Sharp Quantitative Stability of the Brunn-Minkowski Inequality

Alessio Figalli[1], Peter van Hintum [2], Marius Tiba [3]

[1]ETH, Zurich, Switzerland.

[2]New College, University of Oxford, United Kingdom.

[3]Mathematical Institute, University of Oxford, United Kingdom

Introduction

The Brunn-Minkowski inequality states that for bounded measurable sets A and B in $\mathbb{R}^d$ of equal volume and $t \in (0, 1/2]$, we have

$$|tA + (1-t)B| \geq |A|.$$

Equality holds exactly if $A = B$ is convex (up to a zero-set). The stability question asks if A and B are close to achieving equality, how close A and B are to each other and to being convex. Hence, we're looking for quantitative relations between δ, γ, and ω, defined as follows.

Bonus Result!
For convex sets $X, Y \subset \mathbb{R}^d$, we have

$$\frac{|\operatorname{co}(X \cup Y)|}{\min\{|X|, |Y|\}} - 1 \leq O_d\left(\frac{|X \triangle Y|}{|X \cap Y|}\right).$$

Parameters

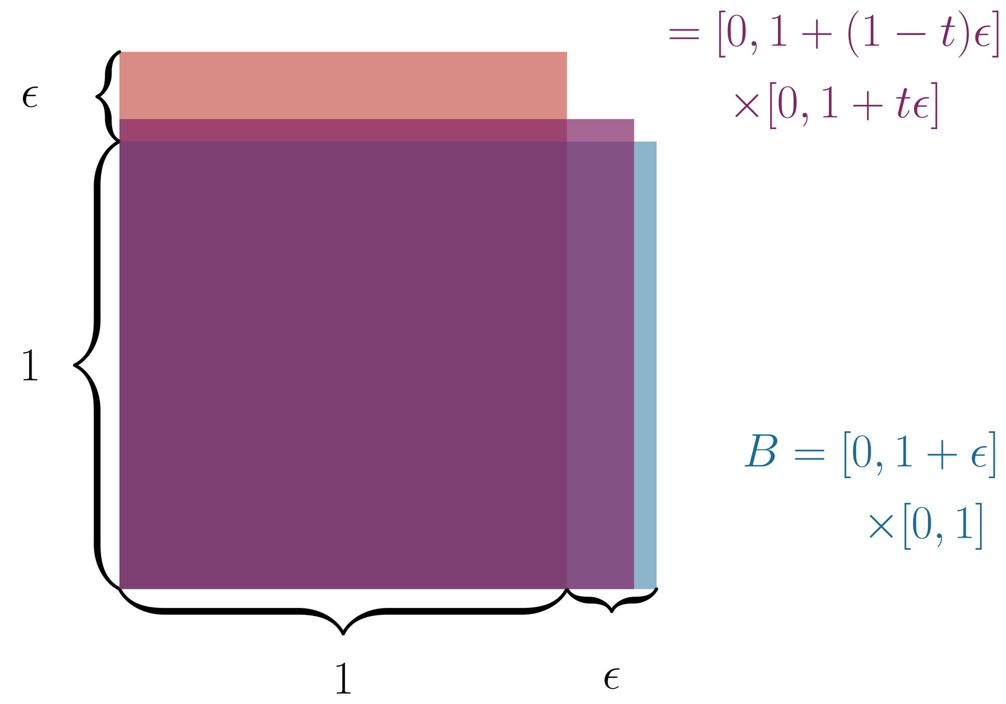
$$\delta := \frac{|tA + (1-t)B| - |A|}{|A|},$$

$$\omega := \min_{x \in \mathbb{R}^d} \frac{|\operatorname{co}(A \cup (x + B)) \setminus A|}{|A|},$$

$$\gamma := \frac{|\operatorname{co}(A) \setminus A| + |\operatorname{co}(B) \setminus B|}{|A|}.$$

Example 1: Quadratic Stability

$A = [0,1] \times [0, 1+\epsilon]$ $\quad tA + (1-t)B = [0, 1 + (1-t)\epsilon] \times [0, 1 + t\epsilon]$



In this example,

$$\delta \approx |tA + (1-t)B| - |A|$$
$$= (1 + t\epsilon)(1 + (1-t)\epsilon) - (1+\epsilon) \approx t\epsilon^2, \text{ and}$$
$$\omega \approx |\operatorname{co}(A \cup B) \setminus A|$$
$$= (1 + 2\epsilon + \epsilon^2/2) - (1+\epsilon) \approx \epsilon.$$

Hence;

$$\omega \approx \sqrt{\frac{\delta}{t}}.$$

Results

Theorem 1 *For all $d \in \mathbb{N}$ and $t \in (0, 1/2]$, there are $c_d, \Delta_{d,t} > 0$ such that the following holds. Assume $\delta \in [0, \Delta_{d,t}]$ and let $A, B \subset \mathbb{R}^d$ be measurable sets with equal volume satisfying*

$$|tA + (1-t)B| = (1+\delta)|A|.$$

Then, up to translation, there is a convex set $K \supset A \cup B$ such that

$$|K \setminus A| + |K \setminus B| \leq c_d \sqrt{\frac{\delta}{t}} |A|.$$

Theorem 2 *Under the same assumptions, we moreover have*

$$|\operatorname{co}(A) \setminus A| + |\operatorname{co}(B) \setminus B| \leq t^{-c_d} \delta |A|.$$

Hence,

$$\omega \leq c_d \sqrt{\frac{\delta}{t}} \quad \text{and} \quad \gamma \leq t^{-c_d}\delta.$$

Steps of the Proof of Theorem 1

1. By Theorem 2 and the Bonus Result, it suffices to show that after some translation $|A \triangle B| \leq c_d \sqrt{\delta / t} |A|$.
2. We may assume for some convex $K \subset \mathbb{R}^d$; $B(o, 1/100) \subset 0.99K \subset A, B \subset K \subset B(o, 100)$.
3. Partition into cones so narrow that they register the atomic structure of A and B (see Proposition 3) and reduce to the stability problem inside each cone.
4. In a cone C, if $\min_{x \in \mathbb{R}^d} |(A \cap C) \triangle x + (B \cap C)|$ is small, then so is $|(A \cap C) \triangle (B \cap C)|$ (using that $A \cap C$ and $B \cap C$ are almost convex).
5. The narrowness of the cones gives them an approximately two-dimensional structure.

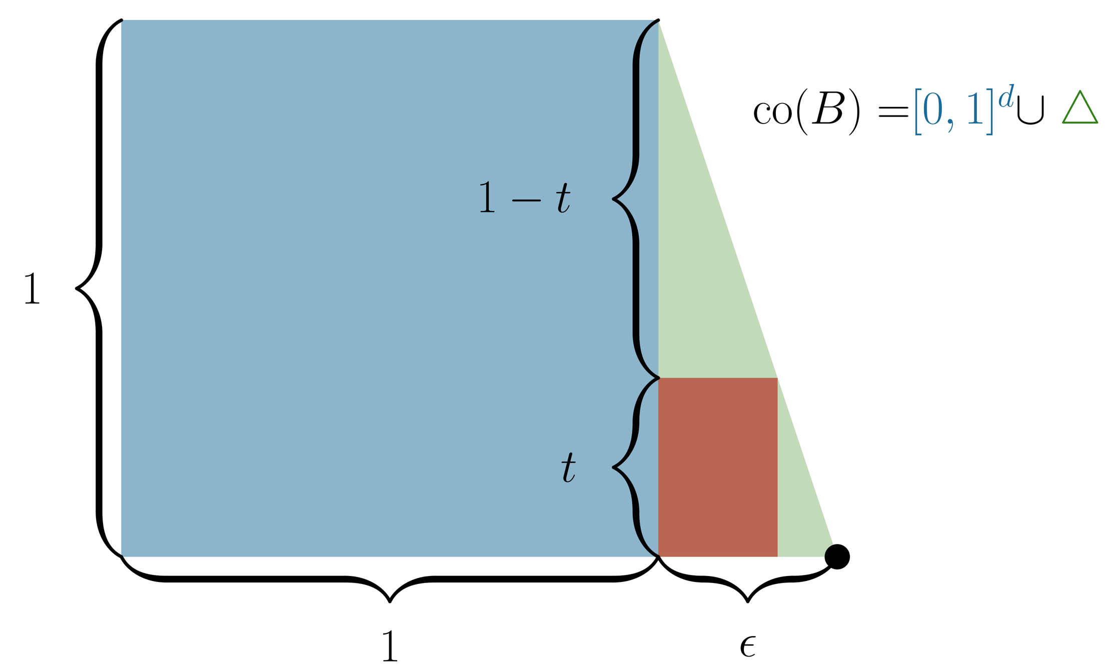
Cones

Proposition 3 *Consider $A, B \subset \mathbb{R}^d$ with $|A| = |B|$, then (up to translation) there exists a partition $\mathcal{C}$ of $\mathbb{R}^d$ into convex cones C with apex at the origin so that:*

- $|C \cap A| = |C \cap B|$ *for all* $C \in \mathcal{C}$,
- $\sum_{C \in \mathcal{C}} |t(A \cap C) + (1-t)(B \cap C)| \leq |tA + (1-t)B|$,
- *C is narrow in all but at most one direction for almost all $C \in \mathcal{C}$, and*
- *C is essentially the convex hull of few lines for almost all $C \in \mathcal{C}$.*

Example 2: Linear stability

$A = \operatorname{co}(A) = [0,1]^d \qquad B = [0,1]^d \cup \{(1+\epsilon, 0, \dots, 0)\}$



$$tA + (1-t)B = [0,1]^d \cup [1, 1 + (1-t)\epsilon] \times [0, t]^{d-1}$$

In this example,

$$\delta = |tA + (1-t)B| - |A| = t^{d-1}(1-t)\epsilon, \text{ and}$$
$$\gamma = |\operatorname{co}(A) \setminus A| + |\operatorname{co}(B) \setminus B| = 0 + \epsilon/d.$$

Hence;

$$\gamma = \Omega_d(t^{d-1}\delta).$$

Ideas of the Proof of Theorem 2

We consider the case $\operatorname{co}(A) = \operatorname{co}(x_0, \dots, x_d)$ is a simplex and show that $|\operatorname{co}(A) \setminus A| = O_{d,t}(\delta)$.

1. Note that $|\operatorname{co}(A)| \leq (1 + \epsilon_d)|A|$ (by weak stability).
2. Find $v \in A$ central in $\operatorname{co}(A)$ so that:
   - $\max_i |vx_i| \leq 0.9 \max_{i,j} |x_i x_j|$, and
   - letting $A_i := A \cap \operatorname{co}(x_0, \dots, x_{i-1}, v, x_{i+1}, \dots, x_d)$, then $|\operatorname{co}(A_i)| \geq \frac{1}{2(d+1)} |\operatorname{co}(A)|$.
3. Find a partition $B = B_0 \sqcup \dots \sqcup B_d$ so that
   - $|A_i| = |B_i|$, and
   - $|(tA_i + (1-t)B_i) \cap (tA_j + (1-t)B_j)| = 0$.
4. Note that $\sum_i |\operatorname{co}(A_i) \setminus A_i| = |\operatorname{co}(A) \setminus A|$ and $\sum_i |tA_i + (1-t)B_i| - |A_i| \leq |tA + (1-t)B| - |A|$, so that we can iterate in each subsimplex.
5. Iterate (keep finding central points and subsimplices) as long as $|\operatorname{co}(A_i)| \leq (1 + \epsilon_d)|A_i|$.
6. Distinguish three classes of simplices $\operatorname{co}(A_i)$, all of which satisfy $\sum_i |\operatorname{co}(A_i) \setminus A_i| \leq O_{d,t}(\delta)|A|$:
   - Low density: $|\operatorname{co}(A_i)| > (1 + \epsilon_d)|A_i|$, these have large doubling.
   - High density, small radius: $|\operatorname{co}(A_i)| \leq (1 + \epsilon_d)|A_i|$ and $\max\{|xy| : x, y \in \operatorname{co}(A_i)\} \leq \rho$, these lie close to the boundary of the atoms of A.
   - High density, big radius: $|\operatorname{co}(A_i)| \leq (1 + \epsilon_d)|A_i|$ and $\max\{|xy| : x, y \in \operatorname{co}(A_i)\} > \rho$, these have vanishing combined volume.

References

[1] Marco Barchiesi and Vesa Julin. Robustness of the Gaussian concentration inequality and the Brunn-Minkowski inequality. *Calc. Var. Partial Differential Equations*, 56, 05 2017. doi: 10.1007/s00526-017-1169-x.

[2] Eric Carlen and Francesco Maggi. Stability for the Brunn-Minkowski and Riesz rearrangement inequalities, with applications to Gaussian concentration and finite range non-local isoperimetry. *Canad. J. Math.*, 69(5):1036–1063, 2017.

[3] Michael Christ. Near equality in the Brunn–Minkowski inequality. *arXiv preprint arXiv:1207.5062*, 2012.

[4] Alessio Figalli, Peter van Hintum, and Marius Tiba. Sharp quantative stability of the Brunn–Minkowski inequality. *Forthcoming*, .

[5] Alessio Figalli, Peter van Hintum, and Marius Tiba. Optimal transport methods for quantative stability of the Brunn–Minkowski inequality. *Forthcoming*, .

[6] Alessio Figalli, Peter van Hintum, and Marius Tiba. Symmetric difference and the common convex hull for convex sets. *Forthcoming*, .

[7] Alessio Figalli and David Jerison. Quantitative stability for sumsets in $\mathbb{R}^n$. *J. Eur. Math. Soc. (JEMS)*, 17(5):1079–1106, 2015.

[8] Alessio Figalli and David Jerison. Quantitative stability for the Brunn–Minkowski inequality. *Adv. Math.*, 314:1–47, 2017.

[9] Alessio Figalli and David Jerison. A sharp Freiman type estimate for semisums in two and three dimensional Euclidean spaces. *Ann. Sci. Éc. Norm. Supér.*, 54(4):235–257, 2021.

[10] Alessio Figalli, Francesco Maggi, and Aldo Pratelli. A refined Brunn–Minkowski inequality for convex sets. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 26–6:2511–2519, 2009.

[11] Alessio Figalli, Francesco Maggi, and Aldo Pratelli. A mass transportation approach to quantitative isoperimetric inequalities. *Invent. Math.*, 182(1):167–211, 2010.

[12] Peter van Hintum and Peter Keevash. The sharp doubling threshold for approximate convexity. *arXiv preprint arXiv:2304.01176*, 2023.

[13] Peter van Hintum and Peter Keevash. Locality in sumsets. *arXiv preprint arXiv:2304.01189*, 2023.

[14] Peter van Hintum, Hunter Spink, and Marius Tiba. Sharp stability of Brunn–Minkowski for homothetic regions. *J. Eur. Math. Soc. (JEMS)*, 24(12):4207–4223, 2022.

[15] Peter van Hintum, Hunter Spink, and Marius Tiba. Sharp L^1 Inequalities for Sup-Convolution. *Discrete Analysis*, 7:16pp, 2023.

[16] Peter van Hintum, Hunter Spink, and Marius Tiba. Sharp quantitative stability of the planar Brunn–Minkowski inequality. *J. Eur. Math. Soc. (JEMS)*, 2023+.

Contact information:
Marius Tiba and Peter van Hintum
marius.tiba@maths.ox.ac.uk and
peter.vanhintum@new.ox.ac.uk