New interplay between the volume product and regularizing property of diffusion flow

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Message

- New way of understanding the volume product of a convex body as a regularising property of diffusion flow (analytic perspective): *hypercontractivity*.
- Blaschke–Santaló inequality and Mahler's conjecture are embedded into the *Brascamp–Lieb theory* which concerns about inequalities for multilinear functional.
  For instance,
  $$\begin{gathered}\text{Inverse Brascamp}-\text{Lieb inequality beyond Barthe}-\text{Wolff's condition}\\ \Downarrow \\ \text{Reverse hypercontractivity beyond Nelson's time}\\ \Downarrow \\ \text{Blaschke}-\text{Santaló inequality.}\end{gathered}$$
- (A wealth of our viewpoint on convex geometry)
  - New lower bounds of $v(K)$ for K whose boundary is "well-curved". $\rightsquigarrow$ A quantitative result to an observation by Stancu and Reisner–Schütt–Werner: *if ∂K has a point at which the Gauss curvature > 0 then $v(K) \neq$ local minimum.*
  - A monotonicity of the functional volume product along Fokker–Planck flow. $\rightsquigarrow$ Flow monotonicity proof of the Blaschke–Santaló inequality.
- (A wealth of our viewpoint on hypercontractivity and Brascamp–Lieb theory)
  - Regularizing property of the flow can be improved if the input has symmetry (improvement of hypercontractivity w.r.t Nelson's time)
  - Give an example of the inverse Brascamp–Lieb inequality due to Barthe–Wolff beyond their non-degenerate condition.

Convex geometry

- Convex body $K \in \mathbb{R}^n \rightsquigarrow$ Polar body of K: $K^\circ := \{x \in \mathbb{R}^n : \sup_{y\in K}\langle x, y\rangle \le 1\}$.
- Volume product: $v(K) := |K||K^\circ| \rightsquigarrow$ What is the maximum and minimum of v?
- (Upper bound: Blaschke–Santaló inequality)
  $$\max_{K:\text{Symmetric}} v(K) = v(\mathbb{B}_2^n).$$
  Here $\mathbb{B}_p^n := \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : \sum_{i=1}^n |x_i|^p \le 1\}$ for $p \in [1, \infty]$.
- (Lower bound: Mahler's conjecture)
  $$\min_K v(K) = v(\Delta_n), \qquad \min_{K:\text{Symmetric}} v(K) = v(\mathbb{B}_\infty^n)?$$
  Here Δ_n denotes the non-degenerate simplex.
  - When $n = 2$, the problem of symmetric and non-symmetric case are both solved by Mahler.
  - When $n = 3$, the problem of symmetric case is recentrly solved by Iriyeh–Shibata. Short proof by Fradelizi–Hubard–Meyer–Roldán-Pensado–Zvavitch.
  - Otherwise the problem is open despite of several partial answers.
- A problem of the uniqueness of minimizer of the volume product:
  - For the symmetric Mahler's conjecture, the minimizer is known to be essentially unique when $n = 2, 3$. However, when $n \ge 4$, the minimizer is known to be non-unique! (Source of difficulty)
  - For the non-symmetric Mahler's conjecture, the minimizer is expected to be unique to the simplex but this is also open problem except $n = 2$.
  - Towards this problem, Stancu and Reisner–Schütt–Werner gave an observation: if ∂K has a point at which the (generalized) Gauss curvature is positive then $v(K)$ can NOT be a local minimum.
  - In other words, the minimizer must have a *flat* boundary! Consistent with the intuition that $v(K)$ detects the *roundness* of K.
- Functional volume product: for a log-concave function $f : \mathbb{R}^n \to [0, \infty)$,
  $$v(f) := \Big(\int_{\mathbb{R}^n} f\,dx\Big)\Big(\int_{\mathbb{R}^n} f^\circ\,dx\Big)$$
  where
  $$f^\circ := \inf_{y\in\mathbb{R}^n} \frac{e^{-\langle x,y\rangle}}{f(y)} = e^{-(-\log f)^*}, \quad \phi^*(x) := \sup_{y\in\mathbb{R}^n} \langle x, y\rangle - \phi(y).$$
- For $f_K(x) = e^{-\frac{1}{2}\|x\|_K^2}$ where $\|x\|_K := \inf\{r > 0 : x \in rK\}$,
  $$\int_{\mathbb{R}^n} f_K\,dx = \frac{(2\pi)^{\frac{n}{2}}}{|\mathrm{B}_2^n|}|K|, \quad \Big(\frac{1}{2}\|\cdot\|_K^2\Big)^* = \frac{1}{2}\|\cdot\|_{K^\circ}^2$$
  and so $v(f_K)$ coincides to $v(K)$ up to constant.
  In particular, $v(\gamma) \leftrightarrow v(\mathrm{B}_2^n)$: Gaussian is "functional Euclidean ball".
- (Functional Blaschke–Santaló inequality)
  Theorem 0.1. *(Ball, Artstein-Avidan–Klartag–Milman, Lehec)*
  For any symmetric $f : \mathbb{R}^n \to [0, \infty)$,
  $$v(f) \le v(\gamma) = (2\pi)^n.$$
  The symmetric assumption can be weakened to $\int_{\mathbb{R}^n} x f(x)\,dx = 0$.
- (Functional Mahler's conjecture)
  Conjecture 0.2. *(Fradelizi–Meyer)*
  For any log-concave function f,
  $$v(f) \ge v(f_*) = e^n, \quad f_*(x) := e^{-(x_1+x_2+\cdots+x_n)}\mathbf{1}_{[-1,\infty)^n}(x).$$
  For any log-concave and symmetric function f,
  $$v(f) \ge v(f_{**}) = 4^n, \quad f_{**}(x) := e^{-(|x_1|+|x_2|+\cdots+|x_n|)}.$$

Diffusion flow: Ornstein–Uhlenbeck flow and Fokker–Planck flow

- (Ornstein–Uhlenbeck flow and Fokker–Planck flow) For $f : \mathbb{R}^n \to [0, \infty)$,
  $$u_t(x) = P_t f(x) := \int_{\mathbb{R}^n} f(e^{-t}x + \sqrt{1-e^{-2t}}y)\,d\gamma(y), \quad (t, x) \in (0, \infty) \times \mathbb{R}^n,$$
  where $d\gamma(x) := (2\pi)^{-n/2}e^{-\frac{1}{2}|x|^2}dx$, solves the heat equation $\partial_t u_t = \mathcal{L}u_t := (\Delta - x\cdot\nabla)u_t, \quad u_0 = f$.
  Also $w_t(x) = P_t^* f(x)$ solves $\partial_t w_t = \mathcal{L}^* w_t := (\Delta + x \cdot \nabla + n)w_t, \quad w_0 = f$.
- OU flow P_t regularizes an input function (possibly very rough). For instance $f_0 = \delta \rightsquigarrow P_t\delta = \gamma_{\beta_t}$ for appropriate $\beta_t > 0$.
- This regularizing property is quantified by the following functional inequalities:
  Theorem 0.3. *(Nelson, Borell)*
  Let $t > 0$ and $p, q \in \mathbb{R}$ satisfy Nelson's time condition: $\frac{q-1}{p-1} \le e^{2t}$.
  - *When $p, q > 1$, we have the forward hypercontractivity:*
    $$\big\|P_t f\big\|_{L^q(\gamma)} \le \|f\|_{L^p(\gamma)}, \quad \forall f : \mathbb{R}^n \to [0, \infty).$$
  - *When $-\infty < p, q < 1$, we have the reverse hypercontractivity:*
    $$\big\|P_t f\big\|_{L^q(\gamma)} \ge \|f\|_{L^p(\gamma)}, \quad \forall f : \mathbb{R}^n \to [0, \infty).$$
  - *Moreover, Nelson's time condition is best possible:*
    $$\frac{q-1}{p-1} > e^{2t} \quad \Rightarrow \quad \sup_f \big\|P_t f\big\|_{L^q(\gamma)}/\|f\|_{L^p(\gamma)} = +\infty, \ \inf_f \big\|P_t f\big\|_{L^q(\gamma)}/\|f\|_{L^p(\gamma)} = 0.$$

Main results

New observation: the Brascamp–Lieb inequality $\rightsquigarrow$ functional volume product. For $p_t := 1 - e^{-2t}$,
$$\lim_{t\downarrow 0}\Big(\int_{\mathbb{R}^{2n}} e^{\frac{e^{-t}}{p_t}\langle x_1,x_2\rangle} f_1(x_1)^{\frac{1}{p_t}} f_2(x_2)^{\frac{1}{p_t}}\,dx_1dx_2\Big)^{p_t} \to \sup_{x_1,x_2} e^{\langle x_1,x_2\rangle} f_1(x_1)f_2(x_2) = \sup_{x_1} \frac{f_1(x_1)}{f_2^\circ(x)}.$$
Take $f_1 = f$ and $f_2 = f^\circ$! This link is guided by the duality between BL and HC:
$$\int_{\mathbb{R}^{2n}} e^{-\pi\langle x,\mathfrak{Q}_t x\rangle} \prod_{i=1,2} f_i(x_i)^{\frac{1}{p_t}}\,dx_1dx_2 = C_t\big\|P_t\big[f^{\frac{1}{p_t}}\big]\big\|_{L^{q_t}(\gamma)}, \quad \mathfrak{Q}_t = \frac{1}{2\pi}\begin{pmatrix} 0 & -e^{-t} \\ -e^{-t} & 0 \end{pmatrix},$$
for $f_1 = f \cdot \gamma$ and $f_2 = \frac{1}{\big\|P_t\big[f^{\frac{1}{p_t}}\big]\big\|_{L^{q_t}(\gamma)}^{q_t}} P_t\big[f^{\frac{1}{p_t}}\big]^{q_t} \cdot \gamma$, where $q_t = p_t' = 1 - e^{2s} < 0$.

Claim 0.4. (N–Tsuji)

1. (reverse HC $\Rightarrow$ BS) Suppose for all small $t > 0$, there exists a constant $\mathrm{BS}_t > 0$ s.t.
   $$\big\|P_t f\big\|_{L^{-2t+O(t^2)}(\gamma)} \ge \mathrm{BS}_t^{\frac{1}{p_t}}\|f\|_{L^{p_t}(\gamma)}, \quad p_t := 2t + O(t^2)$$
   for all symmetric f. Then
   $$\sup_{f:Symmetric} v(f) \le \big(\lim_{t\downarrow 0}\mathrm{BS}_t\big)^{-1} v(\gamma).$$
2. (forward HC $\Rightarrow$ Mahler) Suppose for all small $t > 0$, there exists a constant $\mathrm{IS}_t > 0$ s.t.
   $$\big\|P_t f\big\|_{L^{-2t+O(t^2)}(\gamma)} \le \mathrm{IS}_t^{\frac{1}{p_t}}\|f\|_{L^{p_t}(\gamma)}, \quad p_t := 2t + O(t^2)$$
   for al f s.t. f/γ: log-concave. Then
   $$\inf_{f:log-concave} v(f) \ge \big(\lim_{t\downarrow 0}\mathrm{IS}_t\big)^{-1} v(\gamma).$$
3. Note that $p = 2t + O(t^2)$ and $q = -2t + O(t^2)$ breaks the Nelson't time condition $\rightsquigarrow$ Importance to go beyond Nelson's time in hypercontractivity!

Theorem 0.5. *(N–Tsuji)*
Let $\kappa \in (0, 1)$. Suppose a convex body K satisfies a curvature type condition
$$\nabla^2\Big(\frac{1}{2}\|\cdot\|_K^2\Big)(x),\ \nabla^2\Big(\frac{1}{2}\|\cdot\|_{K^\circ}^2\Big)(x) \ge \kappa\mathrm{id}, \quad \forall x \in \mathbb{S}^{n-1}.$$
Then
$$v(K) \ge \big(\kappa e^{\frac{1}{2}(1-\kappa^2)}\big)^n v(\mathrm{B}_2^n).$$
In particular, if $\kappa \ge 0.423$ then K satisfying the condition satisfies Mahler's conjecture.

Theorem 0.6. *(N–Tsuji)*
Suppose $1 - e^{2t} \le q < 0 < p \le 1 - e^{-2t}$. Then
$$\big\|P_t f\big\|_{L^q(\gamma)} \ge \|f\|_{L^p(\gamma)}, \quad \forall f : \mathbb{R}^n \to [0, \infty)\ s.t.\ Symmetric,\ \frac{f}{\gamma}:\ log-concave.$$
Moreover, $1 - e^{2t} \le q < 0 < p \le 1 - e^{-2t}$ is necessary.
This in particular implies Blaschke–Santaló inequality and provides an example of inverse Brascamp–Lieb inequality without Barthe–Wolff's non-degenerate condition.

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