Coverage by inhomogeneous Poisson cylinders

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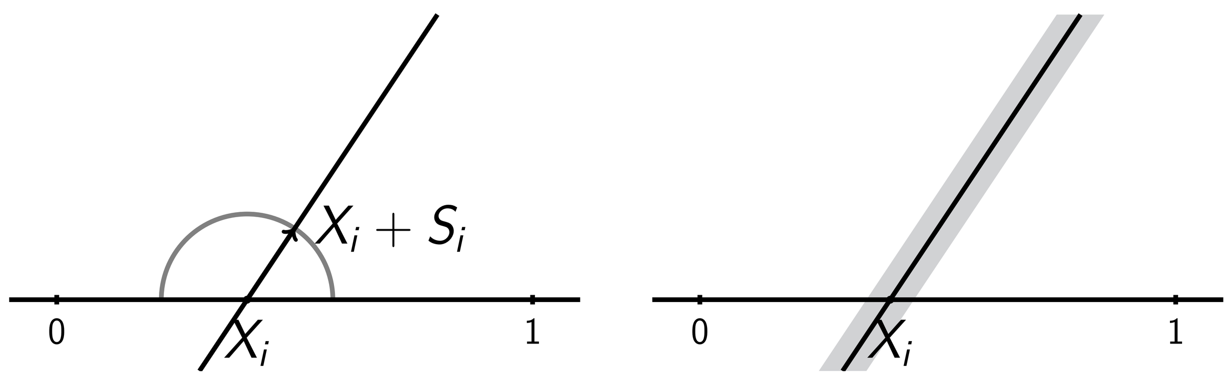
Main result: threshold radius for coverage by cylinders

Definition of cylinder set in $\mathbb{R}^d$ ($d \geq 2$):

- $\xi \sim \mathrm{Poi}(\lambda)$ i.i.d. points $X_1, \dots, X_\xi \sim \mathrm{Unif}([0,1]^{d-1})$
- Random i.i.d. directions $S_1, \dots, S_\xi \in \mathbb{S}^{d-1}$



- Line set $\mathcal{L} = \bigcup_{i \in \{1,\dots,\xi\}} \{X_i + \alpha S_i : \alpha \in \mathbb{R}\}$
- ***Cylinder set*** $\mathcal{C}(K) = \mathcal{L} \oplus K$, with $K \subset \mathbb{R}^d$ convex

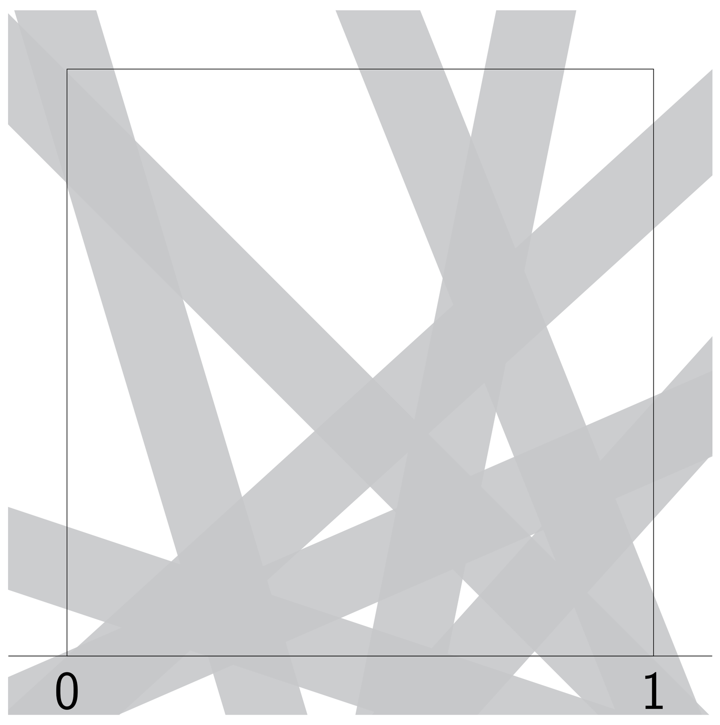


Figure: Cylinder set in $\mathbb{R}^2$

Condition: nonzero density of lines everywhere in $[0,1]^d$

Threshold radius:

$$R_\lambda(K) := \inf\{r : \mathcal{C}(rK) \supset [0,1]^d\}$$

$B_d := B_d(o,1)$ ball of radius 1 around the origin

Theorem For $K = B_d$ and $K = B_{d-1} \times \{0\}$, there exist constants $c_1, c_2 > 0$ such that

$$\mathbb{P}\left(c_1 \leq \frac{R_\lambda(K)}{\sqrt[d-1]{\log \lambda / \lambda}} \leq c_2\right) \to 1 \qquad \text{as } \lambda \to \infty.$$

Homogeneous thickened lines (Chenavier et al., '16)

- Observe line tessellation in $\mathbb{R}^2$ in (fixed) circular window W.
- $\#\{\text{lines crossing } W\} \sim \mathrm{Poi}(\lambda)$
- R_λ is the radius of the largest ball that fits in a cell of the tessellation.
- Lines with width $2R_\lambda$ cover W.

For some constant $c > 0$,

$$\frac{\lambda R_\lambda}{\log \lambda} \xrightarrow{\text{a.s.}} c \qquad \text{as } \lambda \to \infty.$$

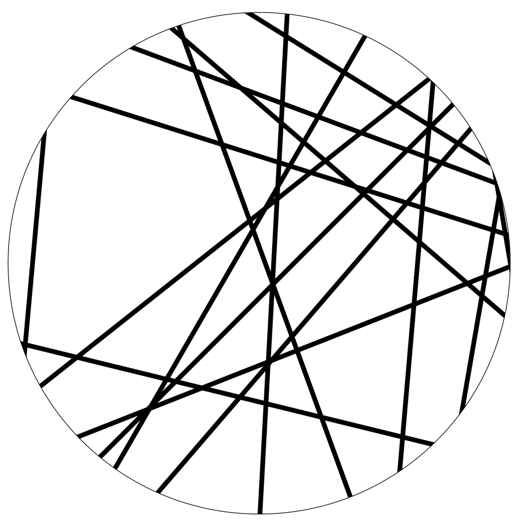


Figure: Homogeneous lines observed in circular window

Coverage in the Boolean model (Penrose, '23)

Similar problem for the Boolean model by Penrose (2023):

- Place n i.i.d. points X_i uniformly in a set $A \subset \mathbb{R}^d$.
- Draw balls $B_d(X_i, r)$.
- $R_n := \inf\{r : \cup_{i \in \{1,\dots,n\}} B_d(X_i, r) \supset A\}$

For a constant $c > 0$,

$$\frac{nR_n^d}{\log n} \xrightarrow{\text{a.s.}} c \qquad \text{as } n \to \infty.$$

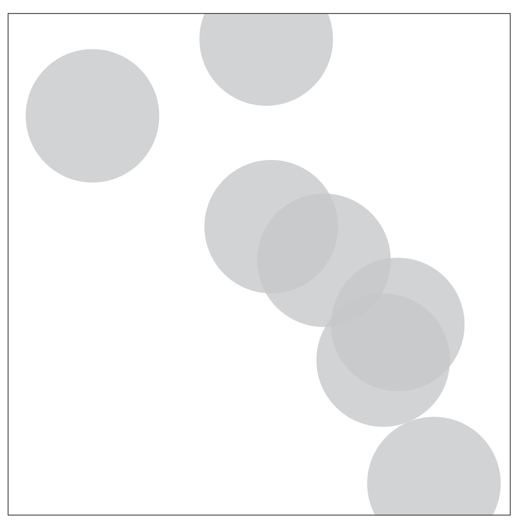


Figure: Boolean model observed in square window

Proof in two dimensions (sketch)

Consider the cylinder set with $d = 2$.

To prove:

$$\mathbb{P}\left(c_1 \frac{\log(\lambda)}{\lambda} \leq R_\lambda(K) \leq c_2 \frac{\log(\lambda)}{\lambda}\right) \to 1 \qquad \text{as } \lambda \to \infty.$$

Note that

$$B_1 \times \{0\} \subset B_2$$
$$\Rightarrow R_\lambda(B_1 \times \{0\}) \geq R_\lambda(B_2)$$

Steps of proof:

1. Lower bound for $K = B_2$
2. Upper bound for $K = B_1 \times \{0\}$

Proof: lower bound for $K = B_2$

Let $r = c\frac{\log(\lambda)}{\lambda}$.

$$\mathbb{P}(R_\lambda(K) < r) \leq \mathbb{P}\left([0,1]^2 \subset \mathcal{C}(rK)\right)$$
$$\leq \mathbb{P}\left([0,1] \times \{1\} \subset \mathcal{C}(rK)\right)$$

$\Rightarrow$ Focus on covering the upper side!

- Divide upper side in intervals of length $\geq \sqrt{2}r$.
- The number of lines intersecting an interval is Poisson distributed with intensity at most $\sqrt{2}r\lambda$.
- Bound $\mathbb{P}(\text{each interval crossed by line})$.
- For c small enough, $\mathbb{P}(R_\lambda(K) < r) \to 0$ as $\lambda \to \infty$.

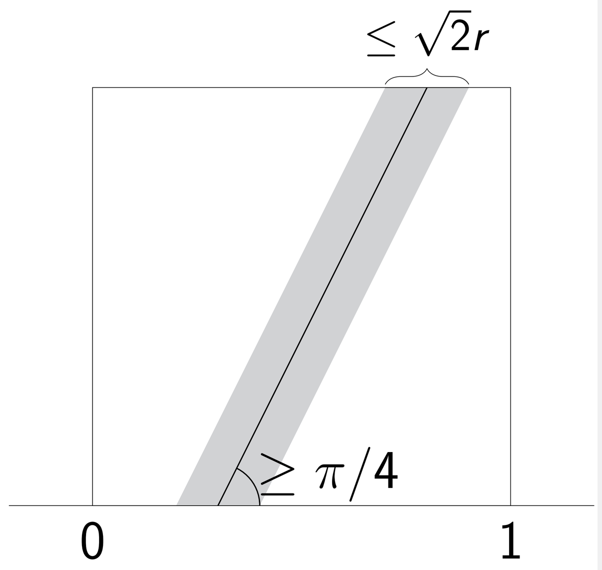


Figure: The horizontal width of the cylinder is at most $\sqrt{2}r$.

Proof: upper bound for $K = B_1 \times \{0\}$

Let $r = c\frac{\log(\lambda)}{\lambda}$ and cover $[0,1]^2$ with squares Q_i with side length r.

$$\mathbb{P}(R_\lambda(K) > r) \leq \frac{a}{r^2} \max_i \mathbb{P}\left(Q_i \not\subset \mathcal{C}(K)\right)$$

$\Rightarrow$ Focus on covering a single square Q_i

Only consider cylinders whose lines...

- have angle $\in [\arctan(2), \pi]$ w.r.t. x-axis;
- cross the lower side of Q_i in its left half.

$\Rightarrow$ These cylinders all cover the square Q_i. The number of such cylinders follows a Poisson distribution.

$\Rightarrow \cdots \Rightarrow$ For c large enough, $\mathbb{P}(R_\lambda(K) > r) \to 0$ as $\lambda \to \infty$.

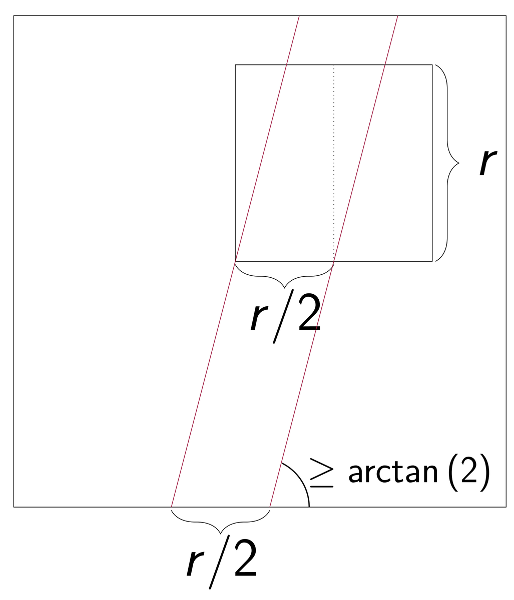


Figure: Two lines with same angle crossing a square Q_i.

References

N. Chenavier and R. Hemsley. Extremes for the inradius in the Poisson line tessellation. 2016.

M.D. Penrose. Random Euclidean coverage from within. 2023.

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