# Normal Approximation of Poisson Functionals

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## via Malliavin-Stein Method

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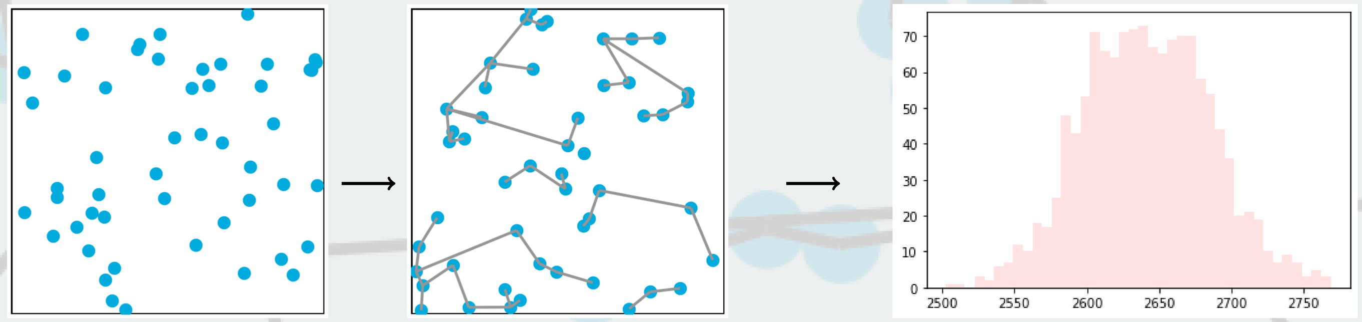
## Goals

- Quantitative CLTs for Poisson functionals
- under minimal assumptions



Poisson measure → Poisson functional → Central Limit Theorem

$F = \sum |\bullet\!-\!\bullet|$

## Tools

Add-one cost operator: for $x \in \mathbb{X}$, it is given by

$$\mathrm{D}_x F(\eta) = F(\eta \cup \{x\}) - F(\eta)$$
$$= |\bullet\!-\!\bullet| - |\bullet\!-\!\bullet|$$

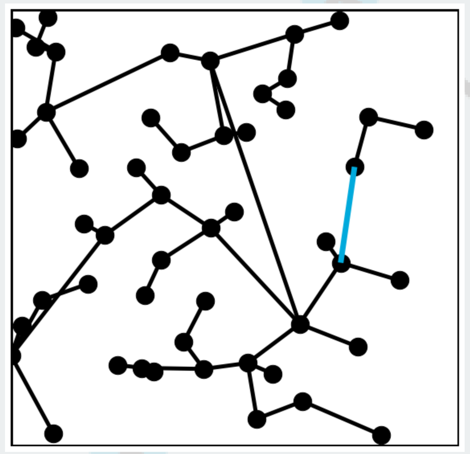


Figure: An ONNG graph before adding the point $x = \bullet$

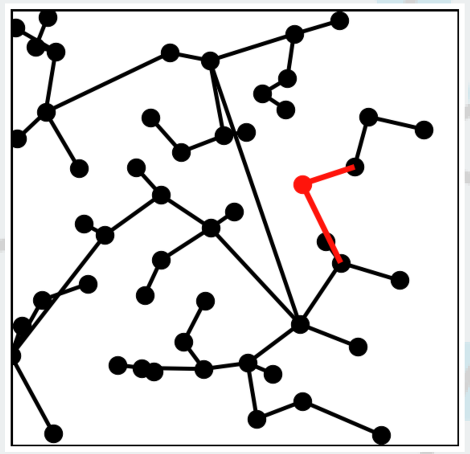


Figure: The graph after adding the point $x = \bullet$

**Theorem (T. 2022)**

$p$-Poincaré inequality: for $F \in L^2(\mathbb{P}_\eta)$ and $p \in [1,2]$,

$$\mathbb{E}|F|^p - |\mathbb{E}F|^p \leq 2^{2-p}\mathbb{E}\int_{\mathbb{X}} |\mathrm{D}_x F|^p \, \lambda(dx).$$

Generalized $p$-Poincaré inequality: let $h \in L^2(\mathbb{P}_\eta \otimes \lambda)$ s.t. $\mathrm{D}\, h$ square-integrable. Let $p \in [1,2]$. Then

$$\mathbb{E}\left[|\delta(h)|^p\right] \leq 2^{2-p}\mathbb{E}\int_{\mathbb{X}} |h(\eta,y)|^p \lambda(dy)$$
$$+ p2^{2-p}\mathbb{E}\int_{\mathbb{X}}\int_{\mathbb{X}} |\mathrm{D}_y h(\eta,x)|\,|\mathrm{D}_x h(\eta,y)|^{p-1}\,\lambda(dy)\lambda(dx)$$
$$+ p2^{3-p}\mathbb{E}\int_{\mathbb{X}}\int_{\mathbb{X}} |\mathrm{D}_y h(\eta,x)| \cdot |h(\eta,y)|^{p-1}\,\lambda(dy)\lambda(dx).$$

## Results

The ingredients:

- a Poisson measure $\eta$ on a $\sigma$-finite space $(\mathbb{X}, \mathcal{X}, \lambda)$;
- a functional $F = F(\eta)$;
- the add-one cost operator $\mathrm{D}_x F$;
- the Wasserstein / Kolmogorov distances $d_{W,K}$.

**Theorem (T. 2022)**

Assume $\mathbb{E}F = 0$, $\mathbb{E}F^2 = 1$ and $\mathbb{E}\int (\mathrm{D}\, F)^2 d\lambda < \infty$. For $p \in [1,2]$, we have

$$d_{W,K}(F,N) \leq 4\left(\int_{\mathbb{X}}\left(\int_{\mathbb{X}} \mathbb{E}\left[|\mathrm{D}_y F|^{2p}\right]^{\frac{1}{2p}} \mathbb{E}\left[|\mathrm{D}^{(2)}_{x,y} F|^{2p}\right]^{\frac{1}{2p}} \lambda(dy)\right)^p \lambda(dx)\right)^{1/p}$$

$+$ similar terms

where $N \sim \mathcal{N}(0,1)$.

→ general quantitative central limit theorems;
→ need only bounds on $\mathrm{Var}(F)$, $\mathrm{D}\, F$ and $\mathrm{D}^{(2)}\, F$;
→ choice of $p$ allows for minimal moment assumptions.

## Applications

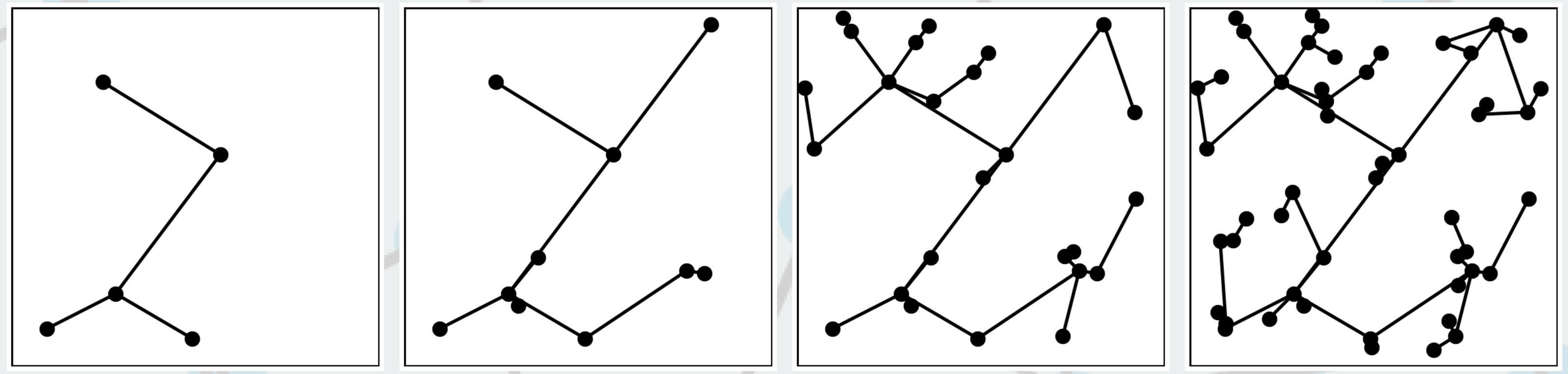


Figure: The Online Nearest Neighbour Graph models networks growing in time

Setting

- Poisson measure on $\mathbb{R}^d \times [0,1]$ in a window of size $t$;
- construct $ONNG_t$ by connecting each point to its nearest neighbour with smaller time coordinate;
- for $\alpha > 0$:

$$F_t^{(\alpha)} := \sum_{e \in ONNG_t} |e|^\alpha.$$

**Theorem (T. 2022)**

For $\alpha \in (0, d/2)$ and any $1 < p < \frac{d}{2\alpha}$, there is a constant $c > 0$ such that

$$d_{W,K}\left(\hat{F}_t^{(\alpha)}, N\right) \leq ct^{-d(1-\frac{1}{p})}.$$

For $\alpha = d/2$, there is a constant $c > 0$ such that

$$d_W\left(\hat{F}_t^{(\alpha)}, N\right) \leq c\log(t)^{-1},$$

where $\hat{F} = \mathrm{Var}(F)^{-1/2}(F - \mathbb{E}F)$.

This result was conjectured in (Wade 2009).

Similar central limit theorems can be derived for other types of graphs when $-\frac{d}{2} < \alpha < 0$.

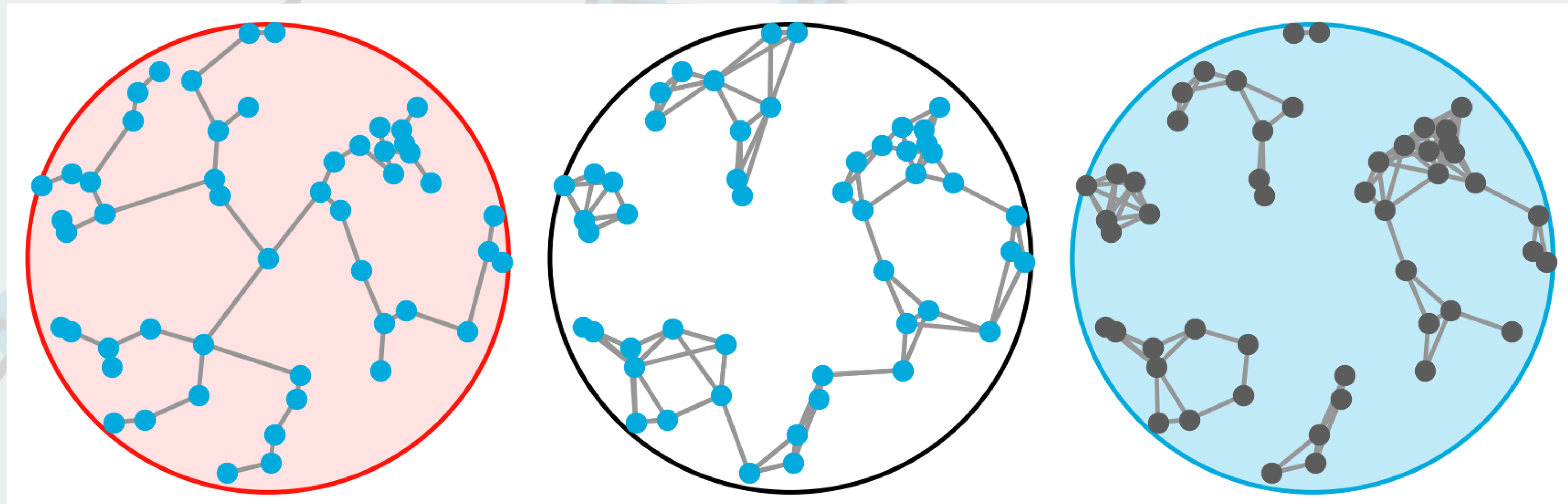


Figure: Radial spanning tree; known: $\alpha \geq 0$ (ST'16)

Figure: $k$-nearest neighbour graph; known: $\alpha \geq 0$ (LPS'16)

Figure: Gilbert graph (dense and thermodynamic case); known: $\alpha > -\frac{d}{4}$ (RST'16)



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