Investigations of general Schatten-norm unit balls

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Setup

For $m,n\in\mathbb{N}$ $(m\le n)$, $\mathbb{K}\in\{\mathbb{R},\mathbb{C}\}$ and $p\in(0,\infty]$ consider $\mathbb{K}^{m\times n}$ endowed with the *Hilbert–Schmidt inner product*

$$\langle x,y\rangle := \Re\operatorname{tr}(xy^*)$$

and the *p-Schatten (quasi-)norm*

$$\|x\|_{S_p} := \begin{cases} \left(\sum_{i=1}^m s_i(x)^p\right)^{1/p} & \text{for } p<\infty,\\ \max\{s_i(x) \,:\, i\in[1,m]\} & \text{for } p=\infty,\end{cases}$$

for $x\in\mathbb{K}^{m\times n}$, where $s_1(x)\ge\cdots\ge s_m(x)\in\mathbb{R}_{\ge 0}$ are the singular values of x.

Objects under investigation are ($\beta:=\dim_{\mathbb{R}}(\mathbb{K})$ always)

- $\mathbb{B}^{m\times n}_{S_p,\beta} := \{x\in\mathbb{K}^{m\times n} \,:\, \|x\|_{S_p}\le 1\}$, the unit ball,
- $\mathbb{S}^{m,n}_{S_p,\beta} := \{x\in\mathbb{K}^{m\times n} \,:\, \|x\|_{S_p}=1\} = \partial\mathbb{B}^{m\times n}_{S_p,\beta}$, the unit sphere.

Geometric results for $p=\infty$

Proposition. *The exact volume of $\mathbb{B}^{m\times n}_{S_\infty,\beta}$ equals*

$$v_{\beta mn}(\mathbb{B}^{m\times n}_{S_\infty,\beta}) = \frac{\prod_{k=0}^{m-1}\Gamma(1+\frac{\beta k}{2})\prod_{k=0}^{n-1}\Gamma(1+\frac{\beta k}{2})}{\prod_{k=0}^{m+n-1}\Gamma(1+\frac{\beta k}{2})}\,\pi^{\beta mn/2}.$$

Proposition. *The isotropy constant $L_{\mathbb{B}^{m\times n}_{S_\infty,\beta}}$ remains bounded in m and n; to be precise,*

$$L^2_{\mathbb{B}^{m\times n}_{S_\infty,\beta}} = \frac{1}{2\pi\mathrm{e}^{3/2}}\,c_\beta(m,n),$$

where we know

$$\mathrm{e}^{1/2}\Big(1+\frac{C_1}{mn}\Big)\le c_\beta(m,n)\le \mathrm{e}^2\Big(1+C_2\Big(\frac{1}{m}+\frac{1}{n}\Big)\Big)$$

for all $m,n\in\mathbb{N}$, with global constants $C_1,C_2\in\mathbb{R}$.

Proposition. *A random variable $X\in\mathbb{K}^{m\times n}$ is distributed uniformly on $\mathbb{B}^{m\times n}_{S_\infty,\beta}$ iff there exist independent random variables $R\in\mathbb{K}^{m\times m}$ and $U\in\mathbb{K}^{n\times m}$ such that $R\sim B^{\mathrm{I}}_{m,\beta}(\frac{\beta n}{2},\frac{\beta(m-1)}{2}+1)$, $U\sim\mathcal{U}(\mathrm{U}_{n,m;\beta})$, and*

$$X\overset{\mathrm{d}}{=}R^{1/2}U^*.$$

Generic geometric results

Proposition. *The unit-volume ball $v_{\beta mn}(\mathbb{B}^{m\times n}_{S_p,\beta})^{-\beta mn}\mathbb{B}^{m\times n}_{S_p,\beta}$ is isotropic.*

Proposition. *For $x\in\mathbb{S}^{m\times n}_{S_p,\beta}$ let $\nu(x)$ denote the outer unit normal vector, then*

$$\langle\nu(x),x\rangle = \|x\|^{1-p}_{S_{2p-2}} \quad (=1 \text{ for } p=\infty).$$

Consequently, the normalized cone measure $\kappa^{m\times n}_{S_p,\beta}$ and surface (Hausdorff) measure on $\mathbb{S}^{m\times n}_{S_p,\beta}$ conicide iff $p\in\{1,2,\infty\}$.

A type of symmetry

Definition. A Borel measure μ on $\mathbb{K}^{m\times n}$ is called $\mathbb{B}^{m\times n}_{S_p,\beta}$*-symmetric* iff there exists a Borel measure ρ on $\mathbb{R}_{\ge 0}$ such that

$$\int_{\mathbb{K}^{m\times n}} f(x)\,\mathrm{d}x = \int_{\mathbb{R}_{\ge 0}}\int_{\mathbb{S}^{m,n}_{S_p,\beta}} f(r\theta)\,\mathrm{d}\kappa^{m\times n}_{S_p,\beta}(\theta)\,\mathrm{d}\rho(r)$$

for all $f\colon\mathbb{K}^{m\times n}\to\mathbb{R}$ measurable and nonnegative.

Clearly $\mathcal{U}(\mathbb{B}^{m\times n}_{S_p,\beta})$ and $\kappa^{m\times n}_{S_p,\beta}$ are $\mathbb{B}^{m\times n}_{S_p,\beta}$-symmetric.

Proposition. *Let a random variable $X\in\mathbb{K}^{m\times n}$ have a $\mathbb{B}^{m\times n}_{S_p,\beta}$-symmetric distribution such that $\mathbb{P}[X=0]=0$, then XX^* and $(XX^*)^{-1/2}X$ are independent, and $X^*(XX^*)^{-1/2}\sim\mathcal{U}(\mathrm{U}_{n,m;\beta})$. (Cf. simulation of $\mathcal{U}(\mathrm{U}_{n,m;\beta})$ by QR-decomposition of $X\sim\mathcal{N}^{\otimes(n\times m)}_\beta$.)*

Proposition. *A random variable $X\in\mathbb{K}^{m\times n}$ has a $\mathbb{B}^{m\times n}_{S_p,\beta}$-symmetric distribution iff*

$$X\overset{\mathrm{d}}{=}R\cdot V\cdot\frac{\operatorname{diag}((Y_i^{1/2})_{i\le m})}{\|Y\|^{1/2}_{p/2}}\cdot U^*,$$

where R, U, V, Y are independent, $R\sim\rho$, $U\sim\mathcal{U}(\mathrm{U}_{n;\beta})$, $V\sim\mathcal{U}(\mathrm{U}_{m;\beta})$, and Y is $\mathbb{R}^m$-valued with Lebesgue-density

$$\frac{1_{\mathbb{R}^m_{\ge 0}}(y)}{Z_{m,n,p,\beta}}\mathrm{e}^{-\beta n\|y\|^{p/2}_{p/2}}\prod_{i=1}^m y_i^{\beta(n-m+1)/2-1}\prod_{1\le i<j\le m}|y_i-y_j|^\beta$$

(interpret $\mathrm{e}^{-\beta n\|y\|^{p/2}_{p/2}} = 1_{\mathbb{B}^m_\infty}(y)$ for $p=\infty$).

Limit results for $p=\infty$

Proposition. *Consider m fixed, let $k\in\mathbb{N}$, and let $(X_n)_{n\ge m}$ have distribution among $\mathcal{U}(\mathbb{B}^{m\times n}_{S_\infty,\beta})$ or $\kappa^{m\times n}_{S_\infty,\beta}$. Denote the projection of X_n onto the first k columns by $X_n^{(k)}$. Then*

$$(n^{1/2}X_n^{(k)})_{n\ge\max\{m,k\}}\xrightarrow{\mathrm{d}}\mathcal{N}^{\otimes(m\times k)}_\beta.$$

Proposition. *Let $(X_n)_{n\ge m}$ and $(Y_n)_{n\ge m}$ be sequences of random variables with distribution among $\mathcal{U}(\mathbb{B}^{m\times n}_{S_\infty,\beta})$ or $\kappa^{m\times n}_{S_\infty,\beta}$, and let $\{X_n,Y_n\}$ be independent for each $n\ge m$. Then*

$$(n^{1/2}X_nY_n^*)_{n\ge m}\xrightarrow{\mathrm{d}}\mathcal{N}^{\otimes(m\times m)}_\beta.$$

Work in progress

Derive Sanov-type LDPs for

$$\mu_n := \frac{1}{m}\sum_{i=1}^m\delta_{m^{1/p}s_i(X^{(n)})},$$

the empirical measure of singular values, as $n\to\infty$ and $\frac{m}{n}\to c\in(0,1]$, analogous to those of [2] for the quadratic case.

Method: Use the probabilistic representation for $\mathbb{B}^{m\times n}_{S_p,\beta}$-symmetric measure, where $(R^{(n)})^{\beta mn}\sim\mathcal{U}([0,1])$ for $X^{(n)}\sim\mathcal{U}(\mathbb{B}^{m\times n}_{S_p,\beta})$ and $R^{(n)}=1$ for $X^{(n)}\sim\kappa^{m\times n}_{S_p,\beta}$, derive an LDP for $\nu_n:=\frac{1}{m}\sum_{i=1}^m\delta_{Y_i^{(n)}}$, and transfer to μ_n.

Obstacles: Miminizer $\mu_{c,p}$ of rate function for $(\nu_n)_{n\in\mathbb{N}}$ not identified in case $p<\infty$ and $c<1$; transfer not simple because contraction principle not applicable (moment map not continuous w.r.t. weak topology).

Knowledge of $\mu_{c,p}$ allows determination of asymptotics of the volume and isotropic constant of $\mathbb{B}^{m\times n}_{S_p,\beta}$. In the quadratic case the former is known from [1] and the latter from [3].

References

[1] Zakhar Kabluchko, Joscha Prochno, and Christoph Thäle. Exact asymptotic volume and volume ratio of schatten unit balls. *Journal of Approximation Theory*, 257:105457, 2020.

[2] Zakhar Kabluchko, Joscha Prochno, and Christoph Thäle. Sanov-type large deviations in Schatten classes. *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques*, 56(2):928–953, 2020.

[3] Hermann König, Mathieu Meyer, and Alain Pajor. The isotropy constants of the Schatten classes are bounded. *Math. Ann.*, 312(4):773–783, 1998.