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Haar Null Sets and Convexity

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Davide Ravasini   davide.ravasini@uibk.ac.at

The Definition

In $\mathbb{R}^d$ we have the following characterisation of sets with zero Lebesgue measure.

Proposition

*A Borel set $E \subseteq \mathbb{R}^n$ has **zero Lebesgue measure** if and only if there is a **Borel probability measure** μ on $\mathbb{R}^n$ such that $\mu(x+E)=0$ for every $x \in \mathbb{R}^n$.*

In an infinite-dimensional, separable Banach space there is no Lebesgue measure, but this characterisation can be used to define null sets.

Definition

*A Borel set E in a separable Banach space X is **Haar null** if there exists a Borel probability measure μ on X such that $\mu(x+E)=0$ for every $x \in X$. E is said to be **Haar positive** if it is not Haar null.*

Facts

Haar null sets are a translation-invariant σ-ideal.

- Countable unions of Haar null sets are also Haar null.
- Borel subsets of Haar null sets are Haar null.
- Translates of Haar null sets are Haar null.

In $\mathbb{R}^d$, Haar null sets and Borel sets with zero Lebesgue measure coincide.

Haar null convex sets

In $\mathbb{R}^d$, a closed and convex set has zero Lebesgue measure if and only if its interior is empty. More generally, the following holds.

Theorem (Matoušková)

*A closed and convex set in a separable, **reflexive** Banach space is Haar null if and only if its interior is empty.*

Question

*Is a **measure-free** characterisation possible in **every** separable Banach space?*

Compactivorous sets I

Definition

*A set E in a Banach space X is **compactivorous** if for every nonempty compact set $K \subset X$ there are an open set $V \subseteq X$ and $x \in X$ with $K \cap V \neq \varnothing$ and $x + K \cap V \subseteq E$.*

Borel compactivorous sets are always Haar positive. It follows from the inner regularity of Borel probability measures.

Question

Let C be a closed and convex set in a separable Banach space X. If C is Haar positive, must C be compactivorous?

Compactivorous sets II

Being compactivorous has some equivalent characterisations.

Theorem (R., 2022)

Let E be a set in a Banach space X. The following assertions are equivalent.

1. *For every nonempty compact set $K \subset X$ there are an open set $V \subseteq X$ and $x \in X$ with $K \cap V \neq \varnothing$ and $x + K \cap V \subseteq E$.*
2. *For every compact set $K \subset X$ there are $\delta > 0$ and $x \in X$ with $x + \delta K \subseteq E$.*
3. *There is $r > 0$ such that for every compact set $K \subseteq rB_X$ there is $x \in X$ with $x + K \subseteq E$.*

The proof is based on quotients of ℓ_1 and Michael's selection theorem.

The answer

Theorem (R., 2023)

A closed and convex set C in a separable Banach space X is Haar positive if and only if C is compactivorous.

The key tool in the proof is the space of C-fit sequences.

Definition

*Let C be a closed, convex and bounded set in a Banach space X. A sequence $(x_n)_{n=1}^{\infty} \in c_0(X)$ is **C-fit** if there are $z \in X$ and $\lambda > 0$ such that $z \pm x_n \in \lambda C$ for every n.*

- The space $F_C(X)$ of C-fit sequences is a linear subspace of $c_0(X)$ and has a natural norm which turns it into a Banach space.
- The inclusion $I : F_C(X) \to c_0(X)$ is continuous.
- If $F_C(X) = c_0(X)$, i.e. if I is onto, it follows from Grothendieck's compactness principle that C is compactivorous.
- If C is Haar positive, then one can prove that $I^* : \ell_1(X^*) \to F_C(X)^*$ is an embedding and the theorem follows.

A simpler result

Closed, convex Haar null sets can be characterised via their weak* closures in the second dual.

Theorem (R., 2023+)

*A closed and convex set C in a separable Banach space X is Haar null if and only if the weak* closure of C in X^{**} has **empty interior** with respect to the **norm topology** of X^{**}.*

Some consequences

Corollary

*Let $(e_n)_{n=1}^{\infty}$ be an unconditional, normalised Schauder basis in a Banach space X. If the positive cone of $(e_n)_{n=1}^{\infty}$ is Haar positive, then $(e_n)_{n=1}^{\infty}$ is equivalent to the **standard basis of** c_0.*

Corollary

*Let X be a separable Banach lattice. The positive cone of X is Haar positive if and only if X is lattice-isomorphic to an **AM-space**.*

Corollary

Let $\mathcal{C}(X)$ be the space of all nonempty, closed, convex, bounded subsets of a separable Banach space X, endowed with the Hausdorff metric. Then a convergent sequence $(C_n)_{n=1}^{\infty}$ of Haar null sets has a Haar null limit.

References:

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