Generating Functions of Minkowski Valuations

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Summary

Minkowski valuations can be represented by their generating functions.

- We show new results on their regularity.
- We provide necessary and sufficient conditions for generating functions.
- We describe the action of the Alesker–Hard–Lefschetz operators.

Introduction

A *Minkowski valuation* is an operator on the space $\mathcal{K}^n$ of convex bodies (that is, convex, compact sets) in $\mathbb{R}^n$ such that

$$\Phi(K \cup L) + \Phi(K \cap L) = \Phi K + \Phi L \qquad \text{if } K, L, K \cup L \in \mathcal{K}^n.$$

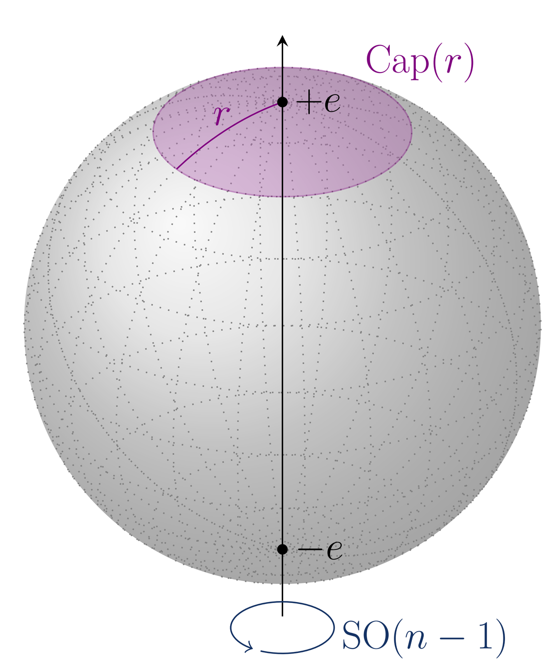
We consider the space $\mathbf{MVal}_i$ of continuous, translation invariant, $\mathrm{SO}(n)$ equivariant Minkowski valuations on $\mathbb{R}^n$ homogeneous of degree $i \in \{1, \dots, n-1\}$.

Schuster, Wannerer [3], and Dorrek [4] showed the following Hadwiger-type theorem for the space $\mathbf{MVal}_i$ involving the spherical convolution and area measures:

Theorem. For every $\Phi_i \in \mathbf{MVal}_i$, there exists a unique centered, $\mathrm{SO}(n-1)$ invariant function $f_{\Phi_i} \in L^1(\mathbb{S}^{n-1})$ such that

$$h_{\Phi_i K} = S_i(K, \cdot\,) * f_{\Phi_i}, \qquad K \in \mathcal{K}^n.$$

The function f_{Φ_i} is called the *generating function* of Φ_i.



Since f_{Φ_i} is $\mathrm{SO}(n-1)$ invariant, it corresponds to a function $\bar{f}_{\Phi_i}$ on the interval $[-1, 1]$.

Examples.

- The support function h_L of a convex body of revolution L generates Minkowski valuations for every degree $i \in \{1, \dots, n-1\}$.
- The *j-th mean section body* $\mathrm{M}_j K$ of a convex body K is defined by
$$h_{\mathrm{M}_j K}(u) := \int_{\mathrm{AGr}(n,j)} h_{K \cap E}(u)\, dE, \qquad u \in \mathbb{S}^{n-1},$$
where $\mathrm{AGr}(n, j)$ denotes the affine Grassmannian of j-dimensional affine subspaces of $\mathbb{R}^n$.

  Up to a translation, $\mathrm{M}_j \in \mathbf{MVal}_i$, where $i + j = n + 1$. Their generating functions (which are not support functions) were determined by Goodey and Weil [5].

Regularity

The regularity of generating functions is important for several applications, such as isoperimetric type problems and the classification of fixed points.

In the top degree case $i = n - 1$, generating functions are continuous on $\mathbb{S}^{n-1}$. For $i < n - 1$, merely integrability has been shown.

By investigating $\Box_n f_{\Phi_i}$, where $\Box_n$ is the second order differential operator

$$\Box_n := \frac{1}{n-1}\Delta_{\mathbb{S}^{n-1}} + \mathrm{Id},$$

we obtain new information on the regularity of f_{Φ_i}.

Theorem

Let $\Phi_i \in \mathbf{MVal}_i$. Then

- $\Box_n f_{\Phi_i}$ is a signed measure (in the distributional sense) and
- f_{Φ_i} is locally Lipschitz outside the poles.

If, in addition, Φ_i is monotone, then $\Box_n f$ is non-negative and

$$|\Box_n f_{\Phi_i}|(\mathrm{Cap}(r)) \leq C r^{i-1}, \qquad r \geq 0.$$

This shows that $\bar{f}_{\Phi_i} \in C(-1, 1)$, confirming a conjecture by Dorrek.

Differential Inequalities

Clearly, the space of generating functions of Minkowski valuations of a given degree is a convex cone. The question arises whether generating functions can be described in terms of differential inequalities on $(-1, 1)$, such as

$$\lambda(1 - t^2)\bar{f}''(t) - t\bar{f}'(t) + \bar{f}(t) \geq ct. \qquad (*)$$

Given a function $\bar{f} \in C[-1, 1]$, we can give a necessary and a sufficient condition for it to generate a Minkowski valuation in $\mathbf{MVal}_{n-1}$:

Theorem

Let $\Phi_{n-1} \in \mathbf{MVal}_{n-1}$. Then for each $\frac{1}{n-1} \leq \lambda \leq 1$, there exists $c \in \mathbb{R}$ (depending on λ) such that $\bar{f}_{\Phi_{n-1}}$ satisfies $(*)$ in the distributional sense.

Theorem

Suppose that $\bar{f} \in C^2[-1, 1]$ satisfies $(*)$ for all $0 \leq \lambda \leq 1$ and some $c \in \mathbb{R}$ independent of λ. Then f generates a Minkowski valuation $\Phi_{n-1} \in \mathbf{MVal}_{n-1}$ (for every dimension $n \geq 2$).

Since these two conditions are quite similar, we conjecture that a classification of generating functions in terms of $(*)$ is possible.

Alesker–Hard–Lefschetz operators

The operator $\Lambda : \mathbf{MVal}_i \to \mathbf{MVal}_{i-1}$ is defined by

$$(\Lambda\Phi_i)(K) := \left.\frac{d}{dt}\right|_{t=0^+} \Phi_i(K + t\mathbb{B}^n), \qquad K \in \mathcal{K}^n.$$

It was shown by Schuster and Parapatits [6] that the operator Λ is well-defined. It essentially preserves the generating function:

$$f_{\Lambda\Phi_i} = i f_{\Phi_i}.$$

The operator $\mathfrak{L} : \mathbf{MVal}_i \to \mathbf{MVal}_{i+1}$ is defined by

$$(\mathfrak{L}\Phi_i)(K) := \int_{\mathrm{AGr}(n,n-1)} \Phi_i(K \cap H)\, dH, \qquad K \in \mathcal{K}^n.$$

It acts on the generating function as follows:

Theorem

There exists a unique $\mathrm{SO}(n-1)$ invariant and positive function $\rho_i \in L^1(\mathbb{S}^{n-1})$ which is smooth outside the north pole such that for all $\Phi_i \in \mathbf{MVal}_i$,

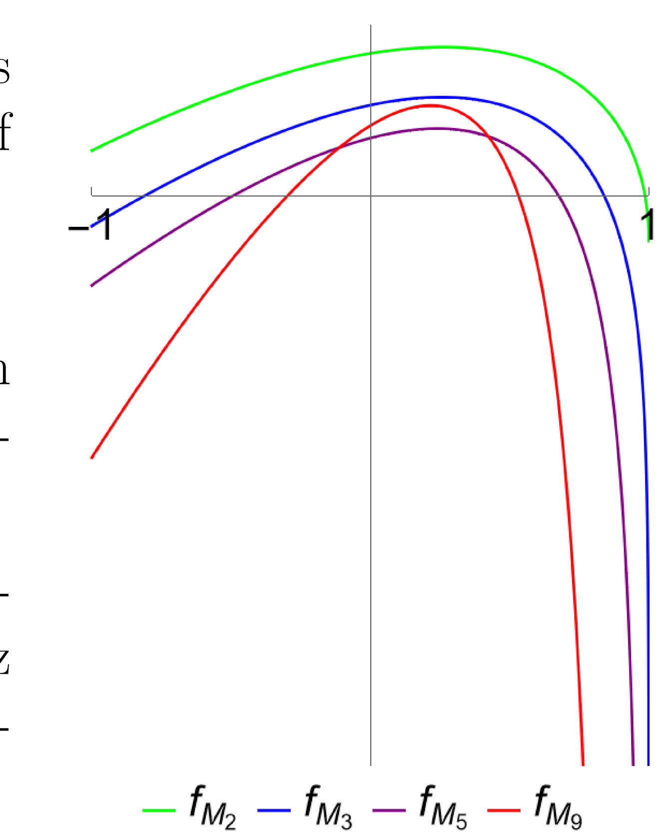
$$f_{\mathfrak{L}\Phi_i} = f_{\Phi_i} * \rho_i.$$

To date, all known Minkowski valuations in $\mathbf{MVal}_i$ are generated by a function of the form

$$f = h_L + f_{\mathrm{M}_j} * \Box_n g$$

where L is a convex body of revolution and g generates some Minkowski valuation in $\mathbf{MVal}_1$.

The theorem above implies that the application of the Alesker–Hard–Lefschetz operators does not yield any new examples of Minkowski valuations.



References

[1] L. Brauner, O. Ortega-Moreno, *Fixed Points of Mean Section operators*, arXiv: 2302.11973

[2] L. Brauner, G. Hofstätter, O. Ortega-Moreno, *Klain–Schneider functions and Hard–Lefschetz operators*, In preparation.

[3] F. Schuster, T. Wannerer, *Minkowski valuations and generalized valuations*, J. Eur. Math. Soc. (JEMS) **20** (2018), no. 8

[4] F. Dorrek, *Minkowski enomorphisms*, Geom. Funct. Anal. **27** (2017), no. 3

[5] P. Goodey, W. Weil, *Sums of sections, surface area measures, and the general Minkowski problem*, J. Differential Geom. **97** (2014), no. 3

[6] L. Parapatits, F. Schuster, *The Steiner formula for Minkowski valuations*, Adv. Math. **230** (2012), no. 3