Crofton Formulae with a Fixed Subspace

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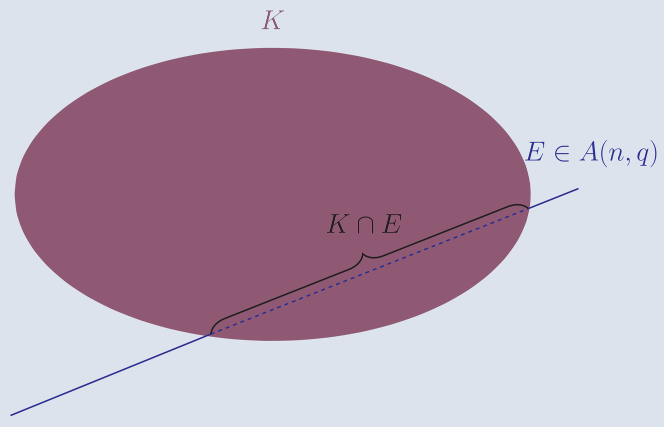
Background

We have the classical Crofton formula for convex sets

$$\int_{A(n,q)} V_{q-j}(K \cap E)\ \mu_q(dE) = c_0 V_{n-j}(K) \qquad (1)$$



for $n \in \mathbb{N}$, $q \in \{0, \dots, n\}$, $j \le q$ and $K \in \mathcal{K}^n$, see [1].
A rotational version is derived in [2],

$$\int_{G(n,k)} \phi_L(K \cap L)\ \nu_k(dL) = V_{n-j}(K), \quad K \in \mathcal{K}^n,$$

where

$$\phi_L(K \cap L) = c_1 \int_{A(L,q)} d(o,E)^{n-k} V_{q-j}(K \cap E) \nu_q^L(dE), \qquad (2)$$

and $k \in \{1, \dots, n-1\}$, $j \le k-1$ and q can be chosen in $\{j, \dots, k-1\}$. Furthermore (2) does not depend on q.

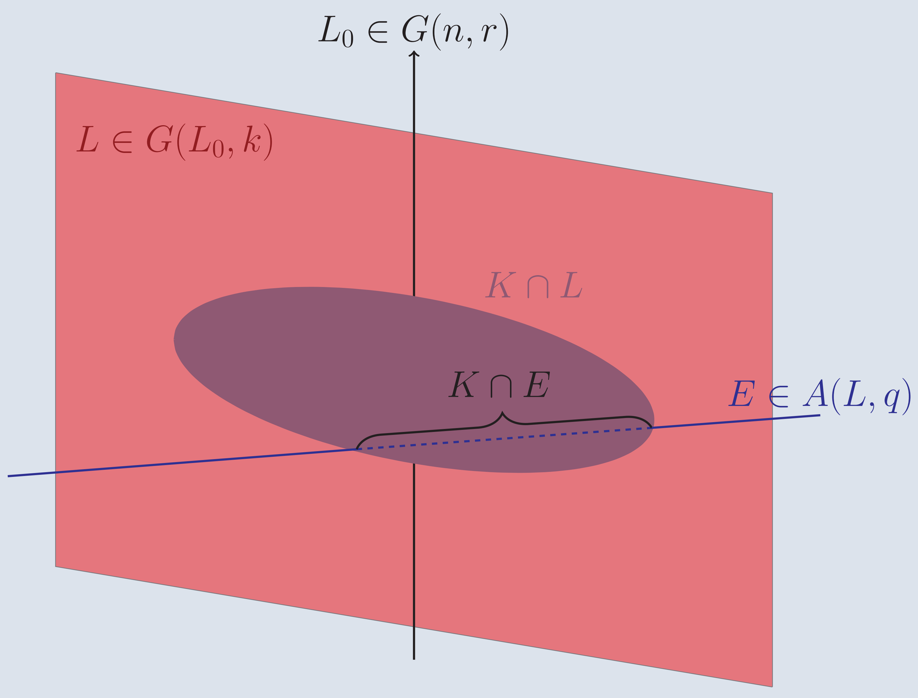
Our aim is to construct Crofton formulae where the integration is over $G(L_0, k)$ instead of $G(n,k)$ for a fixed $L_0 \in G(n,r)$.

New rotational Crofton Formulae

Theorem 1
Let $n, r, k \in \mathbb{N}_0$ with $r+1 \le k \le n$ be given and fix a subspace $L_0 \in G(n,r)$. Then, for $j = 0, \dots, k-(r+1)$ and $K \in \mathcal{K}^n$, we have

$$\int_{G(L_0,k)} \phi_{L,q}^{L_0}(K \cap L) \nu_k^{L_0}(dL) = V_{n-j}(K), \qquad (3)$$

where



$$\phi_{L,q}^{L_0}(K \cap L) = c_2 \int_{A(L,q)} V_{q-j}(K \cap E) \Big(d(0,E)[\mathrm{span}(E), L_0]\Big)^{n-k} \mu_q^L(dE). \qquad (4)$$

Here q can be chosen in $\{j, \dots, k-(r+1)\}$. Furthermore (4) may depend on q.
Sketch of proof
We derive and use the Blaschke-Petkantschin formula

$$\int_{G(L_0,k)} \int_{A(L,q)} f(E) \big(d(0,E)[\mathrm{span}(E), L_0]\big)^{n-k} \mu_q^L(dE) \nu_k^{L_0}(dL)$$
$$= c_3 \int_{A(n,q)} f(E) \mu_q(dE),$$

for $f \ge 0$, and combine it with the classical Crofton formula (1).

Notation

Let $n \in \mathbb{N}$, $k \in \{1, \dots, n\}$ and $r \in \{0, \dots, k\}$.

Intrinsic volumes:
The intrinsic volumes V_i are functionals on $\mathcal{K}^n := \{A \subseteq \mathbb{R}^n : A \neq \emptyset, \text{compact,convex}\}$ defined by

$$\lambda_n(K + \epsilon B^n) = \sum_{i=0}^{n} \epsilon^{n-i} \kappa_{n-i} V_i(K), \quad K \in \mathcal{K}^n.$$

Families of subspaces:
The family of *affine* k-dimensional flats of $\mathbb{R}^n$ is denoted by $A(n,k)$ and the *Grassmannian* of k-dimensional *linear* subspaces by $G(n,k)$.
For $L \in G(n,k)$ we let $A(L,q)$ be the family of *affine* subspaces of dimension $q \le k$ contained in L and $G(L_0,k)$ be *linear* subspaces of dimension k containing $L_0 \in G(n,r)$.

Invariant measures on subspaces:
The unique invariant probability measure on $G(n,k)$ is denoted by ν_k. An invariant measure on $A(n,k)$ is μ_k given by the relation

$$\int_{A(n,k)} f(E)\ \mu_k(dE) =$$
$$\int_{G(n,k)} \int_{L^\perp} f(L+x)\ \lambda_{L^\perp}(dx)\ \nu_k(dL)$$

for $f \ge 0$ measurable and $\lambda_{L^\perp}$ being the Lebesgue measure on $L^\perp$.
Likewise we let $\nu_k^{L_0}$ and $\mu_k^{L_0}$ denote invariant measure on $G(L_0,k)$ and $A(L_0,k)$.

Subspace determinant:
For $M \in G(n,q)$ and $N \in G(n,r)$ such that $q + r \le n$ we let $[M,N]$ denote the subspace determinant defined as the $q+r$ dimensional volume of the set

$$P = \Big\{ \sum_{i=1}^{q} \alpha_i m_i + \sum_{j=1}^{r} \beta_j n_j \ :\ {\scriptstyle 0 \le \alpha_i \le 1 \atop 0 \le \beta_j \le 1} \Big\},$$

where $(m_i)_1^q$ and $(n_i)_1^r$ is an orthonormal basis of M and N, respectively.

The measurement function

Let the assumptions of Theorem 1 be satisfied and fix $q \in \{j, \dots, k-(r+1)\}$. When $L \in G(L_0,k)$ the following statements hold.

(i) $\varphi_{L,q}^{L_0}$ is independent of the choice of q.

(ii) We have

$$\varphi_{L,j}^{L_0}(K') = c_2 \int_{G(L,j)} [M, L_0]^{n-k} \int_{K'|M^\perp} \times d(z, M+L_0)^{n-k}\ \lambda_{M^\perp}(dz)\ \nu_j^L(dM),$$

for all convex bodies $K' \subset L$.

(iii) If $j = 0$ (rotational integral for the volume) this simplifies to

$$\varphi_{L,q}^{L_0}(K') = \frac{\omega_{n-r}}{\omega_{k-r}} \int_{K'} d(x, L_0)^{n-k}\ \lambda_L(dx),$$

for all convex bodies $K' \subset L$.

The integration process

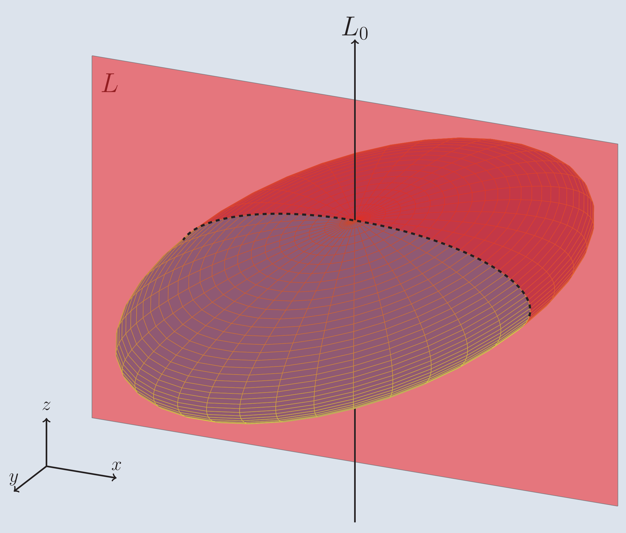


Figure 1: A plane L containing L_0 intersecting an ellipsoid.

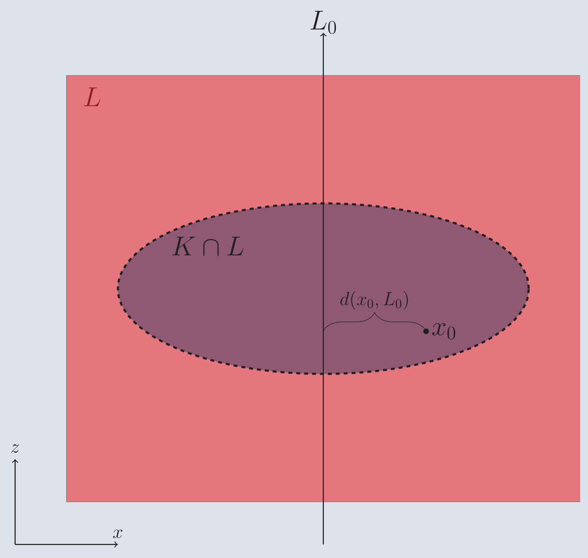


Figure 2: The intersection of L and the ellipsoid.

When $n = 3, k = 2, r = 1$ the integration can be exemplified

1. First, intersect K with a random plane containing L_0 (Figure 1).
2. View the intersection plane as $\mathbb{R}^2$.
3. $\phi_{L,0}^{L_0}(K \cap L)$ is now proportional to the mean distance to L_0 of points in $K \cap L$. (Figure 2).

References

[1] R. Schneider and W. Weil, Stochastic and Integral Geometry. Springer, 2008.

[2] J. Auneau and E. B. V. Jensen, Expressing intrinsic volumes as rotational integrals, *Adv. Appl. Math.*, vol. 45, pp. 1–11, 2010.