p-Means of Convex Bodies and a New Suggestion for the Geometric Mean

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Notation

$\mathcal{C}_0^n := \{C \subset \mathbb{R}^n \mid C$ compact, convex, $0 \in \text{int}(C)\}$; $K, L \in \mathcal{C}_0^n$ always.

Polar: $K^\circ := \{a \in \mathbb{R}^n \mid \forall x \in \mathbb{R}^n \colon a^T x \le 1\} \in \mathcal{C}_0^n$. **Euclidean unit ball:** $\mathbb{B}_2 \in \mathcal{C}_0^n$.

Support function: $h_K \colon \mathbb{R}^n \to [0,\infty), h_K(a) := \max\{a^T x \mid x \in K\}$.

Gauge function: $\|\cdot\|_K \colon \mathbb{R}^n \to [0,\infty), \|x\|_K := \min\{\lambda \ge 0 \mid x \in \lambda K\}$.

$p, q \in [-\infty, \infty]$ always. p-**Mean** of $\alpha, \beta > 0$: $m_p(\alpha,\beta) := \left(\frac{\alpha^p+\beta^p}{2}\right)^{1/p}$ for $p \in \mathbb{R}\setminus\{0\}$,

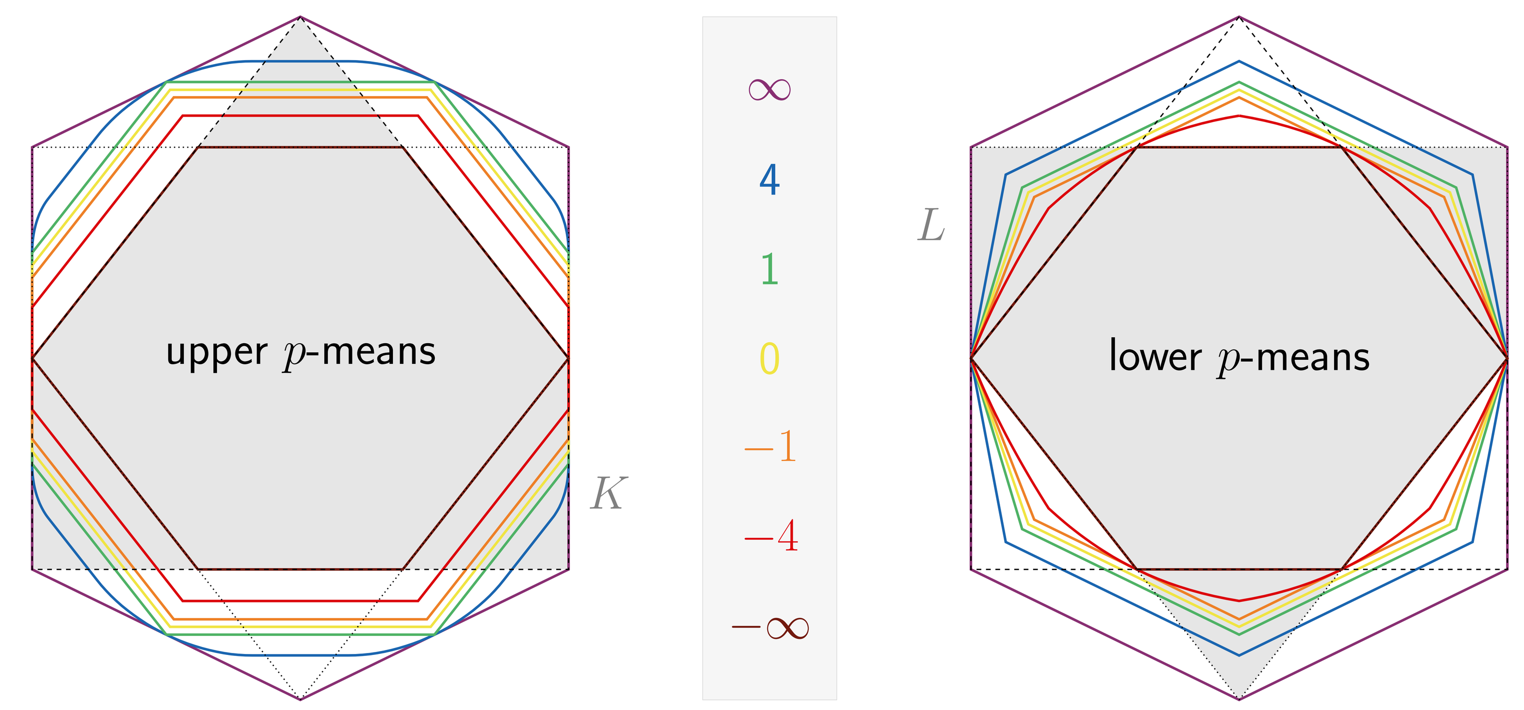
$m_{-\infty}(\alpha,\beta) := \min\{\alpha,\beta\}$, $m_0(\alpha,\beta) := \sqrt{\alpha\beta}$, $m_\infty(\alpha,\beta) := \max\{\alpha,\beta\}$.

special case: $\min\{\alpha,\beta\} \le \left(\frac{\alpha^{-1}+\beta^{-1}}{2}\right)^{-1} \le \sqrt{\alpha\beta} \le \frac{\alpha+\beta}{2} \le \max\{\alpha,\beta\}$

Upper and Lower p-Means

upper p-mean: $\overline{M}_p(K,L) := \left\{x \in \mathbb{R}^n \mid \forall a \in \text{bd}(\mathbb{B}_2) \colon a^T x \le m_p(h_K(a), h_L(a))\right\}$.

lower p-mean: $\underline{M}_p(K,L) := \text{conv}\left(\left\{(m_{-p}(\|x\|_K, \|x\|_L))^{-1} \cdot x \mid x \in \text{bd}(\mathbb{B}_2)\right\}\right)$.



Properties of Upper and Lower p-Means

- $h_{\overline{M}_p(K,L)} \le m_p(h_K, h_L)$, with "=" if $p \in [1,\infty]$.
- $\|\cdot\|_{\underline{M}_p(K,L)} \le m_{-p}(\|\cdot\|_K, \|\cdot\|_L)$, with "=" if $p \in [-\infty,-1]$.

$$\underline{M}_{-\infty}(K,L) = K \cap L \qquad \underline{M}_{-1}(K,L) = \left(\frac{K^\circ + L^\circ}{2}\right)^\circ$$

$$\overline{M}_1(K,L) = \frac{K+L}{2} \qquad \overline{M}_\infty(K,L) = \text{conv}(K \cup L)$$

- If $p < q$: $\overline{M}_p(K,L) \subset \overline{M}_q(K,L)$, $\underline{M}_p(K,L) \subset \underline{M}_q(K,L)$, with "=" iff $K = L$.
- $\underline{M}_p(K,L) \subset \overline{M}_p(K,L)$, with "=" iff $p = \pm\infty$, or $n = 1$, or K and L are dilates.

$$K \cap L \subset \left(\frac{K^\circ + L^\circ}{2}\right)^\circ \subset \frac{K+L}{2} \subset \text{conv}(K \cup L)$$

- $(\overline{M}_p(K,L))^\circ = \underline{M}_{-p}(K^\circ, L^\circ)$.
- If K, L are polytopes, $p \notin (1,\infty)$, $q \notin (-\infty,-1)$: $\overline{M}_p(K,L)$ and $\underline{M}_q(K,L)$ are polytopes.

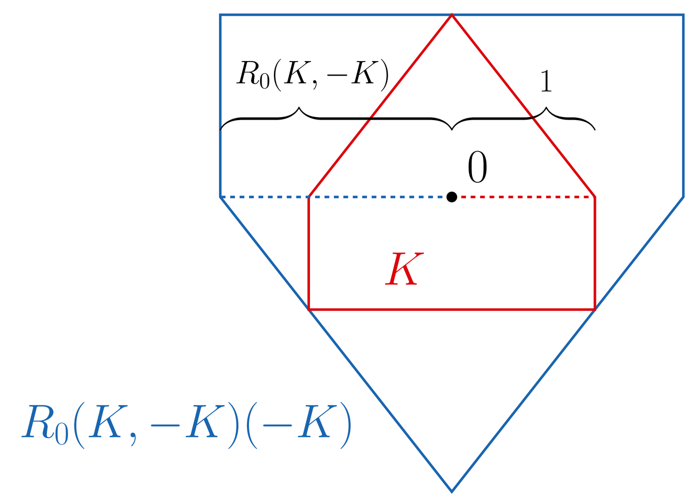
Covering Radii

Define the **covering radius**

$R_0(K,L) := \inf\{\lambda \ge 0 : K \subset \lambda L\}$

and

$R_0^{\max} := \max\{R_0(K,L), R_0(L,K)\} \ge 1$.



Theorem 1.

$$R_0(\overline{M}_p(K,L), \overline{M}_q(K,L)) = \frac{m_p(R_0^{\max},1)}{m_q(R_0^{\max},1)} \qquad \text{if } p \ge q \text{ or } n = 1$$

$$\frac{m_p(R_0^{\max},1)}{m_q(R_0^{\max},1)} \le R_0(\overline{M}_p(K,L), \overline{M}_q(K,L)) \le 1 \qquad \text{if } p \le q \text{ and } n \ge 2$$

Both inequalities are best possible for all $R_0^{\max} \ge 1$.
All results also apply for $R_0(\underline{M}_p(K,L), \underline{M}_q(K,L))$ and $R_0(\underline{M}_p(K,L), \overline{M}_q(K,L))$.

Theorem 2.

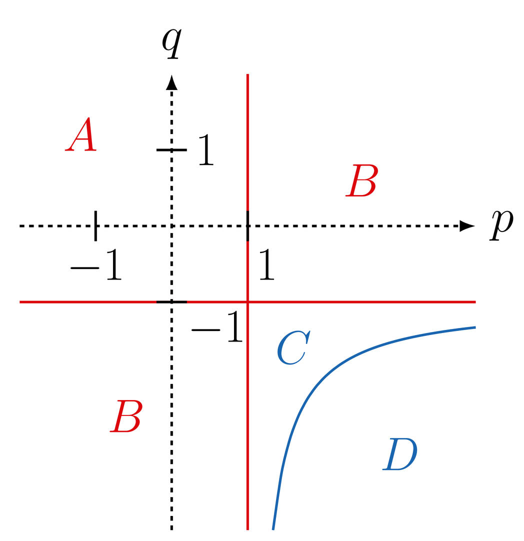
$$\frac{m_p(R_0^{\max},1)}{m_q(R_0^{\max},1)} \le R_0(\overline{M}_p(K,L), \underline{M}_q(K,L)) \le \min\{m_p(R_0^{\max},1), m_{-q}(R_0^{\max},1)\}$$

The lower bound is an equality if $n = 1$, and best possible for all $n \ge 2$ and $R_0^{\max} \ge 1$.
If $(p,q) \in (1,\infty) \times (-\infty,-1)$, it also holds

$$R_0(\overline{M}_p(K,L), \underline{M}_q(K,L)) \le 2^{\frac{1}{q}-\frac{1}{p}} \max_{\lambda \in [0,1]} \frac{\left\|\binom{1}{1} + (R_0^{\max}-1)\binom{\lambda}{1-\lambda}\right\|_{-q}}{\left\|\binom{\lambda}{1-\lambda}\right\|_{\frac{p}{p-1}}}$$

Tightness of the upper bounds for $n \ge 2$:

A : first upper bound for $n = 2$, $R_0^{\max} > 1$ strict.
B : first upper bound best possible for all $R_0^{\max} \ge 1$.
C : second upper bound best possible for all $R_0^{\max} \ge 1$; it simplifies to $\frac{R_0^{\max}+1}{2}$.
D : second upper bound best possible for all $R_0^{\max} \ge 1$; no simpler representation is known.

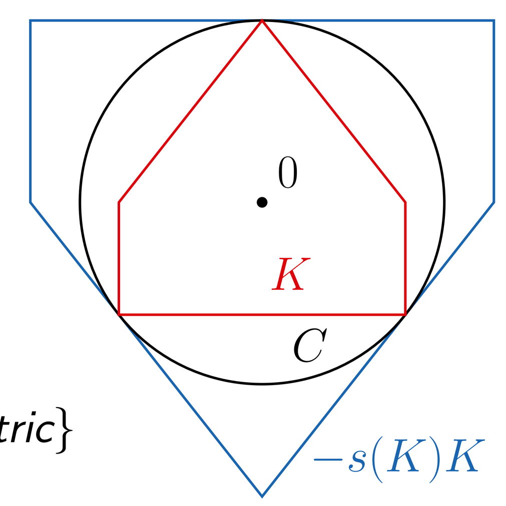


Banach-Mazur Distance

Well-known: $s(K) = d_{BM}\left(K, \frac{K-K}{2}\right) = \min\{d_{BM}(K,S) \mid S \in \mathcal{C}_0^n$ symmetric$\}$

with $s(K) := \min\{\lambda \ge 0 \mid \exists c \in \mathbb{R}^n \colon -K \subset c + \lambda K\}$
the **Minkowski asymmetry** of K.
K is **Minkowski centered** if $s(K)$ is attained for $c = 0$.

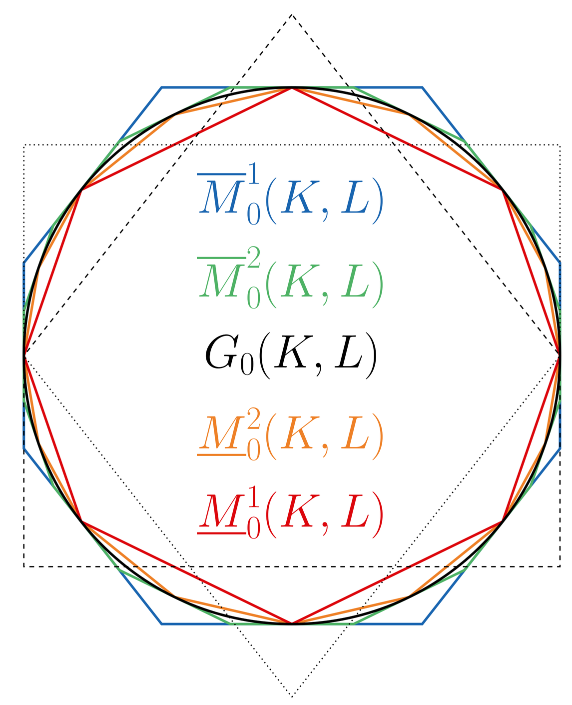
Theorem 3. *For $K \in \mathcal{C}_0^n$ Minkowski centered:*
$\forall C \in \mathcal{C}_0^n$ symmetric, $\underline{M}_p(K,-K) \subset C \subset \overline{M}_p(K,-K)$:
$s(K) = d_{BM}(K,C) = \min\{d_{BM}(K,S) \mid S \in \mathcal{C}_0^n$ *symmetric*$\}$



Geometric Mean

Define iteratively for all $i \in \mathbb{N}$:

$$\overline{M}_0^1(K,L) := \overline{M}_0(K,L),$$
$$\underline{M}_0^1(K,L) := \underline{M}_0(K,L),$$
$$\overline{M}_0^{i+1}(K,L) := \overline{M}_0(\overline{M}_0^i(K,L), \underline{M}_0^i(K,L)),$$
$$\underline{M}_0^{i+1}(K,L) := \underline{M}_0(\overline{M}_0^i(K,L), \underline{M}_0^i(K,L)).$$



Theorem 4. *$(\overline{M}_0^i(K,L))_{i\in\mathbb{N}}$ and $(\underline{M}_0^i(K,L))_{i\in\mathbb{N}}$ converge to a common Hausdorff limit $G_0(K,L) \in \mathcal{C}_0^n$.*

Properties of the Geometric Mean $G_0(K,L)$

Theorem 5. *G_0 satisfies the following (requested) properties:*

1) $G_0(K,K) = K$.
2) *G_0 is symmetric in its arguments: $G_0(K,L) = G_0(L,K)$.*
3) *G_0 is increasing: $\forall K', L' \in \mathcal{C}_0^n, K \subset K', L \subset L' \colon G_0(K,L) \subset G_0(K',L')$.*
4) *G_0 is continuous in both arguments w.r.t. the Hausdorff distance.*
5) *G_0 is invariant under regular linear transformation:*
   $\forall A \in \mathbb{R}^{n\times n}$ regular: $G_0(AK,AL) = AG_0(K,L)$.
6) *G_0 satisfies the scaling property: $\forall \alpha,\beta > 0 \colon G_0(\alpha K, \beta L) = \sqrt{\alpha\beta} \cdot G_0(K,L)$.*
7) *$\forall P \in \mathbb{R}^{n\times n}$ positive definite: $G_0(PK, K^\circ) = P^{1/2} \cdot \mathbb{B}_2$, in particular $G_0(K,K^\circ) = \mathbb{B}_2$.*
8) *G_0 commutes with polarity: $(G_0(K,L))^\circ = G_0(K^\circ, L^\circ)$.*
9) *G_0 satisfies the harmonic-geometric-arithmetic mean inequality:*
   $$\left(\frac{K^\circ + L^\circ}{2}\right)^\circ \subset \underline{M}_0(K,L) \subset G_0(K,L) \subset \overline{M}_0(K,L) \subset \frac{K+L}{2}$$
   with "=" in the second/third inclusion iff $n = 1$ or K and L are dilates, and in the first/forth/every inclusion iff $K = L$.
10) *$G_0(K,-K)$ is 0-symmetric.*
11) *For K, L Minkowski centered: $s(G_0(K,L)) \le \sqrt{s(K)s(L)} \le \max\{s(K), s(L)\}$. The inequality is best possible. In particular, $G_0(K,L)$ is symmetric if K and L are.*
12) *For K Minkowski centered: $d_{BM}(K, G_0(K,-K)) = s(K)$.*

References

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