A FLAG REPRESENTATION FOR AN n-DIMENSIONAL CONVEX BODY

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The support function

The most useful analytic description of compact convex body $\mathbf{B}$ is given by the support function

$$H(x) = \sup_{y \in \mathbf{B}} \langle y, x \rangle, \quad x \in \mathbf{R}^n. \tag{1}$$

Here and below $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathbf{R}^n$.
It is well known that any convex body $\mathbf{B}$ is uniquely determined by its support function.

The cosine transform

The cosine transform plays a fundamental role in convex geometry, integral geometry and a number of related areas.

Theorem (W.Blaschke, R.Schnider [5],[6])

The support function H of a sufficiently smooth origin symmetric convex body $\mathbf{B} \subset \mathbf{R}^n$ has the cosine representation with an even signed measure $\mu(d\Omega) = h(\Omega)\lambda_{n-1}(d\Omega)$ defined on the unit $n-1$ dimensional sphere $\mathbf{S}^{n-1}$:

$$H(\xi) = \int_{\mathbf{S}^{n-1}} \mid \langle \xi, \Omega \rangle \mid h(\Omega)\, \lambda_{n-1}(d\Omega), \quad \xi \in \mathbf{S}^{n-1} \tag{2}$$

with an even continuous function $h(\cdot)$ (not necessarily positive) defined on $\mathbf{S}^{n-1}$. Note that h is unique and is called the generating density of $\mathbf{B}$.

Connection with the geometry of the body

In formula (2) we do not see a direct connection with the geometry of the body.
Of course, the following relation is known

Theorem (W.Blaschke [5])

$$\sum_{i=1}^{n-1} R_i(\Omega) = 2 \int_{\mathbf{S}_\Omega} h_\Omega(u)\, \lambda_{n-2}(du), \quad \Omega \in \mathbf{S}^{n-1} \tag{3}$$

$\mathbf{S}_\Omega \subset \mathbf{S}^{n-1}$ is the great $n-2$ dimensional sphere with the pole at $\Omega \in \mathbf{S}^{n-1}$, $R_i(\Omega)$ is the principal radius of the curvature at the point whose outer normal direction is Ω, $h_\Omega(u)$ is the restriction of h onto $\mathbf{S}_\Omega$.

Formula (3) is the well-known Funk equation (transform) and the inversion formula is known. The inversion formula is complex and has different forms depending on the dimension of the Euclidean space.

Another relation with the geometry of the body

Note that there is another formula that is a disintegration of Blaschke's formula.

Theorem (R.A. 2002)

$$R(\Omega, \xi) = 2 \int_{\mathbf{S}_\Omega} \cos^2 (u \hat{,} \xi) h_\Omega(u) \, \lambda_{n-2}(du), \quad \Omega \in \mathbf{S}^{n-1}, \quad \xi \in \mathbf{S}_\Omega \tag{4}$$

We project the body onto the plane spanned by Ω and ξ. $R(\Omega, \xi)$ is defined as the radius of the curvature of the projection at the point whose outer normal direction is Ω, and is called the projection radius of the curvature of the body.

After integration of (4) with respect to ξ over $\mathbf{S}_\Omega$ we obtain Blaschke's formula. I found the inversion formula for the weighted Funk transform.

We are sure that it will be useful to obtain a representation of the support function of a convex body in terms of geometric characteristics of the body. The well-known problem of geometric characterization of zonoids was posed by W. Blaschke [5] (this problem was repeatedly raised by R. Schneider and W. Weil [8]). W. Weil [7] showed that a local characterization of zonoids does not exist. Also, W. Weil proposed the conjecture about the local equatorial characterization of zonoids. Positive answers exist only for even dimensions (W.Weil, G.Panina) . In [3] was described a subclass of zonoids in $\mathbf{R}^3$ admits a local equatorial characterization.

The concept of a flag

The concept of a flag in $\mathbf{R}^n$ which naturally emerges in Combinatorial integral geometry will be of importance below.

Definition

A flag is an ordered pair of orthogonal unit vectors in $\mathbf{R}^n$, say a_1, a_2. We will use the representations of a flag:

$$(\Omega, \Phi), \tag{5}$$

where $\Omega \in \mathbf{S}^{n-1}$ is the spatial direction of the first vector and Φ is the direction in $\mathbf{S}_\Omega$ which coincides with the direction of the second vector.

$\sin^2$ representation

In $\mathbf{R}^3$ the notion of a flag density was introduced and effectively used in his works by R. V. Ambartzumian [1].

Theorem (Ambartzumian 1987)

There exists the following representation for the support function of an origin symmetric convex body

$$H(\xi) = \int_{\mathbf{S}^2 \times \mathbf{S}^1} \sin^2 \alpha(\Omega, \Phi, \xi)\, m(d\Omega, d\Phi), \tag{6}$$

m-is a measure in the space of flags, α is the angle between Ω and the intersection of planes orthogonal to ξ and Ω respectively.

However, represantation (6) is not unique.

A $\sin^2$ representation

Somewhat later, using stochastic approximation of a convex body, I obtained a specific $\sin^2$ representation of a 2-smooth convex body in terms of geometric characteristics.

Theorem (R.A. 1988)

The support function of an origin symmetric 2-smooth convex body $\mathbf{B} \subset \mathbf{R}^3$ (with positive Gaussian curvature at every point) has the following representation. For $\xi \in \mathbf{S}^2$

$$H(\xi) = \frac{1}{2\pi^2} \int_{\mathrm{S}^2} \int_{\mathrm{S}_\Omega} \sin^2 \alpha(\Omega, \Phi, \xi) \, \frac{\sqrt{K(\Phi)}}{(k(\Phi, \Omega))^2} \, \lambda_1(d\Phi) \, \lambda_2(d\Omega), \qquad (7)$$

here $K(\Phi)$ is the Gaussian curvature at the point with the outer normal Φ on $\partial\mathbf{B}$ and $k(\Phi, \Omega)$ is the normal curvature at the same point in direction $\Omega \in \mathrm{S}_\Phi$.

Next step

The next step is to generalize the cosine transform and find a new representation for an n-dimensional convex body in terms of geometric characteristics of the body.

Definition

The function

$$\rho(\Omega, \Phi, \xi) = \frac{\langle \xi, \Omega \rangle^2}{\sin^{n-1}(\widehat{\xi, \Phi})} \tag{8}$$

defined for $\Omega, \xi \in \mathrm{S}^{n-1}$, $\Phi \in \mathrm{S}_\Omega$ (the great $n-2$ dimensional sphere with pole at Ω) $(\xi \neq \Phi)$ is called *the flag density* function (for $\xi = \Phi$ and $\xi = -\Phi$ we assume that $\rho = 0$).

For $n = 3$ we have $\rho(\Omega, \Phi, \xi) = \sin^2 \alpha(\Omega, \Phi, \xi)$.

A generalization of the cosine kernel

Theorem (R.A.2019)

For $\Omega, \xi \in \mathbf{S}^{n-1}$ we have

$$\int_{\mathrm{S}_\Omega} \rho(\Omega, \Phi, \xi)\, \lambda_{n-2}(d\Phi) = \frac{\pi \sigma_{n-3}(n-4)!!}{(n-3)!!} \mid \langle \xi, \Omega \rangle \mid . \tag{9}$$

Thus the flag density function is a disintegration of the cosine function.

$\sigma_k = \lambda_k(\mathbf{S}^k)$ is the total measure of $\mathbf{S}^k$

The main results

Theorem (R.A. 2019)

The support function of an origin symmetric 2-smooth convex body $\mathbf{B} \subset \mathbf{R}^n$ *(with positive Gaussian curvature at every point) has the following representation . For* $\xi \in \mathbf{S}^{n-1}$ *(see [1])*

$$H(\xi) = \frac{(n-1)!!}{2\,\sigma_{n-2}} \int_{\mathrm{S}^{n-1}} \int_{\mathrm{S}_\Omega} \frac{\langle \xi, \Omega \rangle^2}{\sin^{n-1}(\widehat{\xi, \Phi})} \frac{\sqrt{K(\Phi)}}{(k(\Phi, \Omega))^{\frac{n+1}{2}}} \lambda_{n-2}(d\Phi)\, \lambda_{n-1}(d\Omega), \tag{10}$$

here $K(\Phi)$ *is the Gaussian curvature at the point with the outer normal* Φ *on* $\partial\mathrm{B}$ *and* $k(\Phi, \Omega)$ *is the normal curvature at the same point in direction* $\Omega \in \mathrm{S}_\Phi$. λ_k *is the spherical Lebesgue measure on* $\mathbf{S}^k$ *and* σ_k *is the total measure of* $\mathbf{S}^k$.

The consequence of the Theorem

We propose the following sufficient condition for an origin symmetric convex body to be a zonoid.

Theorem

Let $\mathbf{B}$ *be an origin symmetric 2-smooth convex body in* $\mathbf{R}^n$ $(\mathbf{B} \in \mathcal{B}_o^n)$*. If for any* $\Omega \in \mathbf{S}^{n-1}$ *and* $\xi \in \mathbf{S}^{n-1}$ *the expression*

$$F(\Omega, \xi) = \int_{\mathrm{S}_\Omega} \frac{|\langle \xi, \Omega \rangle|}{\sin^{n-1}(\widehat{\xi, \Phi})} \frac{\sqrt{K(\Phi)}}{(k(\Phi, \Omega))^{\frac{n+1}{2}}} \lambda_{n-2}(d\Phi) \tag{11}$$

does not depend on $\xi \in \mathbf{S}^{n-1}$ *, then* $\mathbf{B}$ *is a zonoid.*

Another consequence of the Theorem

Theorem (R.A. 2019)

Let $\mathbf{B}$ *be an origin symmetric 2-smooth convex body in* $\mathbf{R}^n$. *If for any* $\Omega \in \mathbf{S}^{n-1}$ *and* $\Phi \in \mathrm{S}_\Omega$ *the expression*

$$G(\Omega, \Phi) = \frac{\sqrt{K(\Phi)}}{(k(\Phi, \Omega))^{\frac{n+1}{2}}} \tag{12}$$

does not depend on $\Phi \in \mathrm{S}_\Omega$ *then* $\mathbf{B}$ *is a zonoid.*

Note that for any $\Omega \in \mathbf{S}^{n-1}$ the expressions $F(\Omega, \xi)$ and $G(\Omega, \Phi)$ depend on the boundary of $\mathbf{B}$, which consists of points where the exterior unit vector belongs to a neighborhood of the equator $\mathbf{S}_\Omega$. Hence for any $\Omega \in \mathbf{S}^{n-1}$ the expressions $F(\Omega, \xi)$ and $G(\Omega, \Phi)$ have a local equatorial description.
It was proved in [3] that in $\mathbf{R}^3$ a convex body boundary of which is an ellipsoid satisfies the condition (12) and hence also satisfies the condition (11).

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Thank you for your attention