Fixed points of Minkowski valuations

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Introduction

Fixed points of geometric operators appear in a number of problems in convex geometry.

The projection body ΠK of a convex body K is defined in terms of its support function by

$$h_{\Pi K}(u) = V_{n-1}(K|u^{\perp}), \qquad u \in \mathbb{S}^{n-1}.$$

The conjectured inequality of Petty

Conjecture The volume ratio

$$V_n(\Pi K)/V_n(K)^{n-1},$$

where $K \in \mathcal{K}^n$, is minimized precisely on ellipsoids.

The class-reduction technique

Schneider pointed out that any minimizer K of $V_n(\Pi K)/V_n(K)^{n-1}$ satisfies the fixed point equation $\Pi^2 K = \alpha K$ where α is a positive number.

In particular, this observation implies that minimizers should be zonoids. This is called the *class-reduction technique*.

Fixed Points of Π^2

Weil (1971) classified all possible polytopal solutions of Π^2.

For smooth bodies the question remains open. It is conjectured that these are precisely *ellipsoids*.

Theorem [Saroglou-Zvavitch 2017, Ivaki 2018] There is a C^2 neighbourhood of the unit ball where the only fixed points of Π^2 are ellipsoids.

Fixed Points of Π_i^2

The i-th projection body $\Pi_i K$ of a convex body K is defined by

$$h_{\Pi_i K}(u) = V_i(K|u^\perp), \qquad u \in \mathbb{S}^{n-1}.$$

Theorem [Ivaki 2018] Let $1 < i < n-1$. Then, there is a C^2 neighbourhood of the unit ball where the only fixed points of Π_i^2 are balls.

We want to extend these results to a larger class of geometric operators.

Minkowski Valuations

A Minkowski valuation is a map $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ such that

$$\Phi K + \Phi L = \Phi(K \cap L) + \Phi(K \cup L),$$

whenever $K \cup L \in \mathcal{K}^n$.

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2. positively homogeneous of degree i: $\Phi(\lambda K) = \lambda^i \Phi(K)$, for all $\lambda > 0, K \in \mathcal{K}^n$,
3. $\mathrm{SO}(n)$ equivariant: $\Phi(\vartheta K) = \vartheta \Phi K$ for all $\vartheta \in \mathrm{SO}(n)$.

Hadwiger-type Theorem for Minkowski Valuations

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Theorem [Schuster-Wannerer 2017; Dorrek 2017] For every $\Phi_i \in \mathbf{MVal}_i$, there exists a unique centered, $SO(n-1)$ invariant function $f \in \mathrm{L}^1(\mathbb{S}^{n-1})$ such that

$$h(\Phi_i K, \cdot) = S_i(K, \cdot) * f, \qquad K \in \mathcal{K}^n.$$

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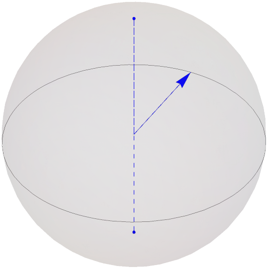
The convolution transform T_f is given by

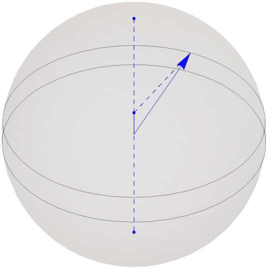
$$T_f \mu = (\mu * f)(u) = \int_{\mathbb{S}^{n-1}} \bar{f}(\langle u, v \rangle)\mu(dv)$$

where μ is a measure on $\mathbb{S}^{n-1}$.



For a fixed $v \in \mathbb{S}^{n-2}$, $\bar{f}$ defined on $[-1, 1]$ is given by

$$\bar{f}(t) = f(te + \sqrt{1 - t^2}v).$$



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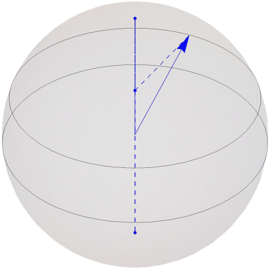

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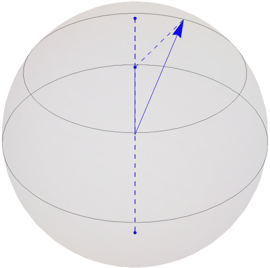

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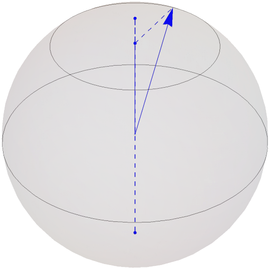

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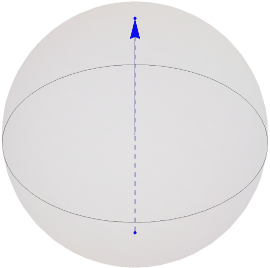

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$$h(\Pi_i K, u) = \frac{1}{2} \int_{\mathbb{S}^{n-1}} |\langle v, u\rangle| S_i(K, dv), \qquad u \in \mathbb{S}^{n-1}.$$

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2. Let $L \in \mathcal{K}^n$ be a convex body of revolution, then

$$h(\Phi_i K, u) = \int_{\mathbb{S}^{n-1}} h_{L(v)}(u) S_i(K, dv), \qquad u \in \mathbb{S}^{n-1}.$$

   Here $L(u)$ denotes the rotated copy of L with axis of revolution the line span by u.

   We say that Φ_i is C^2_+-regular if L is of class C^2_+.

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- **Theorem [Schuster 2007]** If $\Phi_{n-1} \in \mathbf{MVal}_{n-1}$ is even, then its generating function is the support function of a symmetric convex body of revolution.
- The classification of all generating functions is still open even in the top degree case.

Mean section operators

Definition [Goodey-Weil 1992] For $0 \leq j \leq n$, the j-th mean section body $\mathrm{M}_j K$ of a convex body K is defined by

$$h(\mathrm{M}_j K, u) = \int_{\mathrm{AG}(n,j)} h(K \cap E, u) dE, \qquad u \in \mathbb{S}^{n-1},$$

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- M_j is not translation invariant: $\tilde{\mathrm{M}}_j K = \mathrm{M}_j(K - S(K))$.
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- It is not hard to check that $\tilde{\mathrm{M}}_j \in \mathbf{MVal}_i$ where $i + j = n + 1$.
- Finding their generating function is highly non-trivial: This was done by Goodey and Weil [JDG 2014].

$$\Box_n = \frac{1}{n-1}\Delta_{\mathbb{S}^{n-1}} + \mathrm{Id}.$$

It relates the support function of a convex body K to its area measure of degree 1:

$$\Box_n h_K = S_1(K, \cdot).$$

Berg's functions: $g_n \in C^\infty(-1,1)$ such that

$$\Box_n \breve{g}_n = \delta_{\bar{e}} - c_n\langle e, \cdot\rangle.$$

In other words, $\breve{g}_n$ is essentially the Green's function of $\Box_n$.

Generating functions of M_j

Theorem [Goodey-Weil 2014] For $2 \leq j \leq n$, Berg's function $\breve{g}_j$ (as a function on $\mathbb{S}^{n-1}$) is the generating function of $\tilde{\mathrm{M}}_j$.

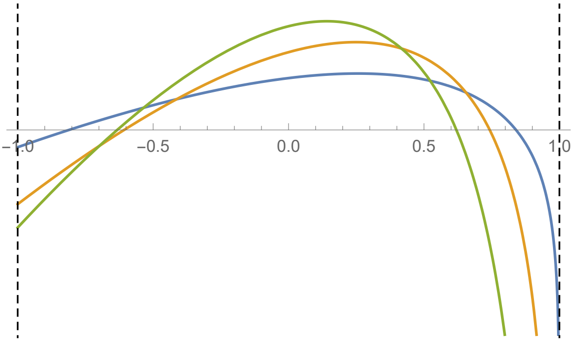
Remarks

1. The g_j's are neither even nor support functions.
2. All generating functions of Minkowski valuations in $\mathbf{MVal}_i$ we know are of the form:

$$f = h_L + g_i * \mu$$

where μ is the generating function of a Minkowski endomorphism.

Berg's functions



Fixed Points of Minkowski Valuations

Theorem [Schuster-OM 2021] Let $\Phi_i \in \mathbf{MVal}_i$ be C^2_+ regular and even. Then, there is a C^2 neighbourhood of the unit ball where the only fixed points of Φ_i^2 are balls.

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Remarks

1. As oppose to Π^2, the only fixed points are balls.
2. There is a gap between Ivaki's results and the theorem above.

Fixed Points of Minkowski Valuations

We unified the previous results by Ivaki, and and Schuster-OM.

Theorem [Brauner-OM 2023] Let $1 < i \leq n-1$ and $\Phi_i \in \mathbf{MVal}_i$ generated by *any* convex body of revolution. Then, there exists a C^2 neighborhood of the unit ball where the only fixed points of Φ_i^2 are Euclidean balls,

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Theorem [Brauner-OM 2023] Let $1 < j \leq n-1$. There exists a C^2 neighborhood of the unit ball where the only fixed points of M_j^2 are Euclidean balls.

Regularity of convolution operators

We need to understand which convolution operators T_f are bounded linear operators from C^0 to C^2.

For the cosine transform we have the following:

Theorem [Martinez-Maure 2001] The cosine transform given by

$$Cg(u) = \frac{1}{2} \int_{\mathbb{S}^{n-1}} |\langle u, v \rangle| g(v) dv$$

is a bounded linear operator from C^0 to C^2. Moreover,

$$D^2 \mathrm{C} f(u) = \frac{1}{2} \int_{\mathbb{S}^{n-1} \cap u^{\perp}} v \otimes v \, f(v) dv.$$

Regularity of convolution operators

Theorem [Brauner-OM 2023+] The following statements are equivalent:

1. $T_f : C^0 \to C^2$ is a bounded linear operator.
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Regularity of convolution operators

Theorem [Brauner-OM 2023+] The following statements are equivalent:

1. $T_f : C^0 \to C^2$ is a bounded linear operator.
2. $D^2 f$ is a matrix-valued signed measure.
3. $\square_n f$ is a signed measure and

$$\int_0^{\frac{\pi}{2}} \frac{1}{r} |(\square_n f)(\{u \in S^{n-1} : |\langle \bar{e}, u \rangle| > \cos r\})| dr < \infty.$$

Regularity of generating functions

Theorem [Brauner-OM 2023+] Let $1 \leq i \leq n-1$ and $\Phi_i \in \mathbf{MVal}_i$ with generating function f. Then, $\Box_n f$ is a signed measurea and f is locally Lipschitz outside the poles.

Moreover, if Φ_i is weakly monotone or $i = n-1$, then there exists a constant $C > 0$ such that

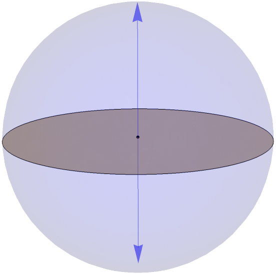
$$|\Box_n f|(\{u \in \mathbb{S}^{n-1} : |\langle e, u\rangle| > \cos r\}) \leq C r^{i-1}, \qquad r > 0.$$

Sketch of the proof

Let D^{n-1} denote the $n-1$ dimensional disk in $\mathbb{R}^n$.

The surface area measure of the disk is given by

$$S_{n-1}(\mathrm{D}^{n-1}, \cdot) = \kappa_{n-1}(\delta_{-e} + \delta_e)$$



for $1 < i < n-1$, D^{n-1} has absolutely continuous i-th area measure:

$$S_i(\mathrm{D}^{n-1}, dv) = \frac{n-1-i}{n-1}(1 - |\langle e, v\rangle|^2)^{-\frac{i}{2}}\, dv$$

Note that

$$S_1(\Phi_i \mathrm{D}^{n-1}, \cdot) = \Box_n h(\Phi_i \mathrm{D}^{n-1}, \cdot) = S_i(\mathrm{D}^{n-1}, \cdot) * \Box_n f.$$

Note that

$$S_1(\Phi_i \mathrm{D}^{n-1}, \cdot) = \square_n h(\Phi_i \mathrm{D}^{n-1}, \cdot) = S_i(\mathrm{D}^{n-1}, \cdot) * \square_n f.$$

Thus, by a simple approximation argument, one can show

$$S_1(\Phi_i \mathrm{D}^{n-1}, C_{2r}(e)) \geq S_i(\mathrm{D}^{n-1}, C_r(e))(\square_n f)(C_r(e))$$

where

$$C_r(e) = \{u \in \mathbb{S}^{n-1} : |\langle e, u \rangle| > \cos r\} \qquad , r > 0$$

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On the other hand,

$$\begin{aligned} S_i(D^{n-1}, C_r(e)) &= \omega_{n-1} \frac{n-1-i}{2} \int_{\cos r}^{1} (1-t^2)^{\frac{n-3-i}{2}} dt \\ &\geq \kappa_{n-1} (\sin r)^{n-1-i} \approx \kappa_{n-1} r^{n-1-i} \end{aligned}$$

So far we have shown,

$$cr^{n-1-i}\square_n f(C_r(e)) \leq S_1(\Phi_i \mathrm{D}^{n-1}, C_{2r}(e))$$

We need an estimate of $S_1(\Phi_i \mathrm{D}^{n-1}, C_{2r}(e))$ from above. However, $\Phi_i \mathrm{D}^{n-1}$ could be *any* convex body in $\mathbb{R}^n$.

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Theorem [Firey, '70] Let $K \in \mathcal{K}^n$ be a convex body. Then, for all $u \in \mathbb{S}^{n-1}$,

$$S_1(K, C_r(u)) \leq C_n(\operatorname{diam} K) r^{n-2}$$

where $\operatorname{diam} K$ denotes the diameter of K.

Thus, $\Box_n f(C_r(e)) \leq c'' r^{i-1}$.

Thank you for your attention!