Complex L_p-Intersection Bodies

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Summary

- Introduction of complex L_p-intersection bodies
- (Pseudo-)convexity of complex L_p-intersection bodies.
- Busemann type inequalities for complex L_p-intersection bodies for certain values of p

Intersection and Centroid Bodies in Real Vector Spaces

Geometric operators on the set of star bodies and convex bodies, respectively, are main objects of interest in convex geometry. A fundamental operator is the intersection body $\mathrm{I}K$ of a star body K defined by Lutwak [9] by

$$\rho_{\mathrm{I}K}(u) = \left|K \cap u^{\perp}\right|, \quad u \in \mathbb{S}^{n-1}.$$

Formulated in different terms, Busemann showed that the operator I maps origin-symmetric convex bodies to convex bodies, as well as established his famous intersection inequality, which states that among convex bodies K containing the origin in their interiors, the ratio

$$|\mathrm{I}K|/|K|^{n-1} \tag{1}$$

is maximized precisely by origin-symmetric ellipsoids. Recently, this notion got extended to the L_p-intersection bodies I_pK (see [5, 7]), which are defined by

$$\rho^{-p}_{\mathrm{I}_pK}(u) = \int_K |(x,u)|^p dx, \quad u \in \mathbb{S}^{n-1},$$

for all nonzero $p > -1$. Note that, if $p \geq 1$, the right hand side is (up to normalization) the polar body of the L_p-centroid body, which was introduced by Lutwak and Zhang [11], who also together with Yang [10], established the important L_p-Busemann–Petty centroid inequalities for them. The crucial connection between these operators and the intersection body is given by

$$\lim_{p\to -1^+} \left(\frac{1}{\Gamma(p+1)}\right)^{-1/p} \mathrm{I}_pK = 2 \cdot \mathrm{I}K.$$

Recently, Berck [3] extended Busemann's convexity theorem to L_p-intersection bodies and Adamczak et al [2] proved Busemann type intersection inequalities for them.

Intersection and Centroid Bodies in Complex Vector Spaces

The operator of the previous section were investigated rigorously in real vector spaces but they have only been considered in complex vector spaces recently. Motivated by an earlier work of Abardia and Bernig [1] on projection bodies in complex vector spaces, Haberl [6] defined the complex centroid body by

$$h_{\Gamma_C K}(u) = \frac{1}{|K|}\int_K h_{Cu}(x)dx, \quad u \in \mathbb{C}^n,$$

where C is a convex body in $\mathbb{C}$ and $Cu := \{cu : c \in C\}$. Haberl also obtained a complex Busemann–Petty centroid inequality.

In [8], Koldobsky, Paouris and Zymonopoulou defined a complex analogue of the intersection body for $\mathbb{S}^1$-invariant star bodies in $\mathbb{C}^n$ by considering complex hyperplanes in the definition of the intersection body, i.e.

$$\pi\rho^2_{\mathrm{I}_cK}(u) = \left|K \cap u^{\perp_{\mathbb{C}}}\right|, \quad u \in \mathbb{S}^{2n-1},$$

and established a complex type convexity theorem, which states that if K is $\mathbb{S}^1$-invariant and convex, then I_cK is convex.

Similar to the case of the L_p-intersection body, we want to find a connection between these geometric operators with analogous properties as in the real setting.

Main Results

In [4], we define the complex L_p-intersection body of a star body K, similarly to Haberl's definition of the complex centroid body,

$$\rho^{-p}_{\mathrm{I}_{C,p}K}(u) = \int_K h_{Cu}(x)^p dx, \quad u \in \mathbb{C}^n, \tag{2}$$

where $C \subseteq \mathbb{C}$ is a convex body containing the origin in its relative interior and nonzero $p \in (-\dim C, 1)$. Clearly, if $C = [-1,1]$, this is the L_p-intersection body. For $p = 1$, it equals (up to normalization) the polar body of the complex centroid body. Note that for $\dim C = 2$, (2) is well-defined for $p > -2$, which extends the range of the real analogue.

As a first result, we establish an interpolation property of complex L_p-intersection bodies between the complex intersection body and complex centroid bodies, in analogy with the real case:

$$\lim_{p\to -2^+} \left(\frac{1}{\Gamma(p+2)}\right)^{-1/p} \mathrm{I}_{C,p}K = k_C \cdot \mathrm{I}_cK,$$

for all admissible $C \subseteq \mathbb{C}$ with $\dim C = 2$, where $k_C > 0$.

Similar to the case of the complex intersection body, we use $\mathbb{S}^1$-invariance to prove (pseudo-)convexity of complex L_p-intersection bodies. It is an open question, whether one can strengthen the statement for $p < -1$.

Theorem

If K is convex and $\mathbb{S}^1$-invariant, then $\mathrm{I}_{C,p}K$ is convex, if $p \geq -1$, and $\operatorname{int} \mathrm{I}_{C,p}K$ is pseudo-convex, if $-2 < p < -1$.

As another analogue to the real case, we establish a sharp upper bound for the volume of $\mathrm{I}_{C,p}K$ for all origin-symmetric convex bodies $C \subseteq \mathbb{C}$ and specific values of p. We deduce these inequalities from Busemann's intersection inequality (1), as well as the coresponding inequalities for the L_p-intersection bodies.

Theorem

Let C be origin-symmetric and $0 < p < 1$ or $-1 \leq p < 0$ and $n/|p| \in \mathbb{N}$. Among all star bodies K containing the origin in their interiors, the ratio

$$V_{2n}\left(\mathrm{I}_{C,p}K\right)/V_{2n}\left(K\right)^{2n+p}$$

is maximized by $\mathbb{S}^1$-invariant ellipsoids. If $p = -1$, these are the only maximizers.

References

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