Upper semicontinuous valuations on the space of functions

Fernanda Moreira Baêta

Technische Universität Wien

Objective

To obtain a classification of upper semicontinuous, translation and dually epi-translation invariant valuations on convex functions whose domain is a compact set of $\mathbb{R}$.

Convex Bodies

Let $\mathcal{K}^n$ denote the set of *convex bodies*, i.e., non-empty, compact, convex subsets of $\mathbb{R}^n$. A functional $Z : \mathcal{K}^n \to \mathbb{R}$ is called a *valuation* if

$$Z(K \cup L) + Z(K \cap L) = Z(K) + Z(L)$$

whenever $K, L, K \cup L \in \mathcal{K}^n$. We say that Z is *translation invariant* if $Z(K + c) = Z(K)$ for every constant $c \in \mathbb{R}$, and Z is *rotation invariant* if $Z(\phi K) = Z(K)$ for all rotation $\phi : \mathbb{R}^n \to \mathbb{R}^n$. We equip $\mathcal{K}^n$ and its subspaces with the topology induced by the Hausdorff metric.

Consider the set

$$\mathcal{W} = \left\{ \zeta : [0, +\infty) \to [0, \infty) : \ \zeta \text{ is concave}, \ \lim_{t \to 0} \zeta(t) = 0, \lim_{t \to +\infty} \zeta(t)/t = 0 \right\}.$$

In [1] M. Ludwig showed that if $\mu : \mathcal{K}^2 \to \mathbb{R}$ is an upper semicontinuous and rigid motion invariant valuation, then there are constants $c_0, c_1, c_2 \in \mathbb{R}$ and a function $\zeta \in \mathcal{W}$ such that

$$\mu(K) = c_0 + c_1 L(K) + c_2 A(K) + \int_{S^1} \zeta(\rho(K, u)) d\mathcal{H}^1(u) \tag{1}$$

for every $K \in \mathcal{K}^2$. Here L and A are the length and the area functions, respectively, S^1 is the unit sphere in $\mathbb{R}^2$, $\rho(K, u)$ is the curvature radius of the boundary of K at the point with normal $u \in S^1$ and $\mathcal{H}^1$ is the 1-dimensional Hausdorff measure.

Convex Functions

The standard space of convex functions is

$$\mathrm{Conv}(\mathbb{R}^n) := \{u : \mathbb{R}^n \to (-\infty, +\infty] : u \text{ is lower semicontinuous}, \ u \not\equiv +\infty\}$$

and one important subset of $\mathrm{Conv}(\mathbb{R}^n)$ is the set

$$\mathrm{Conv}_{\mathrm{cd}}(\mathbb{R}^n) = \left\{ u \in \mathrm{Conv}(\mathbb{R}^n) : \lim_{|x| \to +\infty} \frac{u(x)}{|x|} = +\infty \text{ and } \operatorname{dom} u \text{ is compact} \right\},$$

which is directly related to the set $\mathcal{K}^{n+1}$. It is usually equipped with the topology induced by epi-convergence: we say that a sequence of convex functions $u_k \in \mathrm{Conv}_{\mathrm{cd}}(\mathbb{R}^n)$ is *epi-convergent* to $u \in \mathrm{Conv}_{\mathrm{cd}}(\mathbb{R}^n)$ if $\{u_k \le t\}$ converges to $\{u \le t\}$ for all $t \neq \min_{x \in \mathbb{R}} u(x)$ with respect to the Hausdorff metric. The *domain* of u is the set

$$\operatorname{dom} u = \{x \in \mathbb{R}^n : u(x) < +\infty\}.$$

The *indicator function* of a convex body $K \in \mathcal{K}^n$ is given by

$$\mathrm{I}_K(x) = \begin{cases} 0, & \text{if } x \in K \\ +\infty, & \text{if } x \notin K \end{cases}$$

and belongs to the set $\mathrm{Conv}_{\mathrm{cd}}(\mathbb{R}^n)$. A functional $Z : \mathrm{Conv}_{\mathrm{cd}}(\mathbb{R}^n) \to \mathbb{R}$ is a *valuation* if

$$Z(f) + Z(g) = Z(f \vee g) + Z(f \wedge g)$$

for every $f, g \in \mathrm{Conv}_{\mathrm{cd}}(\mathbb{R}^n)$ such that also their pointwise maximum $f \vee g$ and pointwise minimum $f \wedge g$ belong to $\mathrm{Conv}_{\mathrm{cd}}(\mathbb{R}^n)$. We say that Z is *translation invariant* if $Z(u \circ \tau^{-1}) = Z(u)$ for every $u \in \mathrm{Conv}_{\mathrm{cd}}(\mathbb{R}^n)$ and translation τ on $\mathbb{R}^n$, it is $\mathrm{SL}(\mathbb{R}^n)$ *invariant* if $Z(u \circ \vartheta^{-1}) = Z(u)$ for every $u \in \mathrm{Conv}_{\mathrm{cd}}(\mathbb{R}^n)$ and $\vartheta \in \mathrm{SL}(\mathbb{R}^n)$ and it is *dually epi-translation* invariant if $Z(u + l + c) = Z(u)$ for all $u \in \mathrm{Conv}_{\mathrm{cd}}(\mathbb{R}^n), c \in \mathbb{R}$, and every linear function $l : \mathbb{R}^n \to \mathbb{R}$. Continuity of Z is understood with respect to epi-convergence. We say that $Z : \mathrm{Conv}_{\mathrm{cd}}(\mathbb{R}^n) \to \mathbb{R}$ is an *upper semicontinuous* valuation if for every sequence u_k epi-converging to u,

$$Z(u) \ge \limsup_{k \to +\infty} Z(u_k).$$

Main Result

We have the following functional version of (1) for $n = 1$.

Theorem

Let $Z : \mathrm{Conv}_{\mathrm{cd}}(\mathbb{R}) \to \mathbb{R}$ be an upper semicontinuous, translation and dually epi-translation invariant valuation. Then there are constants $c_0 \in \mathbb{R}, c_1 \le 0$, and a function $\zeta \in \mathcal{W}$ such that

$$Z(u) = c_0 + c_1 L(\operatorname{dom} u) + \int_{\operatorname{dom} u} \zeta(u''(x)) dx \tag{2}$$

for every $u \in \mathrm{Conv}_{\mathrm{cd}}(\mathbb{R})$.

Sketch of the proof:

- There is $c_0 \in \mathbb{R}$ such that $Z(\mathrm{I}_{\{x_0\}}) = c_0$ for every $x_0 \in \mathbb{R}$.
- $Z(\mathrm{I}_J) = c_1 V_1(J)$ for every closed interval $J \subset \mathbb{R}$ and some $c_1 \le 0$.
- Given $a > 0$ and $m \in \mathbb{R}$, the function $\zeta : [0, +\infty) \to \mathbb{R}$ defined by
$$\zeta(a) = \frac{1}{2m}(Z(f + \mathrm{I}_{[-m,m]}) - c_0) - c_1,$$
where $f(x) = \frac{a}{2}x^2$, belongs to $\mathcal{W}$.
- For a given $\zeta \in \mathcal{W}, c_0 \in \mathbb{R}$ and $c_1 \le 0$, there is a unique $\mu : \mathrm{Conv}_{\mathrm{cd}}(\mathbb{R}) \to \mathbb{R}$ with the following properties:
  - μ is upper semi-continuous;
  - μ is translation and dually epi-translation invariant valuation;
  - $\mu(\mathrm{I}_{\{x_0\}}) = c_0$ for every $x_0 \in \mathbb{R}$;
  - $\mu(\mathrm{I}_J) = c_1 V_1(J)$ for all closed interval $J \subset \mathbb{R}$;
  - $\mu(f + \mathrm{I}_{[-m,m]}) = 2m(\zeta(a) + c_1) + c_0$.

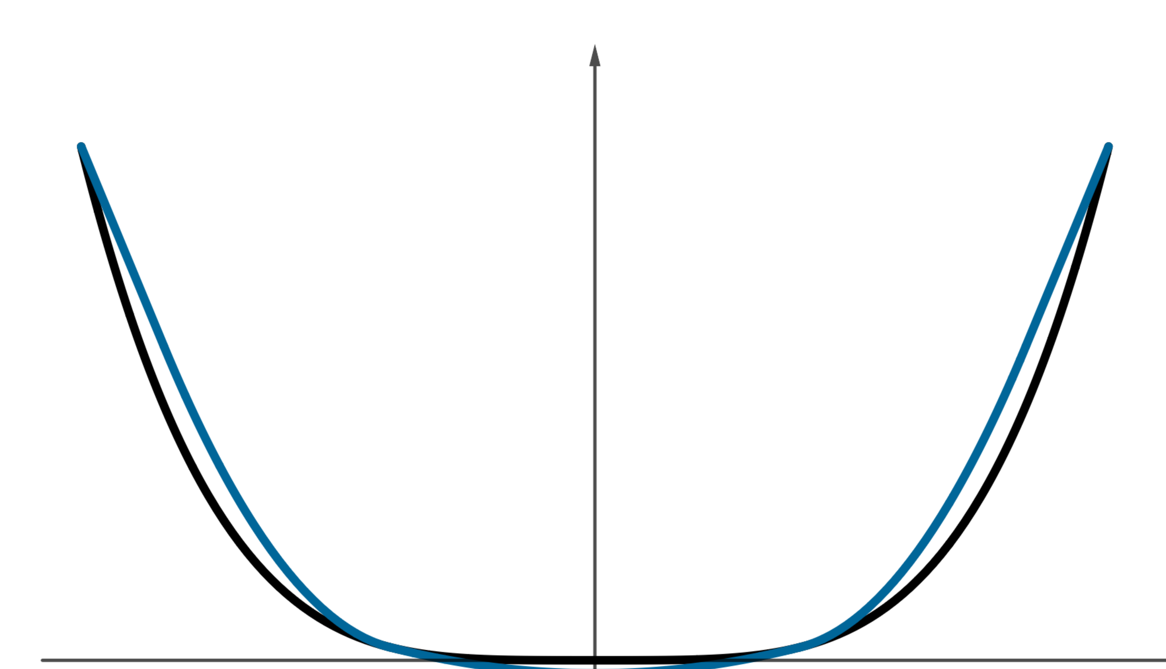

Figure 1:Approximation of a function u (black) by a piecewise linear-quadratic function (blue).

Further Research

In [2] M. Ludwig and M. Reitzner extended the characterization (1) for $n \in \mathbb{N}$ and $\mathrm{SL}(\mathbb{R}^n)$ invariant, and our goal now is to prove that if $Z : \mathrm{Conv}_{\mathrm{cd}}(\mathbb{R}^n) \to \mathbb{R}$ is an upper semicontinuous, $\mathrm{SL}(\mathbb{R}^n)$ invariant, translation and dually epi-translation invariant valuation, then there are constants $c_0 \in \mathbb{R}, c_1 \le 0$, and a function $\zeta \in \mathcal{W}$ such that for every $u \in \mathrm{Conv}_{\mathrm{cd}}(\mathbb{R}^n)$, we have

$$Z(u) = c_0 + c_1 V_n(\operatorname{dom} u) + \int_{\operatorname{dom} u} \zeta(\det \mathrm{D}^2 u(x)) dx.$$

References

[1] Ludwig, M. Upper semicontinuous valuations on the space of convex discs, *Geom. Dedicata* **80** (2000), 263–279.

[2] Ludwig, M., Reitzner, M. A classification of $\mathrm{SL}(\mathbb{R}^n)$ invariant valuations. *Ann. of Math. (2)* **172** (2010), 1219-1267.

fernanda.baeta@tuwien.ac.at