Coarse approximation in convex analysis

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Conference on Convex Geometry and Geometric Probability,
Paris Lodron University Salzburg

September 26, 2023

Disclaimer

Caveat emptor (Let the buyer beware)

Outline

From “Sparse” to “Coarse”

Helly-type results

Open problems

"Sparse" approximation

Consider

$$\mathrm{Id}_d = \sum_{i=1}^{m} c_i u_i \otimes u_i = \sum_{i=1}^{m} c_i u_i {u_i}^T,$$

where $c_1, \ldots, c_m$ are positive weights; $u_1, \ldots, u_m$ are unit vectors in $\mathbb{R}^d$.

“Sparse” approximation

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Problem:

For a given ε, find a smallest index set J such that

$$\frac{\mathrm{Id}_d}{1+\varepsilon} \prec \sum_{j \in J} \tilde{c}_j u_j \otimes u_j \prec (1+\varepsilon)\mathrm{Id}_d$$

for some positive weights $\{\tilde{c}_j\}_{j \in J}$

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Bourgain'95

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Qualitative Question:

What is the size of the smallest index set J such that

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Answer: Trivial, the size is d.

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Answer

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Answer or not:

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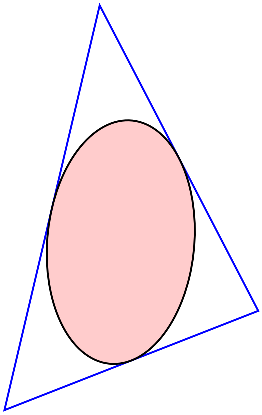
Answer or not:

... take the vectors by the "Dvoretzky–Rogers lemma"...

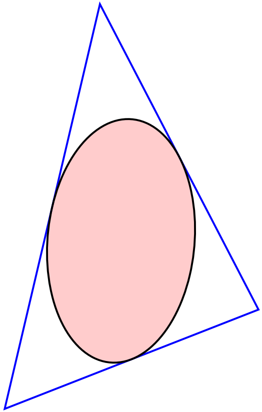
John ellipsoid

1. K is a convex body in $\mathbb{R}^d$



John ellipsoid

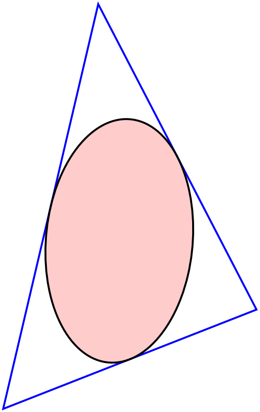
1. K is a convex body in $\mathbb{R}^d$
2. The *John ellipsoid* J_K of K is the maximal volume ellipsoid contained within K



John ellipsoid

1. K is a convex body in $\mathbb{R}^d$
2. The *John ellipsoid* J_K of K is the maximal volume ellipsoid contained within K
3. Let 0 be the center of J_K. Then

$$J_K \subset K \subset d \cdot J_K$$



John condition

$K \subset \mathbb{R}^d$ convex body.

John '48 (+ Ball'92)

Assume $\mathbf{B}^d \subseteq K$. TFAE:

1. $\mathbf{B}^d$ is the maximum volume ellipsoid contained within K
2. there are contact points $u_1, \ldots, u_m \in \operatorname{bd} \mathbf{B}^d \cap \operatorname{bd} K$ and positive weights $c_1, \ldots, c_m$ such that

$$\sum_{i=1}^{m} c_i u_i = 0 \quad \text{and} \quad \sum_{i=1}^{m} c_i u_i \otimes u_i = \mathrm{Id}_d$$

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Helly-type results

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Classical result

Helly'1923*

If the intersection of every $d+1$ sets of a finite family of convex subsets of $\mathbb{R}^d$ is nonempty, then the whole family has a nonempty intersection.

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Notation

For $v \in \mathbb{R}^d \setminus \{0\}$,

$$H_v = \left\{ p \in \mathbb{R}^d \ : \quad \langle p, v \rangle \leq 1 \right\}$$

Coarse approximation

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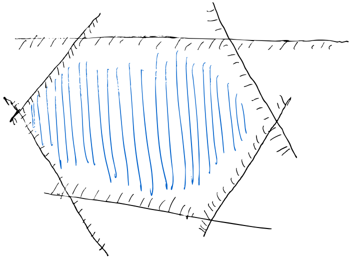

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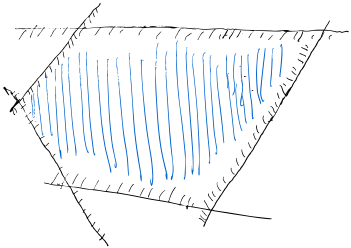

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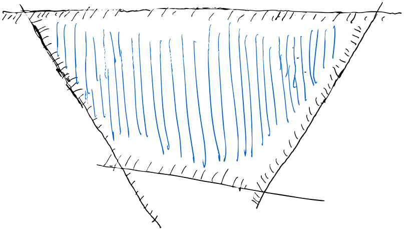

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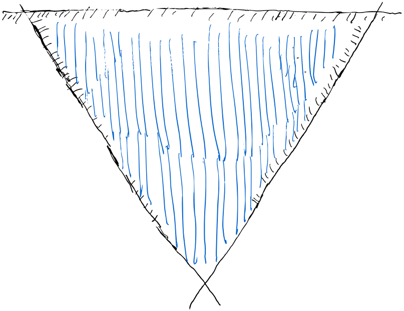

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Qualitative Question:

How many half-spaces (facets) must we retain to ensure that the intersection forms a bounded polytope?

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Helly-number: Sometimes we need $2d$ half-spaces — Cube!

Quantitative Question:

Given a convex polytope P in $\mathbb{R}^d$, can one choose at most $m \leq 2d$ facets of P in such a way that the volume of the intersection of corresponding half-spaces is at most $\nu_d \operatorname{vol}_d P$.

Results

Bárány, Katchalski, Pach'82

Let $\mathcal{F}$ be a finite family of convex subsets of $\mathbb{R}^d$.
Then one can find at most $m \leq 2d$ sets $F_1, \ldots, F_m$ of $\mathcal{F}$ satisfying

$$\mathrm{vol}_d\left(F_1 \cap \cdots \cap F_m\right) \leq \nu_d \,\mathrm{vol}_d \bigcap \mathcal{F}$$

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Extensive use of the John ellipsoid and the Dvoretzky–Rogers...!

Containment in a homothet

Do we need the John ellipsoid in the proofs?

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NO!

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What do we really use?

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1. Duality:

Polar set

The polar S° of a set $S \subset \mathbb{R}^d$ is the set

$$S^\circ = \left\{ p \in \mathbb{R}^d \ : \quad \langle p, x \rangle \leq 1 \qquad \text{for all} \quad x \in S \right\}$$

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$$\Downarrow$$

$$\operatorname{vol}_d \bigcap_{i \in J \subset I} H_{v_i} \le |\Lambda|^d \operatorname{vol}_d K \quad \text{and} \quad \operatorname{diam} \bigcap_{i \in J \subset I} H_{v_i} \le |\Lambda| \operatorname{diam} K$$

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3. "Good center" c:

$$K - c \subset -d(K - c)$$

Containment in a homothet

What do we really use?

1. Duality:

$$K - c = \bigcap_{i \in I} H_{v_i} \qquad \Longleftrightarrow \qquad (K - c)^\circ = \operatorname{conv}\{v_i \ : \ i \in I\}$$

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The center of the John ellipsoid is a good center, but there are many others!

Containment in a homothet: Double-sided inclusion

Naszódi and I.'22

Let $S \subset \mathbb{R}^d$ be such that $\operatorname{conv} S \subset -d \operatorname{conv} S$. Then there are $m \leq 2d$ and $v_1, \ldots, v_m \in S$ such that

$$\operatorname{conv} S \subset \Lambda \operatorname{conv}\{v_1, \ldots, v_m\},$$

where $\Lambda = -15d^3$.

Containment in a homothet: Double-sided inclusion

A-HAK lemma [Almendra–Hernández, Ambrus, Kendall'22]

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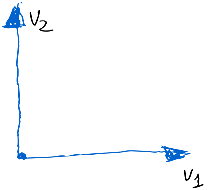
where $\Lambda = -3d^2$.

$v_1, \ldots, v_d$ form a maximum volume o-simplex, i.e.

$$\max_{w_1,\ldots,w_d \in S} \operatorname{vol}_d \operatorname{conv}\{0, w_1, \ldots, w_d\} = \operatorname{vol}_d \operatorname{conv}\{0, v_1, \ldots, v_d\}$$

Containment in a homothet: Double-sided inclusion



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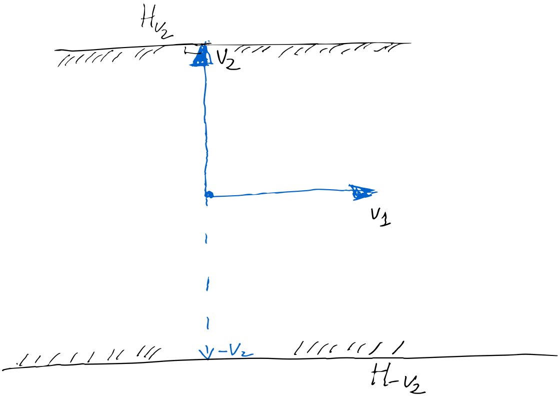
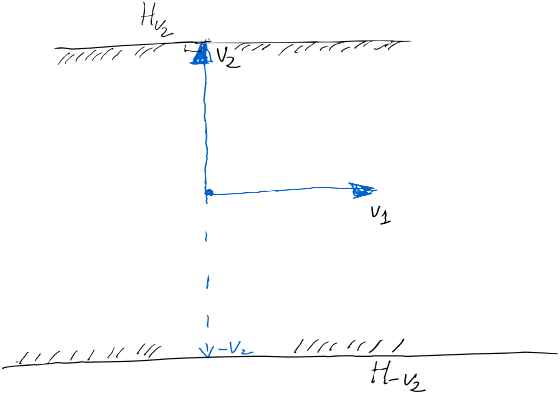


Containment in a homothet: Double-sided inclusion

$$(K - c)^{\circ} \subset H_{v_2} \cap H_{-v_2}$$



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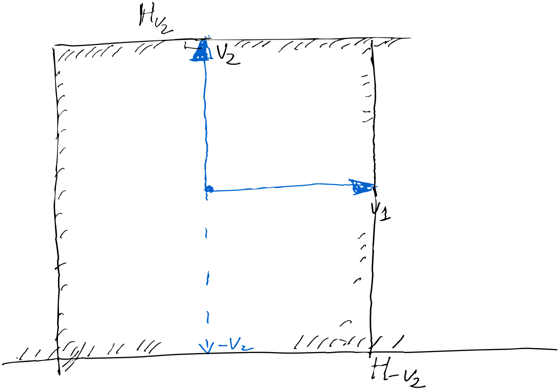
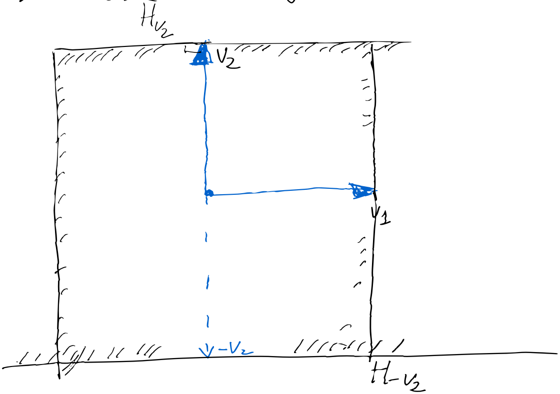


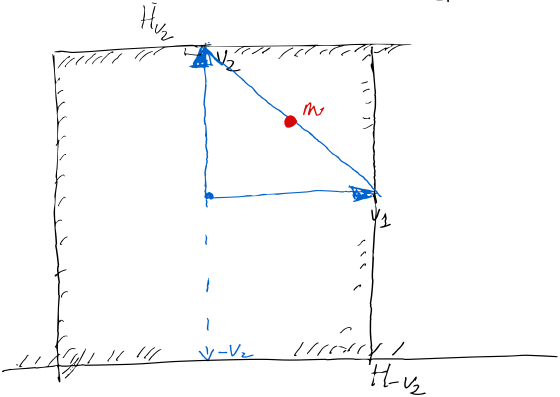
Containment in a homothet: Double-sided inclusion

$$(K - c)^{\circ} \subset \bigcap_{i=1}^{d} \left(H_{v_i} \cap H_{-v_i} \right)$$



Containment in a homothet: Double-sided inclusion

$m = \frac{v_1 + \ldots + v_d}{d}$



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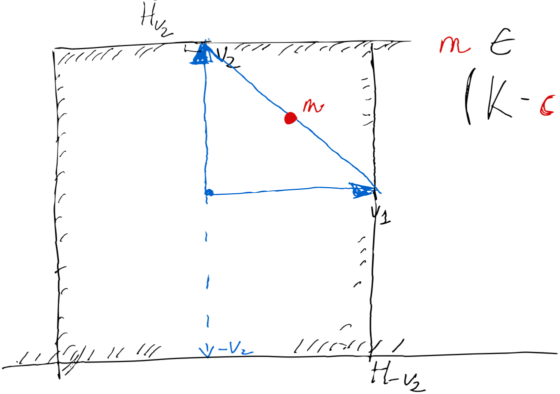
$$m = \frac{v_1 + \ldots + v_d}{d}$$

$$m \in (K - c)^{\circ}$$



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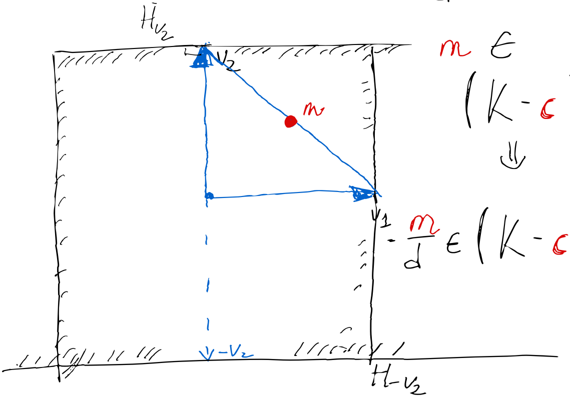

$$m \in (K - c)^\circ$$

$$\Downarrow$$

$$v_1 - \frac{m}{d} \in (K - c)^\circ$$

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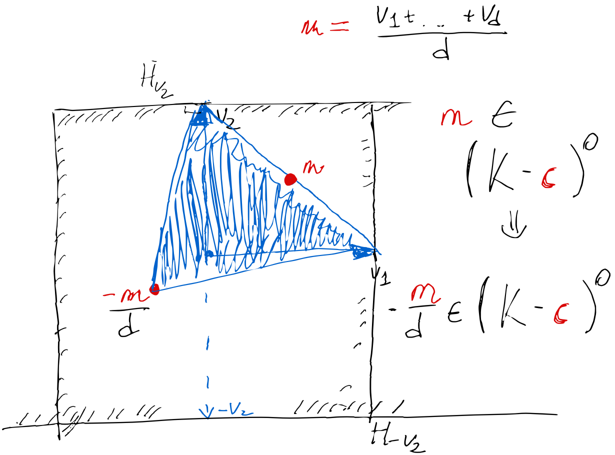

Outline

Extremal arrangement of half-spaces

Conjecture

Let $\{u_1, \ldots, u_{2d}\}$ be unit vectors in $\mathbb{R}^d$. Then there is a point in the set

$$\bigcap_{i=1}^{2d} \{x \in \mathbb{R}^d \ : \ \langle u_i, x \rangle \leq 1\}$$

with norm $\sqrt{d}$.

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Follows from a theorem of Ball and Prodromou'09

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with norm $\sqrt{\frac{d}{2}}$.

Good center: Macbeath point

Macbeath region

Take a point $p \in K \subset \mathbb{R}^d$, the Macbeath region $\mathcal{M}(K, p)$ is the set

$$\mathcal{M}(K, p) = K \cap (-K + 2p)$$

Good center: Macbeath point

Macbeath region

Take a point $p \in K \subset \mathbb{R}^d$, the Macbeath region $\mathcal{M}(K, p)$ is the set

$$\mathcal{M}(K, p) = K \cap (-K + 2p)$$

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Take a point p and a set K, the Macbeath region $\mathcal{M}(K, p)$ is the set

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Proposition

Let K be a convex body in $\mathbb{R}^d$. There is a unique point p maximizing the volume of the Macbeath region $\mathcal{M}(K, p)$.

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Question:

Let p be the Macbeath point of a convex body $K \subset \mathbb{R}^d$. Is it true that

$$K - p \subset -d\,(K - p)?$$

Colorful versions

Qualitative result. Puzzle:

Let finite families $\mathcal{F}_1, \ldots, \mathcal{F}_{2d}$ of convex sussets of $\mathbb{R}^d$ be such that $\mathrm{vol}_d \bigcap \mathcal{F}_1 \leq 1, \ldots, \mathrm{vol}_d \bigcap \mathcal{F}_{2d} \leq 1$.

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Then there is a "rainbow" selection of sets $F_1 \in \mathcal{F}_1, \dots, F_{2d} \in \mathcal{F}_{2d}$ such that the volume of $F_1 \cap \dots \cap F_{2d}$ is bounded.

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We don't know the Helly-number, i.e. the number of color classes!

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The best we have $3d$ color classes (Damásdi, Földvári, Naszódi'21)!

Log-concave functions

Functional Bárány–Katchalski–Pach (Naszódi and I'22)

Let $f_1, \ldots, f_n$ be upper semi-continuous log-concave functions on $\mathbb{R}^d$. For every $\sigma \subseteq \{1, \ldots, n\}$, let f_σ denote the pointwise minimum:

$$f_\sigma(x) = \min\{f_i(x) \ : \ i \in \sigma\}.$$

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Then there is a set $\sigma \in \{1, \ldots, n\}$ of at most $3d + 2$ indices such that

$$\int_{\mathbb{R}^d} f_\sigma \leq 100^d d^{2d} \int_{\mathbb{R}^d} f_{\{1,\ldots,n\}}.$$

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It is surely strictly less than $3d + 2$

Thank you!