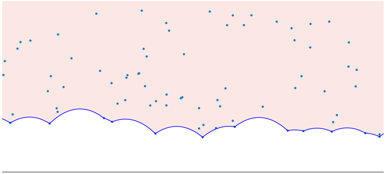
- Burgers: Space derivative of rescaled height function converges to solution of inviscid Burgers' PDE
$u_t + u \cdot u_x = \nu u_{xx},\ \nu \to 0;\quad u(x, t)|_{t=0} = \sum_{(x_i, h_i) \in \mathcal{P}} h_i \delta_{x_i}(x).$

Proof ideas: $K = \mathbb{B}^2$

We need to find the right scaling transform mapping $\partial(tK_t)$ to the upper half plane. The overly simplified approach has these ingredients:

- Define for each t a parabolic scaling transform $T^{(t)} : t\mathbb{B}^2 \to \mathbb{R} \times \mathbb{R}^+$





- $T^{(t)}$ maps the edges of $\partial(tK_t)$ to curves in the upper half-plane which are nearly parabolic, and which become parabolic in the limit $t \to \infty$.
- Via a coupling we show that the t i.i.d. uniform points map to points in a rectangle in the upper half-plane of area t and in the limit this point process converges to a Poisson point process in the upper half-plane.

Summary

- Limit distribution of height of a facet chosen at random has right-sided Tracy-Widom like tails in $d = 2$
- Rescaled one parameter height process of dynamic convex hull tK_t converges as $t \to \infty$ to Burgers' festoon; the triplet 'fluctuations: space: time' exhibits $1 : 2 : 3$ scaling
- The two parameter height process of tK_t belongs to KPZ sub-universality class and converges to the dynamic Burgers' festoon

Thank you for your attention!