MEAN DISTANCE IN POLYHEDRA

Dominik Beck
Department of Mathematics Education, Charles University, 186 75 Prague, Czech Republic

FACULTY OF MATHEMATICS AND PHYSICS
Charles University

MAIN RESULT

> "In any given polyhedron, the mean distance between two of its inner points selected at random can be always expressed in the exact form. The same holds for all the moments."

ABSTRACT

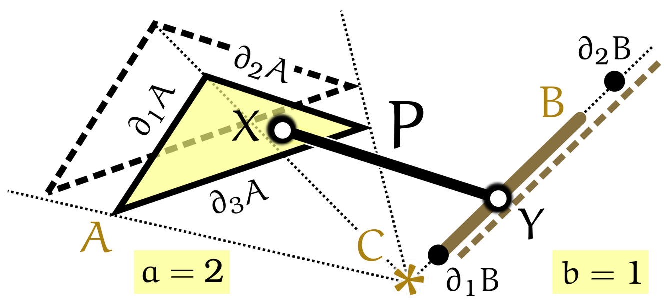
The mean distance between two random points uniformly selected from the interior of a given convex polyhedron is derived for various polyhedra using a modification of the Crofton's reduction technique. It is shown that the method can be easily extended to find the exact value of the mean distance in any convex polyhedra in general.

INTRODUCTION

Let $K \in \mathbb{R}^3$ be a polyhedron (not necessarily convex), from which we select two random points X, Y uniformly. Denote $L = |XY|$ the distance between them and $L_{KK}^{(p)} = \mathbb{E}[L^p]$ its p-th statistical moment. Even-power moments are trivial to compute. Prior to our findings, the value $L_{KK}^{(1)}$ has been known in the exact form only for K being a ball[1] or a cube[2] (the so called Robbins constant). Are there explicit closed form formulae for other polyhedra apart from a cube?

CROFTON'S REDUCTION TECHNIQUE 6 ⟶ 2

Let $P(X, Y)$ be a random variable whose value scales by the factor of p with respect to a given scaling point C. Suppose that the points X and Y are chosen randomly uniformly from the domains A and B with unique dimensions a and b, respectively. Denote $P_{AB} = \mathbb{E}[P \mid X \in A, Y \in B]$. Then, if C lies in the intersection of the affine hulls of A and B, the technique of Crofton reduction provides us with the formula below. In our problem, we simply put $P = L^p$. Note that whenever it is unambiguous, we often write P_{ab} instead of P_{AB}.

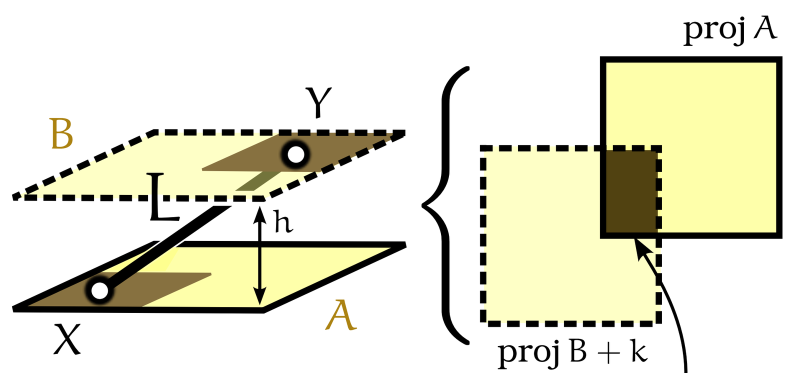


$$pP_{AB} = a(P_{\partial A\,B} - P_{AB}) + b(P_{A\,\partial B} - P_{AB})$$

where $P_{\partial A\,B} = \sum_i w_i P_{\partial_i A\,B}$ with weights $w_i = \frac{u_i}{\sum_i u_i}$,
$u_i = \mu_{\partial_i A}\hat{\mathbf{n}}_{\partial_i A}\cdot\vec{C\partial_i A} = \pm\mu_{\partial_i A}|C\partial_i A|$ outer normal

OVERLAP FORMULA 4 3

If A and B are nonetheless parallel (the intersection of their affine hulls is empty) with separation $h = |AB|$, we use the following overlap formula:



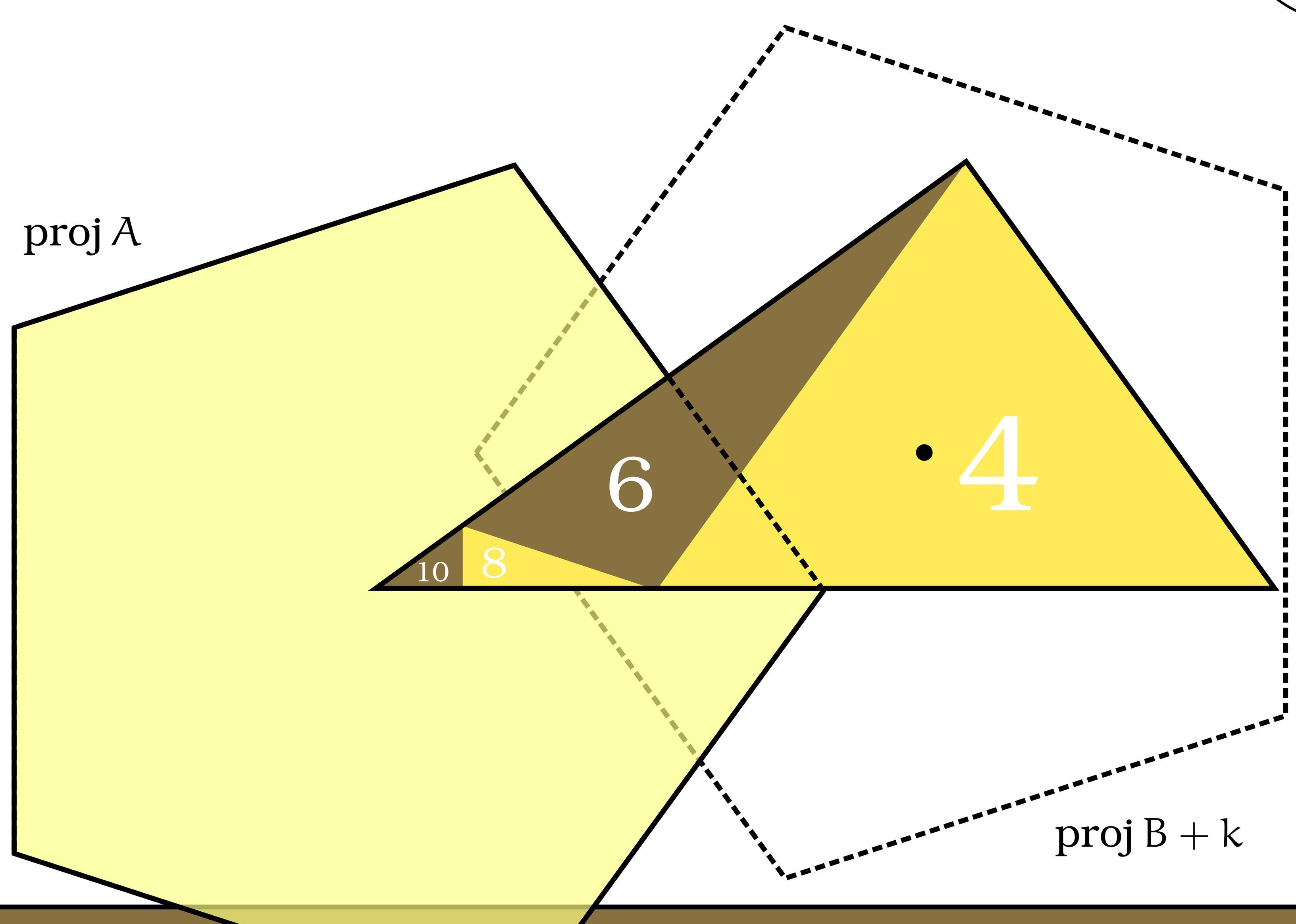
$$L_{AB}^{(p)} = \frac{1}{\mu_A\mu_B}\int_A\int_B |x-y|^p\,dx\,dy = \frac{1}{\mu_A\mu_B}\int_{\mathbb{R}^2}(h^2+k^2)^{\frac{p}{2}}\mu_{\text{proj}\,A\cap(\text{proj}\,B+k)}\,dk$$

DIRECT INTEGRATION 2

With help of polar coordinates, the problem of finding the mean length beween a point and a paralelogram or between any nonparallel two line segments (which is equivalent with the former by using translation argument) can be expressed in terms of elementary functions.

$$L_{20}^{(p)} = \frac{16}{3h^2\sqrt3}\int_0^{\frac{\pi}{3}}\int_{\frac{h}{\sqrt2\cos\varphi}}^{\frac{h}{2\sqrt2\cos\varphi}}(h^2+r^2)^{\frac{p}{2}}\,r\,dr\,d\varphi$$

$$L_{11}^{(p)} = \frac{8}{3h^2\sqrt3}\int_0^{\frac{\pi}{3}}\left(2\int_0^{\frac{h}{2\sqrt2\cos\varphi}}+\int_0^{\frac{h}{\sqrt2\cos\varphi}}\right)(h^2+r^2)^{\frac{p}{2}}\,r\,dr\,d\varphi$$



MEAN OCTAHEDRON DISTANCE (DEMONSTRATION)

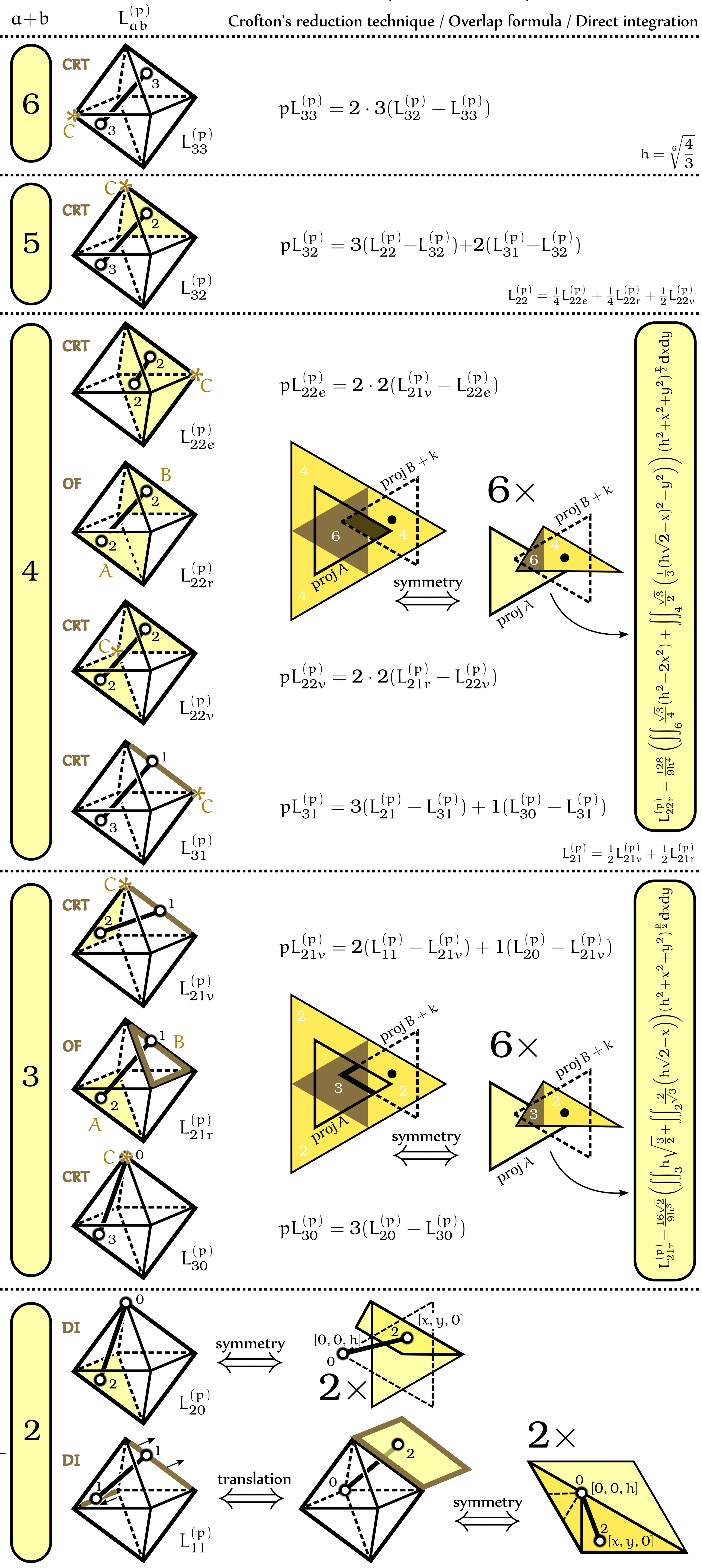


TABLE OF NEW RESULTS

(all the solids have unit volume)

ball[1]	0.63807479	$\frac{18}{35}\sqrt[3]{\frac{6}{\pi}}$
icosahedron	0.64131249	(right margin)
dodecahedron	0.64252068	(left margin)
octahedron	0.65853073	$\sqrt[3]{\frac34}\left(\frac{4}{105}+\frac{13\sqrt2}{105}-\frac{4\pi}{45}+\frac{109\ln3}{630\sqrt2}+\frac{16\,\mathrm{arccot}\sqrt2}{315}+\frac{158\,\mathrm{arccoth}\sqrt2}{315}\sqrt2\right)$
cube[2]	0.66170718	$\frac{4}{105}+\frac{17\sqrt2}{105}-\frac{2\sqrt3}{35}-\frac{\pi}{15}+\frac15\,\mathrm{arccoth}\sqrt2+\frac45\,\mathrm{arccoth}\sqrt3$
tetrahedron	0.72946242	$\sqrt[3]{3}\left(\frac{\sqrt2}{7}-\frac{37\pi}{315}+\frac{4}{15}\arctan\sqrt2+\frac{113\ln3}{210\sqrt2}\right)$

Left margin (dodecahedron):

$$\frac{1}{\sqrt[3]{30+14\sqrt5}}\left(\frac{1516}{1575}+\frac{2\sqrt{\frac25}}{45}-\frac{124\sqrt{\frac35}}{175}-\frac{71\sqrt2}{1575}-\frac{12\sqrt3}{35}+\frac{342}{175\sqrt5}+\frac{493\pi}{23625}+\frac{67\pi}{945\sqrt5}+\frac{(397-244\sqrt5)\,\mathrm{arccot}\,2}{18900}+\frac{(24023+11788\sqrt5)\left(\arccos\frac23-\arccos\frac13\right)}{94500}-\frac{(461+212\sqrt5)\left(\arccos\frac{23}{41}+\arccos\frac{39}{41}\right)}{1000}-\frac{(1031+521\sqrt5)\,\mathrm{arccosh}\,\frac{13}{3}}{75600}+\frac{(367+163\sqrt5)\,\mathrm{arccosh}\,9}{16800}+\frac{(22197+8149\sqrt5)\left(\mathrm{arccosh}\,\frac{121}{41}-\mathrm{arccosh}\,\frac{57}{41}\right)}{84000}+\frac{(15763+7063\sqrt5)\left(\mathrm{arccosh}\,\frac73-\mathrm{arccosh}\,3\right)}{21000}+\frac{(288889+129739\sqrt5)\ln3}{378000}+\frac{2(423+187\sqrt5)\left(\mathrm{arccosh}\,4-\mathrm{arccosh}\,2\right)}{875}+\frac{(109-3143\sqrt5)\ln5}{151200}\right)$$

Right margin (icosahedron):

$$\frac12\sqrt[3]{\frac95-\frac{3}{\sqrt5}}\left(\frac{197}{525}+\frac{239}{525\sqrt5}-\frac{44}{525}\sqrt{\frac{2(5+\sqrt5)}{5}}-\frac{(17226+6269\sqrt5)\pi}{157500}-\frac{(2186+1413\sqrt5)\,\mathrm{arccot}\,\phi}{15750}+\frac{(82-75\sqrt5)\,\mathrm{arccot}(\phi^2)}{5250}+\frac{4(2139+881\sqrt5)\,\mathrm{arcsch}\,\phi}{7875}+\frac{(15969+7151\sqrt5)\,\mathrm{arccoth}\,\phi}{12600}+\frac{(4449-1685\sqrt5)\ln3}{42000}-\frac{(75783+37789\sqrt5)\ln5}{252000}\right)$$

REFERENCES

[1] trivial

[2] David Robbins and Theodore Bolis. "Average Distance between Two Points in a Box." In: Amer. Math. Monthly 85 (1978), p. 278.

RSJ