Vector-Valued Valuations On The Space Of Convex Functions

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Abstract

In this poster, we discuss the progress that has been made so far in classifying vector-valued continuous, translation invariant and rotation equivariant valuations on the space of all convex super-coercive functions on $\mathbb{R}^n$. We remark that these valuations vanish on indicator functions.

Introduction and Preliminaries

Currently, a geometric valuation theory on function spaces is being developed. On a space X of real-valued functions, a vector-valued functional Z in $\mathbb{R}^n$ is called a *valuation* if

$$Z(u \vee v) + Z(u \wedge v) = Z(u) + Z(v)$$

where $u, v \in X$ such that the pointwise maximum $u \vee v$ and the pointwise minimum $u \wedge v$ are in X. This definition is motivated from the study of vector-valued valuations on convex bodies (see [4]). We restrict our attention to the space of *convex super-coercive* functions which is given by

$$\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) = \left\{ u \in \mathrm{Conv}(\mathbb{R}^n) \mid \lim_{|x| \to +\infty} \frac{u(x)}{|x|} = +\infty \right\}$$

This space is dual to the space of *finite convex* functions by the *Fenchel–Legendre* transform.

A valuation $Z : \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \to \mathbb{R}^n$ is said to be *epi-translation invariant* if

$$Z(u \circ \tau + \alpha) = Z(u)$$

for every $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$, every translation τ on $\mathbb{R}^n$ and every $\alpha \in \mathbb{R}$. It is called *rotation equivariant* if $Z(u \circ \phi^{-1}) = \phi Z(u)$ for all $\phi \in \mathrm{SO}(n)$. We say that a valuation on $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ is *epi-homogeneous of degree* j if $Z(\lambda \cdot u) = \lambda^j Z(u)$ where $\lambda > 0$ and $\lambda \cdot u(x) = \lambda u(\frac{x}{\lambda})$ for all $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$. We denote by $[A]_j$ the j-th elementary symmetric function of the eigenvalues of a symmetric matrix A and use the convention that $[A]_0 = 1$.

Main Results

So far, we were able to establish an existence theorem of vector-valued continuous valuations with the properties stated above on the space of convex super-coercive functions. We begin this by treating the n-homogeneous valuation and we observe that a distinction is made according to the dimension. We denote by $\mathcal{C}_{\mathrm{b}}((0,\infty))$ the continuous functions on $(0,+\infty)$ with bounded support.

Theorem

Let $n \geq 3$. A functional $Z : \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \to \mathbb{R}^n$ is a continuous, epi-translation invariant and $\mathrm{SO}(n)$-equivariant valuation that is epi-homogeneous of degree n, if and only if there exists $\xi \in \mathcal{C}_{\mathrm{b}}((0,\infty))$ with $\lim_{t\to 0^+} \xi(t) = 0$ such that

$$Z(u) = \int_{\mathrm{dom}\, u} \xi(|\nabla u(x)|) \frac{\nabla u(x)}{|\nabla u(x)|} dx.$$

The proof of this result is based on [2, Proposition 27] and using the rotation equivariance, one can deduce the form stated above. The latter step is what differs when one considers the dimension to be equal to 2 since the group SO(2) is commutative. Hence, we get the following result.

Theorem

A functional $Z : \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^2) \to \mathbb{R}^2$ is a continuous, epi-translation invariant and SO(2)-equivariant valuation that is epi-homogeneous of degree 2, if and only if there exists a matrix-valued continuous function ξ with bounded support on $(0,+\infty)$ and $\lim_{t\to 0^+} \xi(t) = 0_{\mathbb{R}^2\times\mathbb{R}^2}$ such that

$$Z(u) = \int_{\mathrm{dom}\, u} \xi(|\nabla u(x)|) \frac{\nabla u(x)}{|\nabla u(x)|} dx.$$

Further Results

To further investigate the j-th degree of homogeneity of these valuations, we introduce the following classes of densities with singularities at the origin, that we denote by T_j^n for $0 \leq j \leq n$. They are given by

$$T_j^n = \left\{ \xi \in \mathcal{C}_{\mathrm{b}}((0,\infty)) \mid \lim_{t\to 0^+} t^{n-j}\xi(t) = 0 \text{ and } \lim_{t\to 0^+} t \int_t^{+\infty} s^{n-j-2}\, \xi(s) ds = 0 \right\}$$

and $T_n^n = \{\xi \in \mathcal{C}_{\mathrm{b}}((0,\infty)) \mid \lim_{t\to 0^+} \xi(t) = 0\}$.

We draw our attention to the existence of valuations of degree of homogeneity up to n by stating the next theorem

Theorem

Let $0 \leq j \leq n$. For $\xi \in T_j^n$, there exists a continuous, $\mathrm{SO}(n)$-equivariant, dually epi-translation invariant valuation $Z_{j,\xi}$ that is homogeneous of degree j such that for every $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \cap \mathcal{C}^2(\mathbb{R}^n)$, it is given by

$$Z_{j,\xi}(u) = \int_{\mathrm{dom}\, u} \xi(|\nabla u(x)|) \frac{\nabla u(x)}{|\nabla u(x)|} [D^2 v(x)]_{n-j} dx$$

The proof of this result relies on passing from mixed Hessian measures to mixed Monge–Ampère measures by use of an integral transform acting on the density functions that we denote by $\mathcal{R}$ and which is given by

$$\mathcal{R}\xi(s) := s\xi(s) + \int_s^{+\infty} \xi(t) dt$$

It was defined in [3]

Future Aims

- Our first aim is to establish a Hadwiger type theorem for these valuations. Hence, we need to show that the proceeding valuations represent all the possible valuations of different degrees.
- The second aim is a characterization of continuous, vertical translation invariant, translation covariant and rotation equivariant valuations such that a valuation $Z : \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) \to \mathbb{R}^n$ is said to be vertical translation invariant if $Z(u + \alpha) = Z(u)$ for every $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ and every $\alpha \in \mathbb{R}$ and it is said to be translation covariant if $Z(u \circ \tau_{x_0}^{-1}) = Z(u) + x_0 Z_0(u)$ where τ_{x_0} is a translation in $\mathbb{R}^n$ and Z_0 is a continuous, translation and rotation invariant valuation. In other words, Z will represent a functional version of the intrinsic moment vectors for convex bodies whereas Z_0 is a functional intrinsic volume of convex functions. The latter is fully characterized by Colesanti, Ludwig and Mussnig in [1], see also [5].

References

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