Higher Order Affine Isoperimetric Inequalities Part I

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1 Introduction: Convex bodies, Duality, & Volume.

$\mathbb{R}^n$ - the Euclidean n-dimensional vector space ($|\cdot|$ denotes length, $B_2^n = \{x \in \mathbb{R}^n : |x| \leq 1\}$ the Euclidean ball with boundary $\mathbb{S}^{n-1}$ is the unit sphere and $\langle , \rangle$ is the standard inner-product). K and L will denote convex bodies; K contains the origin, has non-empty interior, is compact and $x, y \in K \to tx + (1-t)y \in K$ for every $t \in [0,1]$. K is symmetric if $K = -K$. The Minkowski sum of K and L is the set defined given by $K + L = \{x + y : x \in K, y \in L\}$. K is uniquely determined by its radial function:

$$\rho_K(x) = \max\Big\{t > 0 : tx \in K\Big\}.$$

Fact: for $f \in L^1(K)$, one can write $\int_K f(x)dx = \int_{\mathbb{S}^{n-1}} \int_0^{\rho_K(\theta)} f(r\theta) r^{n-1} dr d\theta$. Setting $f = 1$ yields the Lebesgue measure, or volume, of K can be written as

$$\mathrm{Vol}_n(K) = \frac{1}{n}\int_{\mathbb{S}^{n-1}} \rho_K(\theta)^n d\theta.$$

We all know that $\mathrm{Vol}_n(\lambda K) = \lambda^n \mathrm{Vol}_n(K)$ for $\lambda \geq 0$, i.e. volume is a homogeneous measure of degree of homogeneity n. But there is much more!

Mixed Volumes

K and L convex bodies in $\mathbb{R}^n$ and $t \geq 0$. Then: $\mathrm{Vol}_n(K + tL)$ is a homogeneous polynomial (in t) of degree n and

$$\boxed{\mathrm{Vol}_n(K + tL) = \sum_{i=0}^{n} t^i \binom{n}{i} V(K[n-i], L[i])}$$

The coefficients $V(K[n-i], L[i])$ are called the mixed volumes of K $[n-i]$ times and L $[i]$ times. When $i = 1$, we write $V(K[n-1], L)$. Facts about mixed volume:

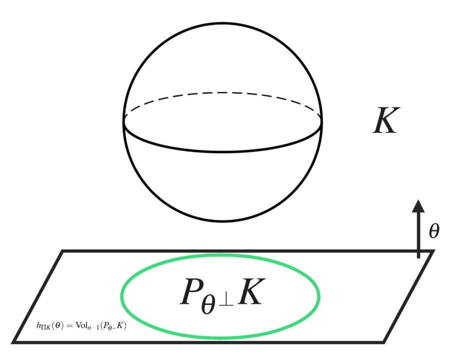
1. $V(K[n-i], K[i]) = \mathrm{Vol}_n(K)$ for all $i = 0, 1, \dots, n$.
2. Mixed volume is translation invariant: $V(K[n-1], L + a) = V(K[n-1], L)$, for $a \in \mathbb{R}^n$.
3. For $T \in GL_n(\mathbb{R}^n)$: $V(TK[n-i], TL[i]) = |\det T| V(K[n-i], L[i])$. In particular: $V(K[n-1], L) = V(-K[n-1], -L)$.
4. The *mean width* of K is given by $w_n(K) = \frac{1}{\mathrm{Vol}_n(B_2^n)} V(B_2^n[n-1], K)$.

A set L containing the origin is a star body if it has a continuous radial function and it is star shaped (i.e. $[o, x] \subset L$ for every $x \in L$). Lutwak introduced the dual Mixed volume for star bodies K and L:

$$\widetilde{V}_i(K[n-i], L[i]) = \frac{1}{n}\int_{\mathbb{S}} \rho_K(\theta)^{n-i} \rho_L(\theta)^i d\theta.$$

When $i = -1$ we write $\widetilde{V}(K[n+1], L)$.

How symmetric is a convex body?



$P_{\theta^\perp}K$: the orthogonal projection of $K \subset \mathbb{R}^n$ onto the hyperplane through the origin orthogonal to $\theta \in \mathbb{S}^{n-1}$. *Cauchy's integral formula* states

$$\mathrm{Vol}_{n-1}(P_{\theta^\perp}K) = \frac{n}{2} V(K[n-1], [-\theta, \theta]) = nV(K[n-1], [o, \theta]).$$

The **polar projection body** of K, denoted $\Pi^\circ K$, is the symmetric convex body given by $\rho_{\Pi^\circ K}(\theta)^{-1} = \mathrm{Vol}_{n-1}(P_{\theta^\perp}K)$.
Why symmetric? Translation invariance!

$$\rho^{-1}_{\Pi^\circ K}(\theta) = nV(K[n-1], [o, \theta]) = nV(K[n-1], [o, -\theta]) = \rho^{-1}_{\Pi^\circ K}(-\theta).$$

The fact that

$$\rho^{-1}_{\Pi^\circ(-K)}(\theta) = nV(-K[n-1], [o, \theta]) = nV(K[n-1], [o, -\theta]) = \rho^{-1}_{\Pi^\circ K}(-\theta)$$

shows

$$\Pi^\circ(-K) = \Pi^\circ K.$$

Petty's isoperimetric inequality: Letting ∂K denote the topological boundary of K

$$\mathrm{Vol}_n(\Pi^\circ K)\mathrm{Vol}_{n-1}(\partial K)^n \geq \mathrm{Vol}_n(B_2^n)\left(\frac{\mathrm{Vol}_n(B_2^n)}{\mathrm{Vol}_{n-1}(B_2^{n-1})}\right)^n,$$

with equality if, and only if, ΠK is a dilate of B_2^n. (Note: follows from Jensen's inequality and Aleksandrov's formula for mixed volume).

$$\boxed{\frac{1}{n^n}\binom{2n}{n} \leq \mathrm{Vol}_n(K)^{n-1}\mathrm{Vol}_n(\Pi^\circ K) \leq \left(\frac{\mathrm{Vol}_n(B_2^n)}{\mathrm{Vol}_{n-1}(B_2^{n-1})}\right)^n.} \quad (1)$$

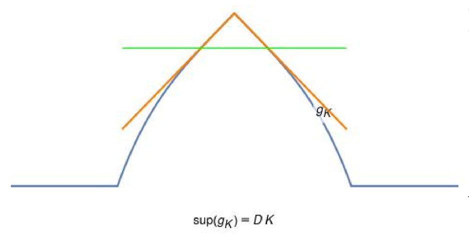
Right-hand side of (1): **Petty's projection inequality**, proven by Petty in 1971; equality occurs in Petty's inequality if, and only if, K is an ellipsoid. Combining with Petty's isoperimetric inequality yields the classical isoperimetric inequality. The left-hand side of (1): **Zhang's inequality**, proven by Zhang in 1991; equality holds if, and only if, K is a simplex (convex hull of $n+1$ affinely independent points).

The **covariogram** of K is given by $g_K(x) = \mathrm{Vol}_n(K \cap (K + x))$ and is supported on the difference body of K

$$DK = \{x : K \cap (K + x) \neq \emptyset\} = K + (-K).$$

$$\boxed{2^n \leq \frac{\mathrm{Vol}_n(DK)}{\mathrm{Vol}_n(K)} \leq \binom{2n}{n}.}$$

Left-hand side: follows from the Brunn-Minkowski inequality ($1/n$-concavity of the Lebesgue measure over all compact sets); equality if, and only if, K is symmetric. Right-hand side: **the Rogers-Shephard inequality**, proven by Rogers and Shephard in 1957; equality holds if, and only if, K is a simplex.



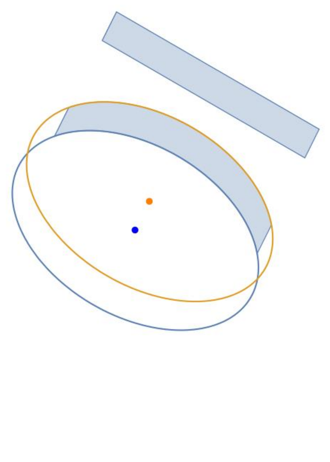
Hint towards the proof:

$$\left.\frac{\mathrm{d}g_K(r\theta)}{\mathrm{d}r}\right|_{r=0^+} = -\rho_{\Pi^\circ K}(\theta)^{-1}.$$

Brunn-concavity: $g_K(x)^{1/n}$ is concave on its support. Thus, for $\theta \in \mathbb{S}^{n-1}$ and $r \in [0, \rho_{DK}(\theta)]$:

$$\boxed{0 \leq \left[1 - \frac{r}{\rho_{DK}(\theta)}\right] \leq \left(\frac{g_K(r\theta)}{\mathrm{Vol}_n(K)}\right)^{1/n} \leq \left[1 - \frac{r}{n\mathrm{Vol}_n(K)\rho_{\Pi^\circ K}(\theta)}\right].} \quad (2)$$

Implies: $\rho_{DK}(\theta) \leq n\mathrm{Vol}_n(K)\rho_{\Pi^\circ K}(\theta)$, that is $DK \subset n\mathrm{Vol}_n(K)\Pi^\circ K$.



Translation invariance and symmetry of the Lebesgue measure, and then Fubini's:

$$\mathrm{Vol}_n(K) = \frac{1}{\mathrm{Vol}_n(K)}\int_K \mathrm{Vol}_n(y - K)dy = \frac{1}{\mathrm{Vol}_n(K)}\int_{DK} g_K(y)dy.$$

Then, use polar coordinates and (2) and the beta function to obtain both Zhang's inequality and the Rogers-Shephard inequality.

2 Rolf Schneider's Higher-Order Generalization

Definition 1. Given a convex K in $\mathbb{R}^n$, its mth order covariogram is given by

$$g_{K,m}(\bar{x}) = \mathrm{Vol}_n\left(K \cap \bigcap_{i=1}^{m}(K + x_i)\right),$$

where $\bar{x} = (x_1, \dots, x_m) \in (\mathbb{R}^n)^m \cong \mathbb{R}^{nm}$. Fact: $g_{K,m}(\bar{x})^{1/n}$ is concave.
The difference body of order m of K, $D^m(K) := \mathrm{supp}(g_{K,m})$. *Schneider's higher-order Rogers Shephard inequality:*

$$\mathrm{Vol}_n(K)^{-m}\mathrm{Vol}_{nm}(D^m(K)) \leq \binom{nm+n}{n},$$

with equality if, and only if, K is a n-dimensional simplex. If $n = 2$, then the lower bound is obtained for all symmetric bodies. For $n \geq 3$ $m \geq 2$, **Schneider's conjecture** is that the lower bound is obtained for ellipsoids.

Operator Hopping (we start here)

Theorem 2. *Let K be a convex body in $\mathbb{R}^n$ and $m \in \mathbb{N}$. For every direction $\bar{\theta} = (\theta_1, \dots, \theta_m) \in \mathbb{S}^{nm-1}$, let $C_{-\bar{\theta}} = \mathrm{conv}_{0 \leq i \leq m}[o, -\theta_i]$. Then:*

$$\boxed{\left.\frac{d}{dr} g_{K,m}(r\bar{\theta})\right|_{r=0^+} = -nV(K[n-1], C_{-\bar{\theta}}).}$$

We define the mth order polar projection body of K as the convex body in $\mathbb{R}^{nm}$ whose radial function is given by

$$\rho^{-1}_{\Pi^{\circ,m}K}(\bar{\theta}) = nV(K[n-1], C_{-\bar{\theta}})$$

$\Pi^{\circ,m}K$ contains the origin as an interior point. For $u \in \mathbb{S}^{n-1}$, let $u_j = (o, \dots, o, u, o, \dots, o) \in \mathbb{S}^{nm-1}$.

$$\rho_{\Pi^{\circ,m}K}(u_j)^{-1} = nV(K[n-1], [o, -u]) = \rho_{\Pi^\circ K}(u)^{-1}.$$

For $m \geq 2$, $\Pi^{\circ,m}K$ is symmetric if, and only if, a translate of K is symmetric ($-\Pi^{\circ,m}K = \Pi^{\circ,m}(-K)$)

Theorem 3 (Higher-Order Petty's isoperimetric inequality)**.** *Let K be a convex body and $m \in \mathbb{N}$. Then, one has the following inequality:*

$$\mathrm{Vol}_{nm}(\Pi^{\circ,m}K)\mathrm{Vol}_{n-1}(\partial K)^{nm} \geq \mathrm{Vol}_{nm}(\Pi^{\circ,m}B_2^n)\mathrm{Vol}_{n-1}(\mathbb{S}^{n-1})^{nm} \geq \mathrm{Vol}_{nm}(B_2^{nm})\left(\frac{n\mathrm{Vol}_n(B_2^n)}{w_{nm}(\Pi^m B_2^n)}\right)^{nm}.$$

Equality in the first inequality holds if, and only if, ΠK is an Euclidean ball. If $m = 1$, there is equality in the second inequality, while for $m \geq 2$, the second inequality is strict.

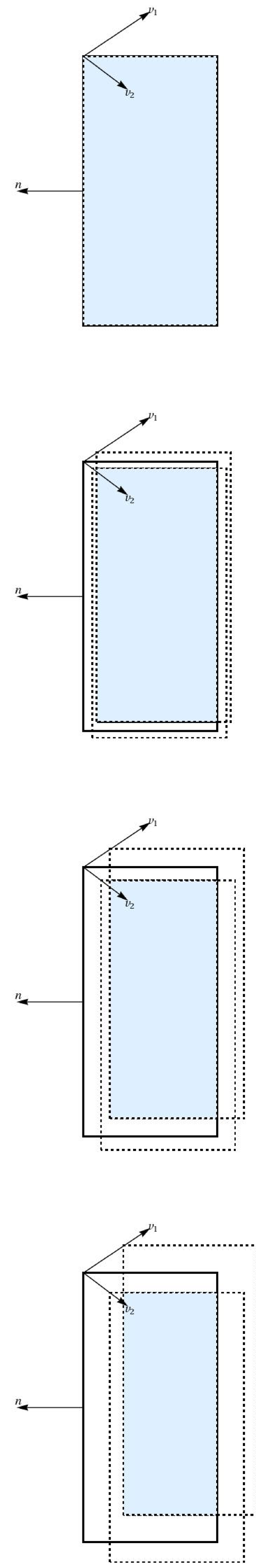


Figure 1: A rectangle and two directions in the plane

Radial Mean Bodies

Let $\psi : [0, \infty) \to [0, \infty)$ be an integrable function that is right continuous and differentiable at 0. Then, the map given by

$$\boxed{\mathcal{M}_\psi : p \mapsto \begin{cases} \int_0^\infty t^{p-1}(\psi(t) - \psi(0))dt, & p \in (-1, 0), \\ \int_0^\infty t^{p-1}\psi(t)dt, & p > 0 \text{ such that } t^{p-1}\psi(t) \in L^1(\mathbb{R}^+), \end{cases}}$$

is piece-wise continuous. This map is known as the Mellin transform.

Definition 4. For $m \in \mathbb{N}$ and $p > -1$, we define the (m, p) *radial mean bodies* $R_p^m K$, to be the star bodies (convex if $p \geq 0$) in $\mathbb{R}^{nm}$ whose radial functions are given by, for $\bar{\theta} \in \mathbb{S}^{nm-1}$:

$$\rho_{R_p^m K}(\bar{\theta}) = \left(p\mathcal{M}_{\frac{g_{K,m}(r\bar{\theta})}{\mathrm{Vol}_n(K)}}(p)\right)^{\frac{1}{p}} = \left(\frac{1}{\mathrm{Vol}_n(K)}\int_K \min_{1 \leq i \leq m} \rho_K(-\theta_i)^p dx\right)^{\frac{1}{p}}$$

for $p \neq 0$. The case $p = 0$ follows from continuity of the pth average.

What follows generalizes $m = 1$ by Gardner and Zhang. From Jensen's inequality: for $-1 < p \leq q \leq \infty$

$$\{o\} = R_{-1}^m K \subset R_p^m K \subset R_q^m K \subset D^m(K).$$

However, by adjusting for asymptotics, we obtain

$$\mathrm{Vol}_n(K)\Pi^{\circ,m}K = \lim_{p \to -1}(1+p)^{\frac{1}{p}} R_p^m K \subset (1+p)^{\frac{1}{p}} R_p^m K \subset (1+q)^{\frac{1}{q}} R_q^m K \subset D^m(K).$$

We can reverse the above: for $-1 < p \leq q \leq \infty$:

$$\boxed{D^m(K) \subseteq \binom{n+q}{n}^{\frac{1}{q}} R_q^m K \subseteq \binom{n+p}{n}^{\frac{1}{p}} R_p^m K \subseteq n\mathrm{Vol}_n(K)\Pi^{\circ,m}K,}$$

with equality if, and only if, K is a n-dimensional simplex. This is established with the following generalization of Berwald's inequality by Fradelizi, Li and Madiman.

Lemma 5. *For every non-increasing, s-concave, $s > 0$, function ψ, the function*

$$G_\psi(p) := \left(\frac{\mathcal{M}_\psi(p)}{\mathcal{M}_{\psi_s}(p)}\right)^{1/p} = \left(p\binom{p + \frac{1}{s}}{p}\mathcal{M}_\psi(p)\right)^{1/p}$$

is decreasing on $(-1, \infty)$ (here, $\psi_s(t) = (1-t)^{1/s}$). Additionally, if there is equality for any two $p, q \in (-1, \infty)$, then $G_\psi(p)$ is constant. Furthermore, $G_\psi(p)$ is constant if, and only if, ψ^s is affine on its support.
(note: version for $s \leq 0$ also exists)

The fact that $\mathrm{Vol}_{nm}(R_{nm}K) = \mathrm{Vol}_n(K)^m$ yields a different proof of Schneider's Rogers-Shephard inequality as well as:

Theorem 6 (Zhang's inequality for higher-order projection bodies)**.** *Fix $m \in \mathbb{N}$ and K be a convex body in $\mathbb{R}^n$. Then, one has*

$$\mathrm{Vol}_n(K)^{nm-m}\mathrm{Vol}_{nm}(\Pi^{\circ,m}K) \geq \frac{1}{n^{nm}}\binom{nm+n}{n},$$

with equality if, and only if, K is a n-dimensional simplex.