Reduced convex bodies in spaces of constant curvature and Pál's isominwidth inequality on the plane

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Introduction

Let $\mathcal{M}^n$ be either the Euclidean space $\mathbb{R}^n$, hyperbolic space H^n or spherical space S^n for $n \geq 2$. We write $V_{\mathcal{M}^n}$ to denote the n-dimensional Lebesgue measure on $\mathcal{M}^n$, and $d_{\mathcal{M}^n}(x,y)$ to denote the geodesic distance between $x, y \in \mathcal{M}^n$. We use $\delta_{\mathcal{M}^n}$ to denote the Hausdorff distance in $\mathcal{M}^n$.
For a convex body $K \subset \mathcal{M}^n$, we use the notation $w(K)$ for the minimal width (or thickness) of K, where the width function depends on the space $\mathcal{M}^n$. For a supporting hyperplane H, the width of K at H, denoted as $w(K, H)$, is the distance of H and the furthest (ultra)parallel supporting hyperplane to H in $\mathbb{R}^n$ and in H^n. On the sphere, $w(K, H)$ is the smallest possible angle enclosed by H and another supporting hyperplane to K. These width functions are all monotone, continuous, and their maximal value is the diameter of K, denoted by $\operatorname{diam}_{\mathcal{M}^n} K$ [3]. We call a convex body $K \subset \mathcal{M}^n$ reduced, if for any convex body $K_0 \subsetneq K$, $w(K_0) < w(K)$.

The isominwidth problem

In 1921, Pál proved for convex bodies in $\mathbb{R}^2$, that for a fixed thickness w, the area is minimal if and only if the convex body is a regular triangle [7]. Clearly, the optimizer of this problem is reduced. In the spherical case, K. Bezdek and Blekherman proved that for $w \leq \frac{\pi}{2}$, area is again minimized by the regular triangle [1]. In these cases, we have the following optimal stability result.

Theorem (Isominwidth stability [4])

Let $\mathcal{M}^2$ denote one of $\mathbb{R}^2$ and S^2. Let $T_w \subset \mathcal{M}^2$ denote a regular triangle of thickness $w > 0$ ($w \leq \frac{\pi}{2}$, if $\mathcal{M}^2 = S^2$). There exists constants $\mathfrak{c}_w, \varepsilon_w > 0$ depending only on w such that for any $\varepsilon \in [0, \varepsilon_w)$ and any convex body $K \subset \mathcal{M}^2$ with $V_{\mathcal{M}^2}(K) \leq V_{\mathcal{M}^2}(T_w) + \varepsilon$ we have $\delta_{\mathcal{M}^2}(K, T) \leq \mathfrak{c}_w \varepsilon$ for a certain regular triangle T of thickness w.

Lassak proved that on the sphere, for $w > \frac{\pi}{2}$, reduced bodies and bodies of constant width are the same. In this case, we have the following inequality with an optimal stability result as well.

Theorem (Isominwidth stability in S^2 for $w > \frac{\pi}{2}$ [4])

Among convex bodies in S^2 with thickness $w \in (\frac{\pi}{2}, \pi)$, the area is minimal if and only if the body is congruent to $U^\circ_{\pi-w}$, where $U_{\pi-w}$ is a Reuleaux triangle of width $w - \frac{\pi}{2}$. Equality holds if and only if K is congruent to $U^\circ_{\pi-w}$. Also, if $K \subset S^2$ is a convex body of constant width $w \in (\frac{\pi}{2}, \pi)$ and $\varepsilon > 0$ such that $V_{S^2}(K) \leq V_{S^2}(U^\circ_{\pi-w}) + \varepsilon$. Then, there exists a Reuleaux triangle $U \subset S^2$ of width $\pi - w$, such that

$$\delta_{S^2}(K, U^\circ) \leq \mathfrak{c}_w \varepsilon$$

for some constant $\mathfrak{c}_w > 0$ depending only on w.

However, in the hyperbolic spaces, there is no minimal volume for a fixed thickness in any dimension.

Theorem [4]

Let $w > 0$ be a fixed positive number. Then,

$$\inf \{V_{H^n}(K) : K \subset H^n \text{ convex body}, w(K) \geq w\} = 0.$$

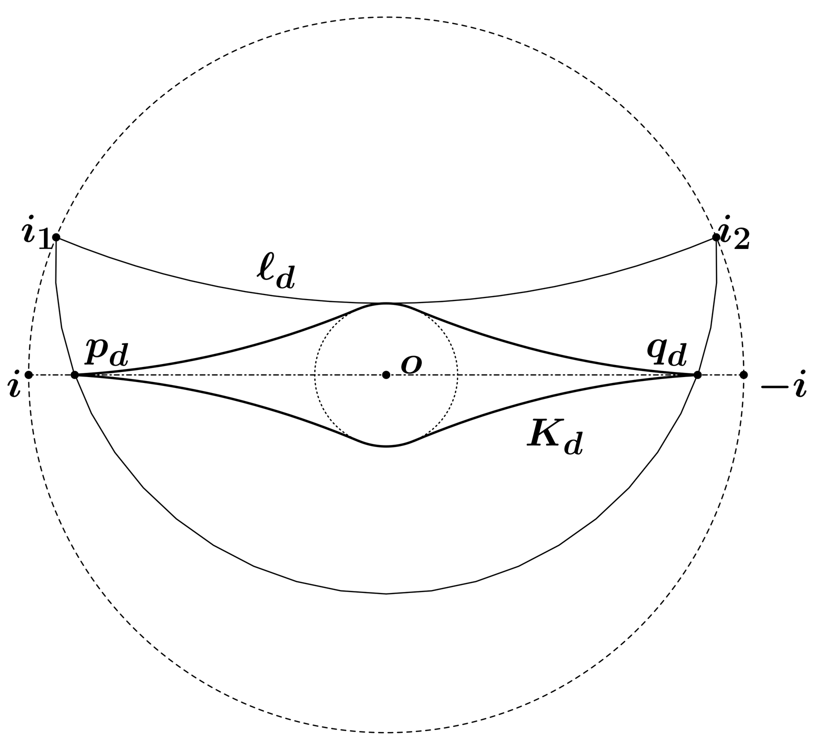

Figure: A convex disc in H^2 with "large" thickness and "small" area

However, among h-convex bodies in the hyperbolic plane, we can find a unique convex body (up to isometries) minimizing the area, if the thickness is fixed. A convex body $K \subset H^n$ is called h-convex, if for any pair of points $x, y \in K$, all connecting hypercyclic arcs are in K (or equivalently, K is an intersection of horoballs). For a hyperbolic regular triangle T, we define the horospherical Reuleaux triangle R_w as the h-convex hull of T, where w is the thickness of R_w.

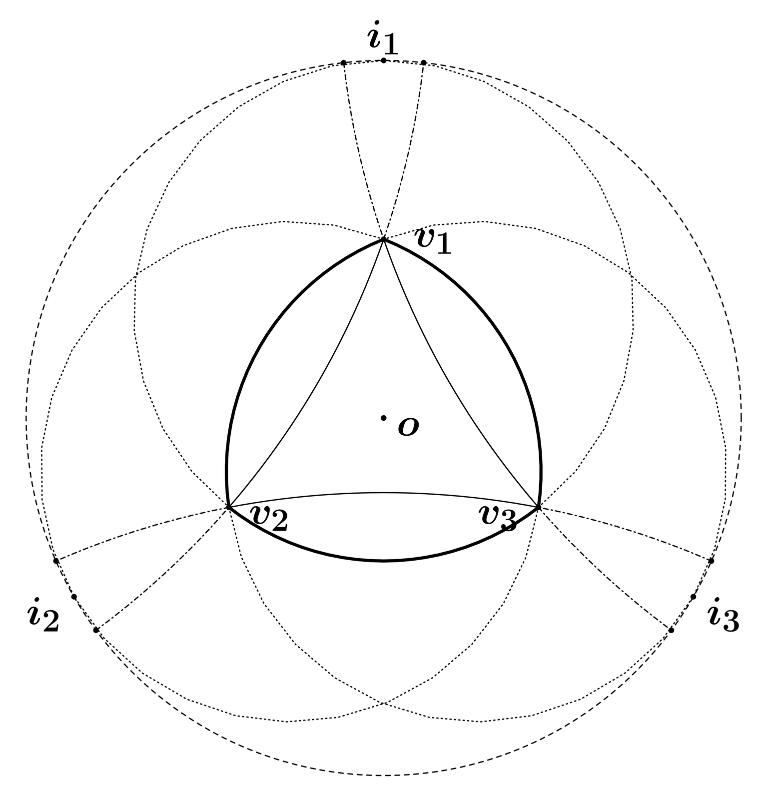

Figure: The horocyclic Reuleaux triangle R_w

Theorem (h-convex isominwidth inequality [4])

For a fixed minimal width $w > 0$, the horocyclic Reuleaux triangle R_w is the unique convex body with the smallest area among h-convex bodies in H^2 up to isometries.

Main ideas behind the results

The key tool for proving the isominwidth problem and its stability on the sphere for thickness $w \in (\frac{\pi}{2}, \pi)$, which coincides with the Blaschke–Lebesgue problem for width w, is the following result.

Theorem (Reverse isodiametric stability [5])

Among bodies of constant width w in $\mathcal{M}^2$, the Reuleaux triangle U_w of diameter w is the unique body with minimal area. Also, if $K \subset \mathcal{M}^2$ is of constant width w such that $V_{\mathcal{M}^2}(K) \leq V_{\mathcal{M}^2}(U_w) + \varepsilon$, then

$$\delta_{\mathcal{M}^2}(K, U) \leq \theta\varepsilon$$

for some positive constant θ and U congruent to U_w.

For the stability, we also use the following lemma by Glasauer [6]:

Lemma [6]

Let $K, L \subset S^2$ be convex bodies such that $\delta_{S^2}(K, L) < \frac{\pi}{2}$. Then,

$$\delta_{S^2}(K, L) = \delta_{S^2}(K^\circ, L^\circ).$$

For the stability in the Euclidean plane, and also in S^2 for $w < \frac{\pi}{2}$, we first need to revisit the main ideas of the proof of Pál's inequality [7, 1]. The following lemma is by Blaschke [2] in $\mathbb{R}^2$, and in S^2 by Bezdek and Blekherman.

Lemma (Blaschke) [2, 1]

Let $K \subset \mathcal{M}^2$ be a convex body of thickness $w > 0$ (in the case $\mathcal{M}^2 = S^2$ we also assume $w \leq \frac{\pi}{2}$). Let $T_w \subset \mathcal{M}^2$ be a regular triangle of thickness w. Then,

$$r(T_w) \leq r(K)$$

with equality if and only if K is a congruent copy of T_w.

We note that in the hyperbolic space there is no Blaschke lemma, as the first figure shows. However, the following h-convex version holds.

Lemma [4]

Let $K \subset H^2$ be an h-convex body of thickness $w > 0$. Then,

$$r(R_w) \leq r(K)$$

with equality if and only if K is a congruent copy of R_w.

If K is different from both the circular disk and the regular triangle T_w, then we can find a spiky disk in K with three congruent, non-overlapping spikes, similarly to the case shown in the following figure.

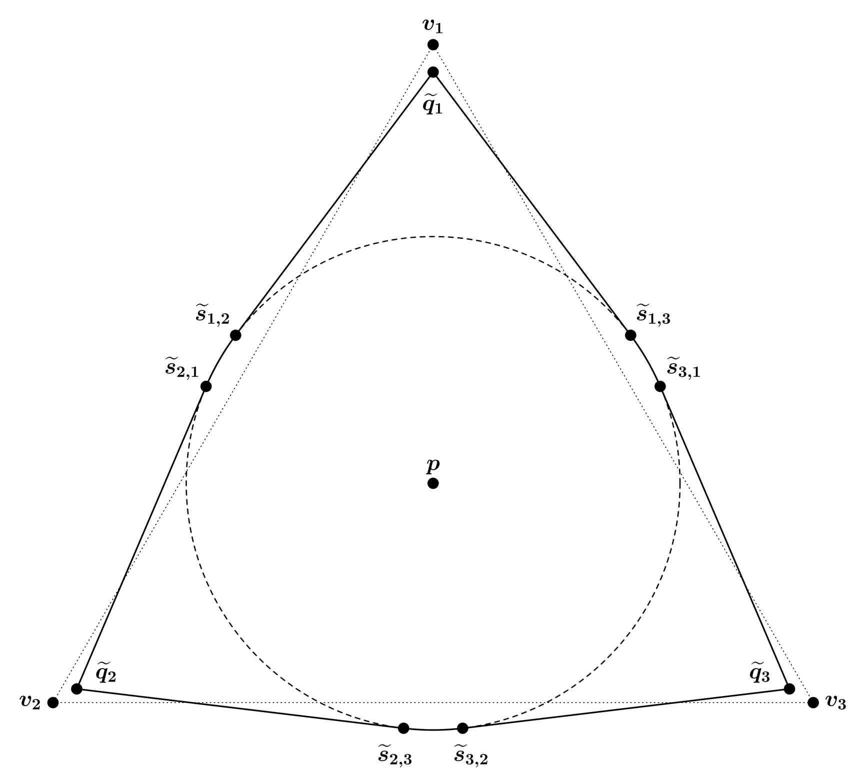

Figure: The symmetric "cap-domain" $C_w(\varrho)$

Then, we can show that the area of the cap-domain increases in the difference η of the inradius and the inradius of the regular triangle. One can use a similar argument for the h-convex case. As for the stability, we first find a suitable copy of T_w. Then, we can bound the distance of K and the cap-domain contained in it, then the distance of the cap-domain in K and the symmetric one in Figure 3, finally we estimate $\delta_{\mathcal{M}^2}(C_w(\varrho), T_w)$.

References

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