On the dimension of the set of minimal projections

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Introduction

This is a joint work with Grzegorz Lewicki. The poster is based on the paper:

- T. Kobos, G. Lewicki, *On the dimension of the set of minimal projections*, Journal of Mathematical Analysis and Applications (2023).

The paper is available also on arXiv. I refer there for the proofs, all the details and the references.

If X is a normed space over $\mathbb{K}$ (where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$) and $Y \subseteq X$ is its proper linear subspace, then by *projection* from X to Y we shall mean a bounded linear operator $P : X \to Y$ such that $P|_Y = \mathrm{Id}_Y$. By $\mathcal{P}(X, Y)$ we denote the set of all projections from X onto Y. The *relative projection constant* of Y is defined as

$$\lambda(Y, X) = \inf\{\|P\| : P \in \mathcal{P}(X, Y)\}.$$

Moreover, if a projection $P : X \to Y$ satisfies $\|P\| = \lambda(Y, X)$ then P is called a *minimal projection*. The set of all minimal projections will be denoted by $\mathcal{P}_{\min}(X, Y)$. By B_X and S_X we will always denote the unit ball and the unit sphere of X, respectively.

Projection is one of the fundamental concepts of the functional analysis and minimal projections have been studied extensively for many years. The following questions are classical, when considered for some specific normed spaces $Y \subseteq X$:

(1) Determine $\lambda(Y, X)$.
(2) Determine a minimal projection $P_0 : X \to Y$, that is some projection P_0 satisfying $\|P_0\| = \lambda(Y, X)$.
(3) Determine if such projection P_0 is unique.

One famous example is Lozinski Theorem about the minimality of the classical Fourier projection. The minimality of Fourier projection was proved by Lozinski in 1948, but it took another 20 years to establish, that it is in fact, the unique minimal projection. It was proved by Cheney, Hobby, Morris, Schurer and Wulbert. This shows that, generally speaking, even if the second question is settled, the third one can still provide an additional and significant challenge. The main goal of our investigation is to consider the third question, but in a much broader sense than traditionally taken.

An affine dimension of the set of minimal projections

We start with a simple observation that the set $\mathcal{P}_{\min}(X,Y)$ is a convex set. Indeed, if $P_1, P_2 \in \mathcal{P}_{\min}(X,Y)$, then for any $t \in [0,1]$ we have

$$\|tP_1 + (1-t)P_2)\| \le t\|P_1\| + (1-t)\|P_2\| = t\lambda(Y,X) + (1-t)\lambda(Y,X) = \lambda(Y,X).$$

On the other hand, the operator: $tP_1 + (1-t)P_2$ is also a projection and hence

$$\|tP_1 + (1-t)P_2)\| \ge \lambda(Y,X).$$

Thus $\|tP_1+(1-t)P_2)\| = \lambda(Y,X)$ and in consequence $tP_1+(1-t)P_2 \in \mathcal{P}_{\min}(X,Y)$.

Because the set of minimal projections is a convex subset of a space $\mathcal{L}(X,Y)$ of all linear operators from X to Y, we can study it in any way that convex sets are usually studied. In particular, it is natural to consider its affine dimension – that is, the minimal possible dimension of an affine subspace containing the set $\mathcal{P}_{\min}(X,Y)$. We will denote this dimension simply as $\dim \mathcal{P}_{\min}(X,Y)$. We will stick only to the real finite-dimensional setting.

Equivalently, the dimension of the set $\mathcal{P}_{\min}(X,Y)$ is the largest d, for which one can find linearly independent operators $L_1, L_2, \ldots, L_d \in \mathcal{L}_Y(X,Y)$ and a projection $P_0 \in \mathcal{P}(X,Y)$ such that:

$$P_0, P_0 + L_1, \ldots, P_0 + L_d \in \mathcal{P}_{\min}(X,Y),$$

where the subspace $\mathcal{L}_Y(X,Y) \subseteq \mathcal{L}(X,Y)$ consists of all linear operators $T : X \to Y$ satisfying $T|_Y \equiv 0$.

Estimates of the dimension of the set of minimal projections

Our main goal is to say something about the dimension of $\dim \mathcal{P}_{\min}(X,Y)$ in terms of the dimensions of X and Y. An old theorem due to Odyniec can be considered as a starting point of our investigation. It states that if $Y \subseteq X$, $\dim X = 3$, $\dim Y = 2$ and $\lambda(Y,X) > 1$ (Y is not a 1-complemented subspace), then the minimal projection $P : X \to Y$ is unique. Or equivalently $\dim \mathcal{P}_{\min}(X,Y) = 0$.

We should note that for $\dim X = 3$ and $\dim Y = 2$ it is immediate from purely algebraic reasons that $\dim \mathcal{P}_{\min}(X,Y) \leq 2$. Theorem of Odyniec shows that under some very general assumption (that Y is not 1-complemented in X) it is possible to improve this bound by 2. This is a motivation for studying the dimension of the set of minimal projections in a more systematic way.

We have the following broad generalization of Theorem of Odyniec:

Theorem 1. Suppose that $Y \subseteq X$, $\dim X = n$, $\dim Y = k$, where $1 \leq k \leq n-1$. Then

(1) $\dim \mathcal{P}_{\min}(X,Y) \leq k(n-k)$.
(2) If $\lambda(Y,X) > 1$, then $\dim \mathcal{P}_{\min}(X,Y) \leq k(n-k) - 2$.

Moreover, both estimates are optimal.

Thus, the second estimate is a broad generalization of the theorem of Odyniec, which corresponds to the case of $n = 3$ and $k = 2$. We note that the first estimate is immediate from the fact that $\mathcal{L}_Y(X,Y) = k(n-k)$. Tis bound can be improved by 2 if Y is not 1-complemented (the second estimate).

To prove that the estimate (1) is the best possible, it is enough to consider $X = \ell_1^n$ and

$$Y = \{((x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \ : x_i = 0 \text{ for } 1 \leq i \leq n-k\}.$$

It can be verified that for such k-dimensional subspace Y we have $\lambda(Y, \ell_1^n) = 1$ and $\dim \mathcal{P}_{\min}(X,Y) = k(n-k)$.

Estimates of the dimension of the set of minimal projections

The second estimate follows from the application of the Chalmers-Metcalf operator. Please refer to the full version of the paper on the arXiv if you are interested in details. Here we provide only example of X and Y, for which the bound in (2) is attained. Interestingly, the construction is a little bit more complicated in this case and the norm depends on k. Clearly, there is no need to consider the case $k = 1$, as 1-dimensional subspaces are always 1-complemented by the Hahn-Banach Theorem. Therefore, we assume that $2 \leq k \leq n-1$ and we define a norm $\|\cdot\|$ in $\mathbb{R}^n$ as

$$\|x\| = \max\left\{ \sum_{i=1}^{n-k+2} |x_i|, |x_{n-k+3}|, \ldots, |x_n| \right\}.$$

Thus the unit ball of $\|\cdot\|$ is a certain Cartesian product of the hypercube and the cross-polytope of suitable dimensions. In the case $k = 2$ we get just a $\|\cdot\|_1$ norm. Now we define also a k-dimensional subspace $Y \subseteq \mathbb{R}^n$ as

$$Y = \{x \in \mathbb{R}^n \ : \ x_1 + x_2 + x_3 = 0, \ x_i = 0 \text{ for } 4 \leq i \leq n-k+2\}.$$

It can be proved that $\lambda(Y, X) = \frac{4}{3}$ and $\dim \mathcal{P}_{\min}(X, Y) = k(n-k) - 2$.

The hyperplane case

In the hyperplane case $k = n - 1$ we can be more precise. In this case the estimates of Theorem 1 read as:

(1) $0 \leq \dim \mathcal{P}_{\min}(X, Y) \leq n - 1$
(2) $0 \leq \dim \mathcal{P}_{\min}(X, Y) \leq n - 3$ if $\lambda(Y, X) > 1$.

Moreover, we already know these upper bounds can be achieved. Our next theorem gives more precise result, stating that in fact all integers in these ranges can be realized as the dimension of $\mathcal{P}_{\min}(X, Y)$.

Theorem 2. Let $0 \leq m \leq n - 1$ be an integer. Then, there exist real normed spaces $Y \subseteq X$ such that $\dim X = n$, $\dim Y = n - 1$ and $\dim \mathcal{P}_{\min}(X, Y) = m$. Moreover, if $0 \leq m \leq n - 3$, then it is possible to choose X and Y, so that we have $\lambda(Y, X) > 1$.

Again we do not present the full proof, but we simply give appropriate examples:

(1) For $m = n - 1$ we take $X = (\mathbb{R}^n, \|\cdot\|_1)$ and $Y = \{x : x_1 = 0\} \subseteq \mathbb{R}^n$.
(2) For $m = n - 2$ we take $X = (\mathbb{R}^n, \|\cdot\|)$ and $Y = \{x : x_1 = 0\} \subseteq \mathbb{R}^n$, where the norm $\|\cdot\|$ is defined in this case as
$$\|x\| = \max\{|x_1| + |x_2| + \ldots + |x_{n-1}|, |x_n|\}.$$
(3) For $m \leq n - 3$ we take $X = (\mathbb{R}^n, \|\cdot\|_\infty)$ and $Y = \{x : x_1 + \ldots + x_k = 0\} \subseteq \mathbb{R}^n$, where $k = n - m$ (then $3 \leq k \leq n$). In this case we have $\lambda(Y, X) = 2 - \frac{2}{k} > 1$.

It would be interesting to know, if the Theorem 1 could be also improved in this way for any dimension k. That is, if all the integers in the considered ranges can be attained as $\dim \mathcal{P}_{\min}(X, Y)$.

Some results in the polyhedral setting

Even if the estimates from Theorem 1 are optimal, one can ask to what degree these estimates reflect what happens on a ”regular basis”. This is a broad question, to which we can provide a partial answer in a specific class of norms. We say that the normed space $X = (\mathbb{R}^n, \|\cdot\|)$ is *polyhedral*, if the unit ball of X is a convex polytope in $\mathbb{R}^n$, i.e. $B_X = \operatorname{conv}\{\pm x_1, \pm x_2, \ldots, \pm x_N\}$ for some $x_1, x_2, \ldots, x_N \in \mathbb{R}^n$. The importance of polyhedral normed spaces is a direct consequence of the importance of the class of convex polytopes when considered as a subset of all convex bodies. It turns out, estimates of Theorem 1 can be significantly improved for polyhedral normed spaces and almost all subspaces Y.

Theorem 3. Let X be an n-dimensional polyhedral normed space. Then for an open and dense subset of k-dimensional linear subspaces $Y \subseteq X$ we have

$$\dim \mathcal{P}_{\min}(X, Y) \leq k(n-k) - n + 1.$$

In particular, for $k = n - 1$, we obtain the following corollary: hyperplane projections are unique in the polyhedral norms for an open and dense set of hyperplanes. This generalizes some previously known results, which have dealt with some particular norms. In these results one could easily observe, that a minimal projection on a hyperplane is not unique only for some quite specific hyperplanes. It is not clear however, if for $2 \leq k \leq n-2$ this estimate is optimal.

We note here that our investigation of the dimension of the set of minimal projections has a surprising by-product related to the norming pairs of minimal projections. If $P : X \to Y$ is a minimal projection, then any point $x_0 \in S_X$ such that

$$\|P(x_0)\| = \|P\| = \lambda(Y, X)$$

is called a *norming point* for P. Furthermore, if x_0 is a norming point for P and $f \in S_{X^*}$ is a functional satisfying

$$f(P(x_0)) = \|P(x_0)\| = \lambda(Y, X),$$

then $(x, f) \in S_X \times S_{X^*}$ is called a *norming pair* for P. Norming points and norming pairs of minimal projections were considered in the literature by different authors, but the results are rather scarce. It is more or less intuitive, that minimal projections should be characterized in some way by having a lot of norming points/pairs (when compared with all linear projections). The following result, concerned with the polyhedral norms, seems to be new. It gives a lower bound for the number of norming pairs but only for **some** minimal projection. It should be noted that the lower bound of n depends only on the dimension of X and not on the dimension of k.

Theorem 4. Let X be an n-dimensional polyhedral normed space and let $Y \subseteq X$ be a k-dimensional subspace, where $1 \leq k \leq n-1$. Then, there exists a projection $P \in \mathcal{P}_{\min}(X, Y)$ with at least n norming pairs.

Concluding remarks

Clearly, there are a lot of questions that can be asked. We propose the following three questions, naturally arising from the previous considerations.

Question 1. Is it true that $\dim \mathcal{P}_{\min}(X, Y)$ can attain all integer values in the ranges given by Theorem 1 (similarly like in Theorem 2 in the hyperplane case)?

The next two questions are related to Theorem 3.

Question 2. Is it true that in any n-dimensional normed space X, the hyperplane projections are unique for some "large" set of hyperplanes?

It should be noted here, that for a general normed space X it should not be expected that uniqueness will hold for an open and dense subset of hyperplanes, like in Theorem 3. However, it could be possible that a minimal projection is not unique only for a meagre set of hyperplanes, when we consider the topology of Grassmanian $\mathrm{Gr}(n-1, n)$. In this case, the answer to Question 2 is affirmative for $n = 3$. However, this situation is rather exceptionally easy, as by Theorem of Odyniec (or Theorem 1) the minimal projection is unique for every non 1-complemented plane. Hence, one needs to deal only with the 1-complemented subspaces, which are quite special. Additionally, if X is a strictly convex norme space (of any dimension), then the minimal projection is unique onto arbitrary hyperplane.

Question 3. Is the estimate of Theorem 3 optimal for $2 \leq k \leq n-2$?

It could be possible that minimal projections are unique for a "large" set of subspaces of any given dimension, similarly like in the hyperplane case, but here the situation is far from being clear. Even in the case $X = \ell_\infty^n$, the description of minimal projections onto subspaces of codimension 2 is already quite complicated. Personally, I would be very interested to know what happens in the case of four dimensional polyhedral normed space X and its two-dimensional subspaces.

Thank you for your attention! I hope that you have learned something interesting. If you have any suggestions, questions or remarks do not hesitate to contact me. See you around!

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