Jung-Type Inequalities and Blaschke-Santalo Diagrams for Different Diameter Variants

René Brandenberg & Mia Runge

Technical University of Munich

Notation

$\mathcal{C}^n := \{C \subset \mathbb{R}^n,\ C$ compact convex$\}$, $\mathcal{C}^n_0 := \{C \subset \mathcal{C}^n,\ 0 \in \operatorname{int} C\}$;
$K \in \mathcal{C}^n, C \in \mathcal{C}^n_0$ always.
Polar: $C^\circ = \{a \in \mathbb{R}^n :\ a^T x \le 1,\ x \in C\}$
Support function: $h_C(\cdot) : \mathbb{R}^n \to \mathbb{R},\ h_C(a) := \max_{x \in C} a^T s$
Gauge function: $\|x\|_C = \min\{\lambda \ge 0 : x \in \lambda C\}$
Circumradius: $R(K,C) := \min\{\rho \ge 0 : \exists t \in \mathbb{R}^n$ such that $K \subset t + \rho C\}$
Inradius: $r(K,C) := \max\{\rho \ge 0 : \exists t \in \mathbb{R}^n$ such that $t + \rho C \subset K\}$

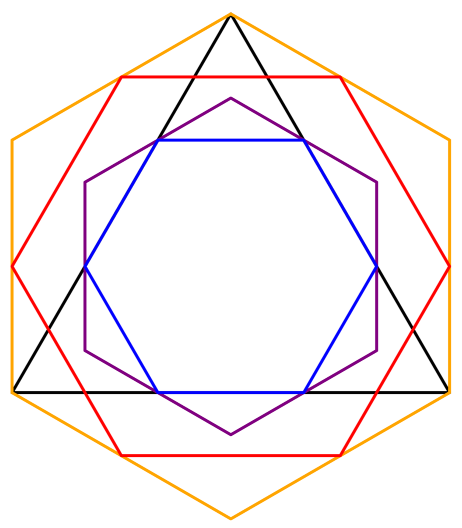
Diameter Definitions

- *Minimum diameter*:
$$D_{\mathrm{MIN}}(K,C) = \max_{x,y \in K} \|x-y\|_C$$
- *Harmonic diameter*:
$$D_{\mathrm{HM}}(K,C) = \max_{x,y \in K} \frac{1}{2}\left(\|x-y\|_C + \|y-x\|_C\right)$$
- *Arithmetic diameter* (standard diameter):
$$D_{\mathrm{AM}}(K,C) = \max_{x,y \in K} 2R([x,y],C) = \max_{s \in \mathbb{R}^n \setminus \{0\}} 2 \cdot \frac{h_K(s) + h_K(-s)}{h_C(s) + h_C(-s)}$$
- *Maximum diameter*:
$$D_{\mathrm{MAX}} = \max_{s \in \mathbb{R}^n \setminus \{0\}} \frac{h_K(s) + h_K(-s)}{\max\{h_C(s), h_C(-s)\}}$$
- If C 0-symmetric
$$D(K,C) = D_{\mathrm{MIN}}(K,C) = D_{\mathrm{HM}}(K,C) = D_{\mathrm{AM}}(K,C) = D_{\mathrm{MAX}}(K,C)$$

Symmetrizations

Symmetrizing the second argument:

- $D_{\mathrm{MIN}}(K,C) = D(K, C \cap (-C))$
- $D_{\mathrm{HM}}(K,C) = D(K, (\frac{C^\circ - C^\circ}{2})^\circ)$
- $D_{\mathrm{AM}}(K,C) = D(K, \frac{C-C}{2})$
- $D_{\mathrm{MAX}} = D(K, \operatorname{conv}(C \cup (-C)))$



Symmetrizations:

- *minimum* $C_{\mathrm{MIN}} := C \cap -C$
- *harmonic mean* $C_{\mathrm{HM}} := \left(\frac{C^\circ - C^\circ}{2}\right)^\circ$
- *arithmetic mean* $C_{\mathrm{AM}} := \frac{C-C}{2}$
- *maximum* $C_{\mathrm{MAX}} := \operatorname{conv}(C \cup -C)$

Notation: $\mathrm{M} \in \{\mathrm{MIN}, \mathrm{HM}, \mathrm{AM}, \mathrm{MAX}\}$

Extended harmonic-arithmetic mean inequality:
$$C_{\mathrm{MIN}} \subset C_{\mathrm{HM}} \subset C_{\mathrm{AM}} \subset C_{\mathrm{MAX}},$$
with equality between any of the means if and only if $C = -C$.

Centering the Gauge

Minkowski-asymmetry: $s(C) := R(C, -C)$
C is *Minkowski-centered* if $C \subset -s(C)C$.

Jung-Type Inequalities

Theorem 1

i) $C \in \mathcal{C}^2$ *with* $0 \in C$.
$$\frac{2}{3} R(K,C) \le D_{\mathrm{MAX}}(K,C).$$
Equality only if C *is a triangle with one vertex at the origin.*

ii) $C \in \mathcal{C}^2_0$*, Minkowski-centered.*
$$R(K,C) \le D_{\mathrm{MAX}}(K,C).$$
Equality if C *is a triangle and* $K = \lambda \frac{1}{2}(-C) + (1-\lambda) C_{\mathrm{MAX}}$ *with* $\lambda \in [0,1]$.

- Consequence: $R(K,C) \le D_{\mathrm{M}}(K,C)$.
- $R(K,C) = D_{\mathrm{AM}}(K,C)$ only for C triangle and $K = -C$.
- Stronger inequality for D_{MIN}:

Theorem 2
$$\frac{n+1}{n} R(K,C) \le D_{\mathrm{MIN}}(K,C)$$
Equality only if K *is a simplex. For every* $s_C \in [1,n]$*, there exists* C *with* $s(C) = s_C$ *and a simplex* K *such that the inequality is tight.*

Additional Inequalities

Triangular gauges (for example with $\mathrm{M} = \mathrm{MAX}$):
For Minkowski-centered triangles S,
$$\left(\frac{D_{\mathrm{MAX}}(K,S)}{R(K,S)} - \frac{1}{2}\right)\left(\frac{3}{2} - \frac{D_{\mathrm{MAX}}(K,S)}{R(K,S)}\right) \le \frac{r(K,S)}{R(K,S)}.$$

General inequalities:
$$D_{\mathrm{M}}(K,C) \le 2R(C_{\mathrm{AM}}, C_{\mathrm{M}}) R(K,C)$$
$$2R(C_{\mathrm{AM}}, C_{\mathrm{M}}) r(K,C) \le D_{\mathrm{M}}(K,C)$$

Completion of the Gauge

- K *complete*: every $\tilde{K} \supset K$ has larger diameter
- *Completion*: $K^* \supset K$, complete, with $D_{\mathrm{M}}(K,C) = D_{\mathrm{M}}(K^*,C)$
- Gauge C always complete for $\mathrm{M} = \mathrm{AM}$ or if $C = -C$.
- C_{M} complete w. r. t. C using D_{M}.

- Let $C \in \mathcal{C}^n_0$ be Minkowski-centered. Then,
  i) C_{MAX} is always a completion of C using D_{MAX},
  ii) C_{MAX} is the unique symmetric completion and
  iii) C_{MAX} is the unique completion iff $C_{\mathrm{MAX}} = \bigcap_{a \in \mathrm{bd}(C^\circ) \cap \mathrm{bd}(-C^\circ)} H^{\le}_{(a,1)}$.
- Let $C \in \mathcal{C}^n_0$ be Minkowski-centered. The following are equivalent:
  i) $\frac{s(C)+1}{2} C_{\mathrm{HM}}$ is a completion of C using D_{HM},
  ii) $R(C_{\mathrm{AM}}, C_{\mathrm{HM}}) = \frac{s(C)+1}{2}$ and
  iii) $D(C, C_{\mathrm{HM}}) = 2R(C, C_{\mathrm{HM}})$.
- ρC_{MIN} is a completion of C using D_{MIN} iff $\rho = s(C) = 1$ $(C = -C)$.

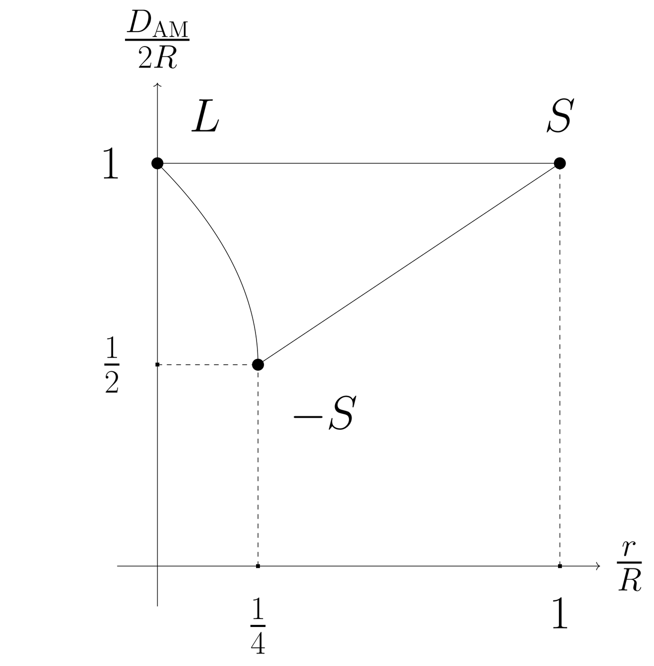
Blaschke-Santaló Diagrams

Given a gauge $C \in \mathcal{C}^n_0$ and values (r, R, D), is there a convex body $K \in \mathcal{C}^n$ such that its inradius w. r. t. C is r, its circumradius is R, and its diameter is D?
Blaschke-Santaló diagram: $f_{\mathrm{M}}(\mathcal{C}^n, C)$ with
$$f_{\mathrm{M}} : \mathcal{C}^n \times \mathcal{C}^n_0 \to \mathbb{R}^2,\ f_{\mathrm{M}}(K,C) = \left(\frac{r(K,C)}{R(K,C)}, \frac{D_{\mathrm{M}}(K,C)}{2R(K,C)}\right)$$

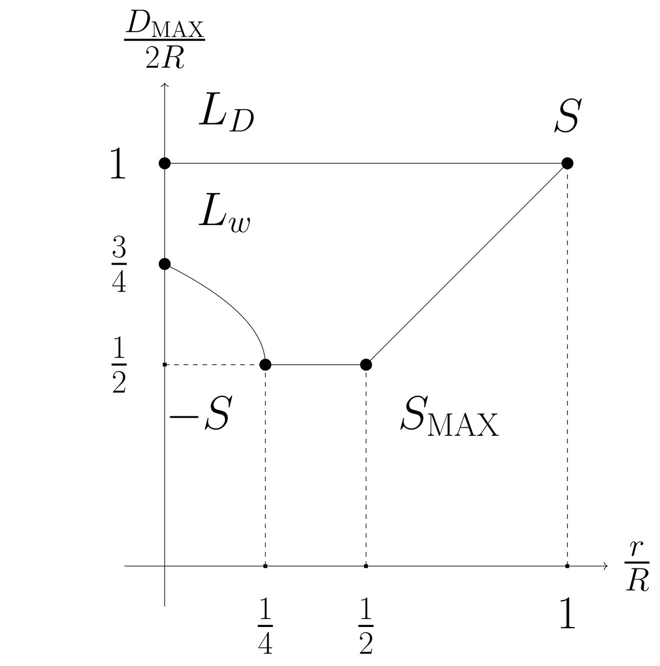
Diagrams for triangular gauges

- $f_{\mathrm{AM}}(\mathcal{C}^2, S) = f_{\mathrm{AM}}(\mathcal{C}^2, \mathcal{C}^2_0)$ given for triangular gauges S in [1].
- $f_{\mathrm{M}}(\mathcal{C}^2, S)$ for the remaining diameters and Minkowski-centered triangular gauges S.

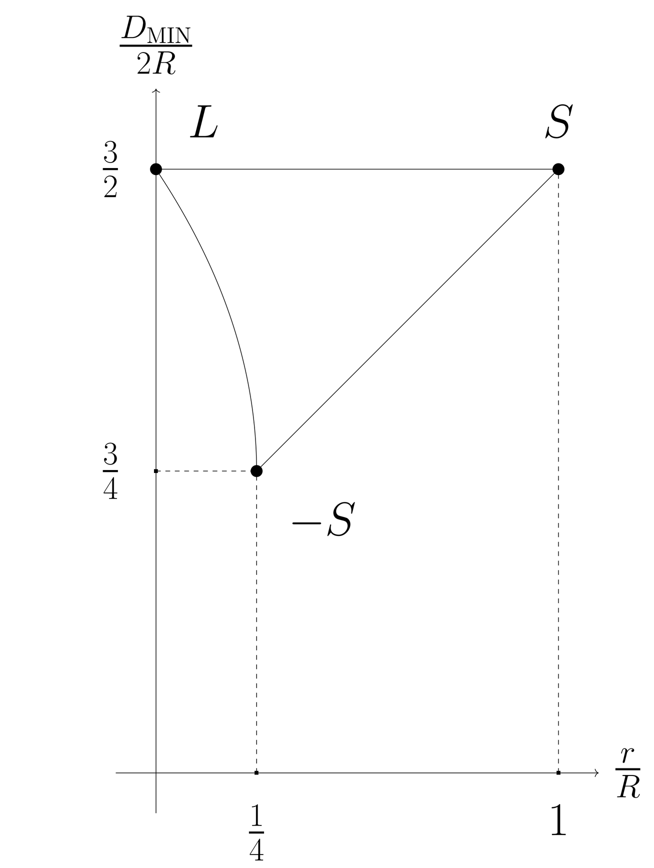
$f_{\mathrm{AM}}(\mathcal{C}^2, S)$



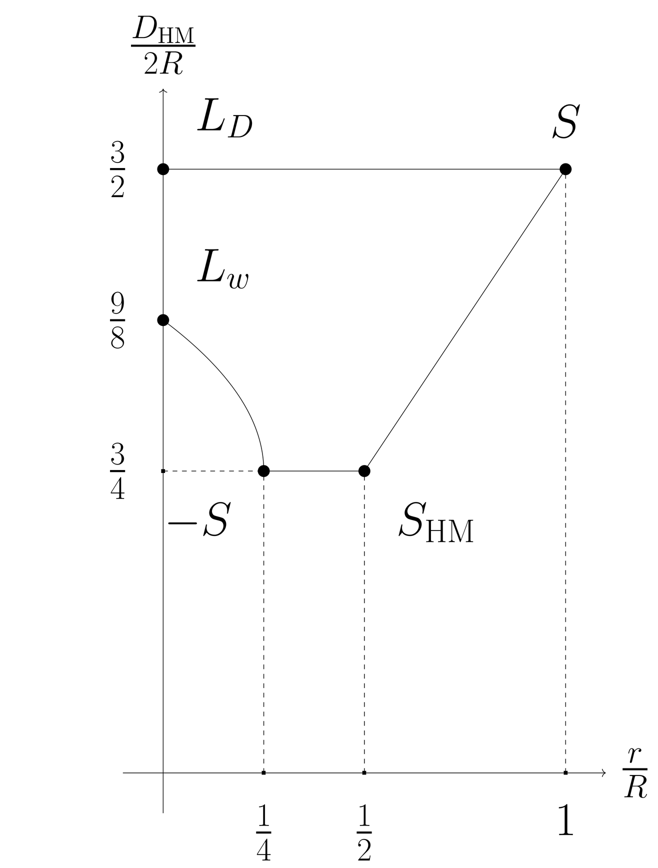
$f_{\mathrm{MAX}}(\mathcal{C}^2, S)$



$f_{\mathrm{MIN}}(\mathcal{C}^2, S)$



$f_{\mathrm{HM}}(\mathcal{C}^2, S)$



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