Higher Order Affine Isoperimetric Inequalities Part II

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1 New Setting and Terminology

Let $\mathrm{M}_{k,l}(\mathbb{R})$ be the space of $k\times l$ matrices with real coefficients. We now identify $\mathbb{R}^{nm}$ with $\mathrm{M}_{n,m}(\mathbb{R})$, $\mathbb{R}^n$ with $\mathrm{M}_{n,1}(\mathbb{R})$ and $\mathbb{R}^m$ with $\mathrm{M}_{1,m}(\mathbb{R})$. Convenient notation:

1. $\mathcal{K}_o^{k,l}$ for convex bodies in $\mathrm{M}_{k,l}(\mathbb{R})$ containing the origin
2. $\mathcal{K}_{(o)}^{k,l}$ for convex bodies in $\mathrm{M}_{k,l}(\mathbb{R})$ containing the origin in their interiors
3. $\mathcal{S}^{k,l}$ for star bodies in $\mathrm{M}_{k,l}(\mathbb{R})$

Let $\{e_{1,i}\}_{i=1}^m$ be the canonical basis in $\mathrm{M}_{1,m}(\mathbb{R})$. The orthogonal simplex in $\mathcal{K}_o^{1,m}$ is then given by

$$\Delta_m = \operatorname{conv}\{o_{1,m}, e_{1,1}, \dots, e_{1,m}\}.$$

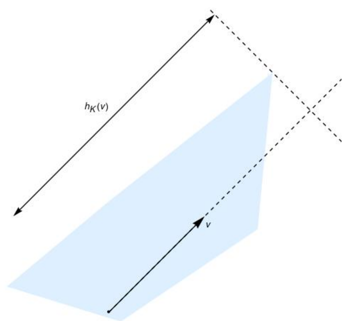
If $x \in \mathrm{M}_{n,m}(\mathbb{R})$ is given by $x = [x_1 \ \dots \ x_m]$, where each x_i is a column vector with n-entries, then

$$x.(-\Delta_m)^t = \operatorname{conv}\{o_{n,1}^t, -x_1, \dots, -x_m\} \cong \operatorname{conv}_{1\le i\le m}[o, -x_i].$$

Thus,

$$\rho_{\Pi^{\circ,m}K}(x)^{-1} = V(K[n-1], x.(-\Delta_m)^t).$$

Intermission: Integral Formulae and Surface Area



A convex body K is uniquely defined by its **support function** $h_K(y) = \sup_{x\in K}\langle x, y\rangle$. Fact: every nonnegative, 1-homogeneous, convex function on the sphere is the support function of a convex body. Firey's L^p Minkowski summation: given $p \ge 1$, convex bodies K and L and $\alpha, \beta > 0$, the convex body $\alpha\cdot K +_p \beta\cdot L$ is defined via the support function

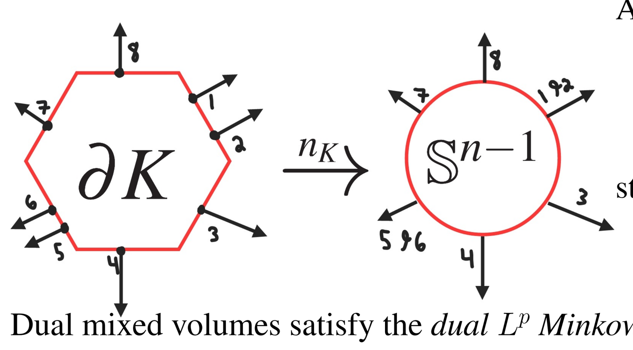
$$h_{\alpha\cdot K+_p\beta\cdot L} = \left(\alpha h_K^p + \beta h_L^p\right)^{1/p}.$$

When $p = 1$, this is $\alpha K + \beta L$. The L^p mixed volumes are given by

$$V_p(K, L) := \frac{p}{n}\lim_{\epsilon\to 0}\frac{\mathrm{vol}_n(K +_p \epsilon\cdot L) - \mathrm{vol}_n(K)}{\epsilon}.$$

In particular: $V_1(K, L) = V(K[n-1], L)$.

The *Gauss map* $n_K(y) : \partial K \to \mathbb{S}^{n-1}$ associates an element y in the boundary of K with its outer unit normal. The *surface area measure* of K is a measure on Borel subsets of $\mathbb{S}^{n-1}$ given by $\sigma_K(E) = \int_{n_K^{-1}(E)} du$. The L^p surface area measure of K, $p \ge 1$, is given by $d\sigma_{K,p}(u) = h_K^{1-p}(u)d\sigma_p(u)$. Aleksandrov's L^p integral formula, established by Lutwak, is precisely



$$V_p(K, L) = \frac{1}{n}\int_{\mathbb{S}^{n-1}} h_L(u)^p d\sigma_{K,p}(u).$$

Lutwak introduced the ith *dual mixed volume* for $p \in \mathbb{R}$, K and L star bodies in $\mathbb{R}^d$:

$$\widetilde{V}_{-p,d}(K, L) = \frac{1}{d}\int_{\mathbb{S}^{d-1}} \rho_K(\theta)^{d-p}\rho_L(\theta)^p d\theta.$$

Dual mixed volumes satisfy the *dual L^p Minkowski first inequality*: for $K, L \in \mathcal{S}^d$, $p \ge 1$

$$\mathrm{vol}_d(K)^{d+p}\,\mathrm{vol}_d(L)^{-p} \le \widetilde{V}_{-p,d}(K, L)^d,$$

with equality if, and only if, K and L are dilates.

Given $\xi \in \mathbb{S}^{d-1}$, *Steiner symmetrization* rearranges K about $\xi^\perp$ to construct the Steiner symmetral $S_\xi K$, with the property that $\mathrm{vol}_d(S_\xi K) = \mathrm{vol}_d(K)$. Fact: there exists a sequence of directions $\{\xi_j\}_{j=1}^\infty \subset \mathbb{S}^{n-1}$ such that, if we define $S_1K = S_{\xi_1}K, S_jK = S_{\xi_j}S_{j-1}K$, then $S_jK \to \kappa_d^{-1/d}\,\mathrm{vol}_d(K)^{1/d}B_2^d$.
Fact: For $u \in \mathbb{S}^{n-1}$ and $K \in \mathcal{K}^{n,1}$, one has $S_u\Pi^\circ K \subseteq \Pi^\circ S_u K$. Thus, Petty's projection equality can be proven via Steiner symmetrization.

The Centroid and Random Simplex inequalities

Given a star body L in $\mathbb{R}^n$, its centroid body ΓL is the unique centrally symmetric convex body whose support function is

$$h_{\Gamma L}(u) = \frac{1}{\mathrm{vol}_n(L)}\int_L |\langle x, y\rangle| dy.$$

Petty's projection inequality implies the **Busemann-Petty centroid inequality**, which says the functional $L \mapsto \mathrm{vol}_n(\Gamma L)\,\mathrm{vol}_n(L)^{-1}$ is minimized when L is a centered ellipsoid. The expected volume of $C_{\bar{X}} = \mathrm{conv}_{1\le i\le n}[o, X_i]$, a *random simplex of* K is given by

$$\mathbb{E}_{K^n}(\mathrm{vol}_n(C_{\bar{X}})) := \mathrm{vol}_n(K)^{-n}\int_K\cdots\int_K \mathrm{vol}_n\left(\mathrm{conv}_{1\le i\le n}[o, x_i]\right) dx_1\dots dx_n.$$

By an observation of Petty, the right-hand side equals $2^{-n}\,\mathrm{vol}_n(\Gamma K)$. Thus, the Busemann-Petty centroid inequality is equivalent to the **Busemann random simplex inequality**:

$$\boxed{\mathbb{E}_{K^n}(\mathrm{vol}_n(C_{\bar{X}}))\,\mathrm{vol}_n(K)^{-1} \ge \left(\frac{\mathrm{vol}_{n-1}(B_2^{n-1})}{(n+1)\,\mathrm{vol}_n(B_2^n)}\right)^n},$$

with equality if, and only if, K is a centered ellipsoid.

The Santaló inequality

Recall the dual of K is given by $K^\circ = \{x : h_K(x) \le 1\}$. The Santaló inequality, proven in a series of works by Saint-Reymond, Petty, Santaló, and Blaschke, can be stated as:

$$\mathrm{vol}_n(K)\,\mathrm{vol}_n((K - s(K))^\circ) \le \mathrm{vol}_n(K)\,\mathrm{vol}_n((K - c(K))^\circ) \le \mathrm{vol}_n(B_2^n)^2,$$

with equality throughout if, and only if, K is a centered ellipsoid, where $s(K) = \mathrm{argmin}_{z\in\mathrm{int}(K)}\,\mathrm{vol}_d((K-z)^\circ)$ and $c(K) := \int_K x dx$ is the center of mass of K. K is said to be in Santaló position if $s(K) = o$.

2 Generalized Higher-Order Bodies

Let $p \ge 1$, $m \in \mathbb{N}$, and fix $Q \in \mathcal{K}_o^{1,m}$.

1. Given $K \in \mathcal{K}_o^{n,1}$ with the property that $\sigma_{K,p}$ is a finite Borel measure on $\mathbb{S}^{n-1}$ (e.g. $K \in \mathcal{K}_{(o)}^{n,1}$), we define the (L^p, Q)-polar projection body of K, $\Pi_{Q,p}^\circ K$ via
$$\rho_{\Pi_{Q,p}^\circ K}(x)^{-p} = \int_{\mathbb{S}^{n-1}} h_Q(v^t.x)^p d\sigma_{K,p}(v), \quad x \in \mathrm{M}_{n,m}(\mathbb{R}).$$
2. Using that $h_Q(v^t.x) = h_{x.Q^t}(v)$, we obtain $\rho_{\Pi_{Q,p}^\circ K}(x)^{-p} = nV_p(K, x.Q^t)$.
3. Given a compact set $L \subset \mathrm{M}_{n,m}(\mathbb{R})$ with positive volume, we define the (L^p, Q)-centroid body of L, $\Gamma_{Q,p}L$, to be the convex body in $\mathrm{M}_{n,1}(\mathbb{R})$ with the support function
$$h_{\Gamma_{Q,p}L}(v)^p = \frac{1}{\mathrm{vol}_{nm}(L)}\int_L h_Q(v^t.x)^p dx.$$

The following lemma also serves as an alternative definition of $\Gamma_{Q,p}L$.

Lemma 1. *Fix $p \ge 1$ and $K \in \mathcal{K}_o^{n,1}$ such that $\sigma_{K,p}$ is a finite Borel measure on $\mathbb{S}^{n-1}$. Then, for every $Q \in \mathcal{K}_o^{1,m}$ and $L \in \mathcal{S}^{n,m}$, we have*

$$\widetilde{V}_{-p,nm}(L, \Pi_{Q,p}^\circ K) = \frac{(nm+p)\,\mathrm{vol}_{nm}(L)}{m}V_{p,n}(K, \Gamma_{Q,p}L).$$

In 1999, McMullen introduced the fibre combination of convex bodies. In 2016, Bianchi, Gardner and Gronchi further generalized the concept of fibre combination and constructed a generalization of Steiner symmetrization. We isolate a particular case of this framework as the natural analogue of Steiner symmetrization in the higher-order setting.

Definition 2. Fix $m, n \in \mathbb{N}$. For $v \in \mathbb{S}^{n-1}$, consider the m-dimensional space $[v] := \{v.t : t \in \mathrm{M}_{1,m}(\mathbb{R})\} \subseteq \mathrm{M}_{n,m}(\mathbb{R})$ and let $V(v)$ be its orthogonal complement, this is

$$V(v) = \{x \in \mathrm{M}_{n,m}(\mathbb{R}) : v^t.x = o \in \mathrm{M}_{1,m}(\mathbb{R})\}.$$

Let $L \subseteq \mathrm{M}_{n,m}(\mathbb{R})$ be a compact set with non-empty interior. We define the mth higher-order Steiner symmetral of L with respect to v

$$\bar{S}_vL = \left\{y + v.\frac{t-s}{2} \in \mathrm{M}_{n,m}(\mathbb{R})\colon y \in V(v), t, s \in \mathrm{M}_{1,m}(\mathbb{R}), (y + v.t), (y + v.s) \in L\right\}.$$

It was shown by Ulivelli that, for $L \in \mathcal{K}_o^{n,m}$, $\mathrm{vol}_{nm}(L) \le \mathrm{vol}_{nm}(\bar{S}_vL)$. The next lemma is used in generalizing Petty's inequality.

Lemma 3. *Fix $v \in \mathbb{S}^{n-1}$, $Q \in \mathcal{K}_o^{1,m}$, and $p \ge 1$. Given $K \in \mathcal{K}_o^{n,1}$ such that $\sigma_{K,p}$ is a finite Borel measure on $\mathbb{S}^{n-1}$ (e.g. $K \in \mathcal{K}_{(o)}^{n,1}$), one has*

$$\bar{S}_v\Pi_{Q,p}^\circ K \subseteq \Pi_{Q,p}^\circ S_vK.$$

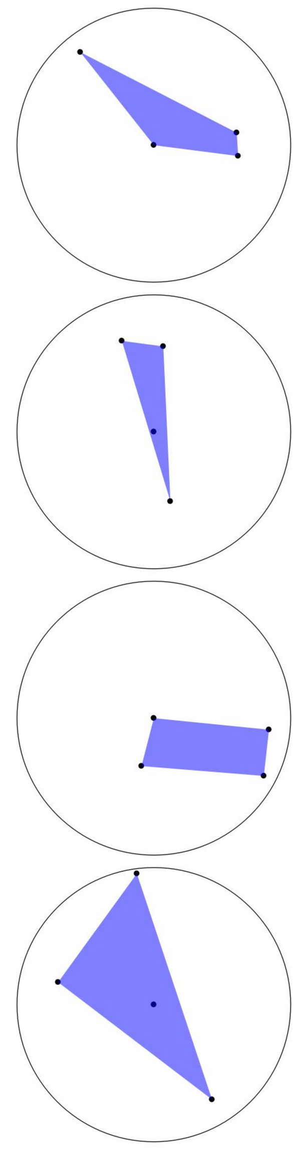


Figure 1: $w_n(\Gamma_{Q,p}L) = \mathbb{E}_L(w_n(C_{\bar{X}}))$ Points chosen independently: $L = \{(x_1, x_2, x_3) \in (\mathbb{R}^2)^3 : |x_1| \le 10\}$

Inequalities

Theorem 4. *Let $m \in \mathbb{N}$, $p \ge 1$ and $Q \in \mathcal{K}_o^{1,m}$.*
Generalized Petty Projection inequality: *for any $K \in \mathcal{K}_{(o)}^{n,1}$ one has*

$$\boxed{\mathrm{vol}_{nm}(\Pi_{Q,p}^\circ K)\,\mathrm{vol}_n(K)^{\frac{nm}{p}-m} \le \mathrm{vol}_{nm}(\Pi_{Q,p}^\circ B_2^n)\,\mathrm{vol}_{nm}(B_2^n)^{\frac{nm}{p}-m}.}$$

If $p = 1$, then there is equality above if, and only if, K is an ellipsoid; if $p > 1$, then there is equality above if, and only if, K is a centered ellipsoid.

Generalized Busemann-Petty Centroid inequality: *Let $L \subset \mathrm{M}_{n,m}(\mathbb{R})$ be a compact domain with positive volume. Then*

$$\boxed{\frac{\mathrm{vol}_n(\Gamma_{Q,p}L)}{\mathrm{vol}_{nm}(L)^{1/m}} \ge \frac{\mathrm{vol}_n(\Gamma_{Q,p}\Pi_{Q,p}^\circ B_2^n)}{\mathrm{vol}_{nm}(\Pi_{Q,p}^\circ B_2^n)^{1/m}},}$$

with equality if, and only if $L = \Pi_{Q,p}^\circ E$ for a centered ellipsoid $E \in \mathcal{K}_{(o)}^{n,1}$.

Theorem 4 generalizes the results by Lutwak, Yang and Zhang, who did the case when $Q = [-1, 1]$, and Haberl and Schuster, who finished the $m = 1$ case. This theorem also holds when sending $p \to \infty$. Fact: $\Gamma_{Q,p}\Pi_{Q,p}^\circ B_2^n \to B_2^n$ for every $Q \in \mathcal{K}_o^{1,m}$. We list as a corollary when $Q = (-\Delta_m)$ and $p = 1$, which is Schneider's setting. We note that

$$\rho_{\Pi^{\circ,m}B_2^n}(\bar{\theta})^{-1} = n\,\mathrm{vol}_n(B_2^n)w_n(C_{\bar{\theta}}).$$

Consider the simple case when $n = m = 2$. Then, for $\bar{\theta} = (\theta_1, \theta_2) \in \mathbb{S}^3$:

$$\rho_{\Pi^{\circ,2}B_2^2}((\theta_1, \theta_2))^{-1} = \mathrm{vol}_1(\partial\,\mathrm{conv}([o, \theta_1], [o, \theta_2])) = |\theta_1| + |\theta_2| + |\theta_1 - \theta_2|,$$

where $\theta_1, \theta_2 \in \mathbb{R}^2$ are such that $|\theta_1|^2 + |\theta_2|^2 = 1$.

Corollary 5. *Let $m \in \mathbb{N}$ be fixed.*

Petty's projection inequality for higher-order projection bodies: For all $K \in \mathcal{K}^n$,

$$\boxed{\mathrm{vol}_n(K)^{nm-m}\,\mathrm{vol}_{nm}(\Pi^{\circ,m}K) \le \mathrm{vol}_n(B_2^n)^{nm-m}\,\mathrm{vol}_{nm}(\Pi^{\circ,m}B_2^n),}$$

with equality if, and only if, K is an ellipsoid.
Busemann-Petty inequality for higher-order centroid bodies: For all $L \in \mathcal{S}^{nm}$,

$$\boxed{\frac{\mathrm{vol}_n(\Gamma^mL)}{\mathrm{vol}_{nm}(L)^{1/m}} \ge \frac{\mathrm{vol}_n(\Gamma^m\Pi^{\circ,m}B_2^n)}{\mathrm{vol}_{nm}(\Pi^{\circ,m}B_2^n)^{1/m}},}$$

with equality if, and only if, $L = \Pi^{\circ,m}E$ for any full-dimensional ellipsoid E.

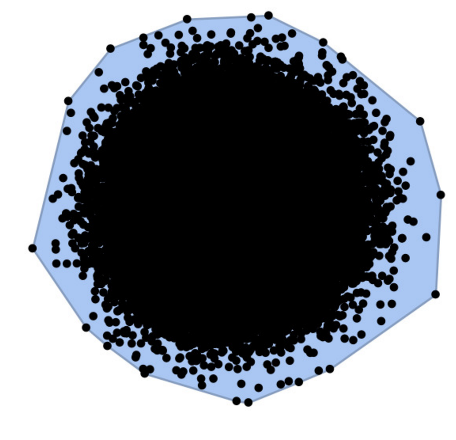


Figure 2: $n = 2$. A typical $C_{\bar{\theta}}$ as $\bar{\theta}$ is chosen randomly from $\mathbb{S}^{nm-1}$

Santaló inequalities

It will be convenient to write $\Gamma_{Q,p}^\circ L = (\Gamma_{Q,p}L)^\circ$. For a compact domain $L \subset \mathrm{M}_{n,m}(\mathbb{R})$ with positive volume, we define its polar with respect to $Q \in \mathcal{K}_o^{1,m}$: $\Gamma_{Q,\infty}^\circ L := \lim_{p\to\infty}(\Gamma_{Q,p}L)^\circ = (\Gamma_{Q,\infty}L)^\circ$. If $\xi \in \Gamma_{Q,\infty}^\circ L$, then $\max_{x\in L} h_Q(\xi^t.x) \le 1$. We can view the map $L \to \Gamma_{Q,\infty}^\circ L$ as a type of duality; it follows from the definition that it is order-reversing. Consider when $Q = [0, 1]$ and $L \in \mathcal{K}_{(o)}^{n,1}$. Then, for every $\xi \in \mathbb{S}^{n-1}$ one has

$$h_{\Gamma_{[0,1],\infty}L}(\xi) = \max_{x\in L}\langle \xi, x\rangle_+ = h_L(\xi).$$

Consequently, for $L \in \mathcal{K}_{(o)}^{n,1}$, $\Gamma_{[0,1],\infty}L = L$.

Theorem 6 (Higher-order Santaló inequality)**.** *Fix $Q \in \mathcal{K}_o^{1,m}$. Consider a compact domain $L \subset \mathrm{M}_{n,m}(\mathbb{R})$ with positive volume. Then, for $p \ge 1$:*

$$\mathrm{vol}_{nm}(L)^{\frac{1}{m}}\,\mathrm{vol}_n((\Gamma_{Q,p}L - s(\Gamma_{Q,p}L))^\circ) \le \mathrm{vol}_n(B_2^n)^2\frac{\mathrm{vol}_{nm}(\Pi_{Q,p}^\circ B_2^n)^{\frac{1}{m}}}{\mathrm{vol}_n(\Gamma_{Q,p}\Pi_{Q,p}^\circ B_2^n)}.$$

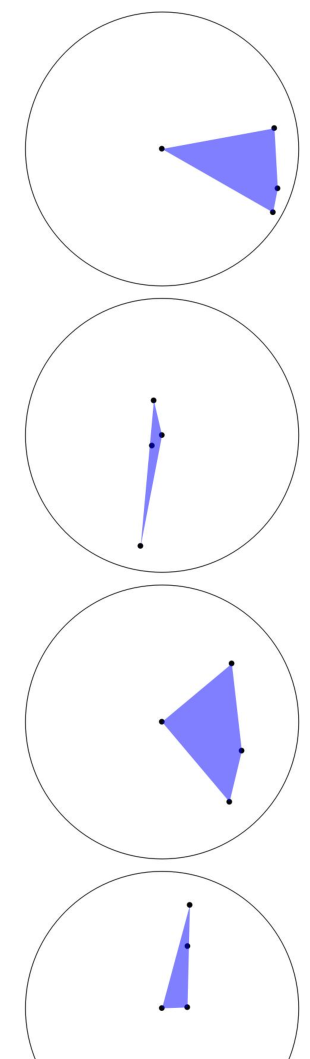
In particular, by sending $p \to \infty$:

$$\mathrm{vol}_{nm}(L)^{\frac{1}{m}}\,\mathrm{vol}_n((\Gamma_{Q,\infty}L - s(\Gamma_{Q,\infty}L))^\circ) \le \mathrm{vol}_n(B_2^n)\,\mathrm{vol}_{nm}(\Pi_{Q,\infty}^\circ B_2^n)^{\frac{1}{m}}.$$

For an example of Theorem 6 that uses the higher-order structure of the results, suppose $Q = [0, 1]^m$ and $L \in \mathcal{K}_{(o)}^{n,m}$. Then, by writing $\mathbb{R}_i^n$ for the ith copy of $\mathrm{M}_{n,1}(\mathbb{R}) \cong \mathbb{R}^n$ in the decomposition of $\mathrm{M}_{n,m}(\mathbb{R}) \cong \mathbb{R}^{nm}$ into m independent products of $\mathbb{R}^n$, one obtains, if $s(\Gamma_{[0,1]^m,\infty}L) = o$,

$$\mathrm{vol}_{nm}(L)^{\frac{1}{m}}\,\mathrm{vol}_n\left(\bigcap_{i=1}^m \left(P_{\mathbb{R}_i^n}L\right)^\circ\right) \le \mathrm{vol}_n(B_2^n)\,\mathrm{vol}_{nm}(\Pi_{Q,\infty}^\circ B_2^n)^{\frac{1}{m}}.$$

Figure 3: $w_n(\Gamma_{Q,p}L) = \mathbb{E}_L(w_n(C_{\bar{X}}))$ points chosen dependently: $L = \{(x_1, x_2, x_3) \in (\mathbb{R}^2)^3 : |x_1| \le 10$ and $\max_i\langle x_i, e_1\rangle - \min_i\langle x_i, e_1\rangle \le 1\}$



The Random Simplex inequality in Schneider's Setting

Proposition 7. *For $L \subset \mathbb{R}^{nm}$ a compact set with positive volume, $K \in \mathcal{K}^{n,m}$, and $\bar{x} \in \mathbb{R}^{nm}$ one has*

$$V(K[n-1], \Gamma^mL) = \frac{1}{\mathrm{vol}_{nm}(L)}\int_L V(K[n-1], C_{-\bar{x}})d\bar{x} = \mathbb{E}_L(V(K[n-1], C_{-\bar{X}})).$$

Using Proposition 7, we establish the following extension of the random simplex inequality.

Theorem 8. *The functional*

$$(K, L) \in \mathcal{K}_o^{n,1}\times\mathcal{S}^{n,m} \mapsto \mathrm{vol}_{nm}(L)^{-\frac{1}{nm}}\,\mathrm{vol}_n(K)^{-\frac{n-1}{n}}\mathbb{E}_L(V(K[n-1], C_{\bar{X}}))$$

is uniquely minimized when K is an ellipsoid and $L = \lambda\Pi^{\circ,m}K$ for some $\lambda > 0$.

A special case of the above theorem is that the functional

$$\mathrm{vol}_{nm}(L)^{-\frac{1}{nm}}\mathbb{E}_L(w_n(C_{\bar{X}})) = \mathrm{vol}_{nm}(L)^{-\frac{nm+1}{nm}}\int_L w_n(C_{\bar{x}})d\bar{x}$$

is minimized for $L = \Pi^{\circ,m}B_2^n$ over $\mathcal{S}^{n,m}$.