Properties of the coordinate ring of a convex polyomino

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Outline

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The coordinate ring of a convex polyomino was introduced by Qureshi.
In order to define polyominoes and polyomino ideals, we give some terminology.

On $\mathbb{N}^2$, we consider the natural partial order defined as follows:

$$(i,j) \leq (k,l) \text{ if and only if } i \leq k \text{ and } j \leq l.$$  

Let $a = (i,j), \ b = (k,l) \in \mathbb{N}^2$ and $a \leq b$.

The set

$$[a, b] = \{c \in \mathbb{N}^2 \mid a \leq c \leq b\}$$

represents an interval in $\mathbb{N}^2$. 
The interval

$$C = [a, a + (1, 1)]$$

is called a cell in $\mathbb{N}^2$ with lower left corner $a$.

Figure: A cell in $\mathbb{N}^2$
Let $\mathcal{P}$ be a finite collection of cells in $\mathbb{N}^2$. Two cells $A$ and $B$ of $\mathcal{P}$ are connected by a path in $\mathcal{P}$, if there is a sequence of cells of $\mathcal{P}$ given by $A = A_1, A_2, \cdots, A_{n-1}, A_n = B$ such that $A_i \cap A_{i+1}$ is an edge of $A_i$ and $A_{i+1}$ for $i \in \{1, \cdots, n-1\}$.

**Definition**

A collection of cells $\mathcal{P}$ is called a polyomino if any two cells of $\mathcal{P}$ are connected by a path in $\mathcal{P}$. 

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Figure: A polyomino
Preliminaries

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The regularity of $K[\mathcal{P}]$
The multiplicity of $K[\mathcal{P}]$

A column convex polyomino

A row convex polyomino

Figure: A convex polyomino
Let $\mathcal{P}$ be a convex polyomino. After a possible translation, we consider $[(1,1), (m,n)]$ to be the smallest interval which contains the vertices of $\mathcal{P}$.

We say that $\mathcal{P}$ is a convex polyomino on $[m] \times [n]$, where $[m] = \{1, \ldots, m\}$ and $[n] = \{1, \ldots, n\}$. 
Let $\mathcal{P}$ be a convex polyomino on $[m] \times [n]$. Fix a field $\mathbb{K}$ and a polynomial ring

$$S = \mathbb{K}[x_{ij} \mid (i,j) \in V(\mathcal{P})],$$

where $V(\mathcal{P})$ is the set of the vertices of $\mathcal{P}$.

The polyomino ideal $I_{\mathcal{P}} \subset S$ is generated by all binomials

$$x_{il}x_{kj} - x_{ij}x_{kl}$$

for which $[(i,j),(k,l)]$ is an interval in $\mathcal{P}$.

The $\mathbb{K}$-algebra $S/I_{\mathcal{P}}$ is denoted $\mathbb{K}[\mathcal{P}]$ and is called the coordinate ring of $\mathcal{P}$. 
Figure: For the "cross", $I_P$ has 11 generators.
The ring

\[ R = \mathbb{K}[x_i y_j \mid (i, j) \in \mathcal{V}(P)] \subset \mathbb{K}[x_1, \ldots, x_m, y_1, \ldots, y_n] \]

can be viewed as an edge ring of a bipartite graph \( G_P \) with vertex set

\[ \mathcal{V}(G_P) = X \cup Y, \text{ where } X = \{x_1, \ldots, x_m\} \text{ and } Y = \{y_1, \ldots, y_n\} \]

and edge set

\[ \mathcal{E}(G_P) = \{\{x_i, y_j\} \mid (i, j) \in \mathcal{V}(P)\}. \]
Preliminaries


Figure: The bipartite graph attached to a cell in $\mathbb{N}^2$

$K[\mathcal{P}]$ can be identified with $K[G_\mathcal{P}]$. 

Let $\mathcal{P}$ be a convex polyomino on $[m] \times [n]$.

**Theorem (A. Qureshi, Theorem 2.2)**

$K[\mathcal{P}]$ is a Cohen-Macaulay domain with $\dim K[\mathcal{P}] = m + n - 1$. 

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Let $X = \{x_1, \ldots, x_m\}$ and $Y = \{y_1, \ldots, y_n\}$.

**Theorem (H. Ohsugi, T. Hibi, Theorem 2.1)**

We consider $G$ to be a bipartite graph on $X \cup Y$ and suppose that $G$ is 2-connected. Then $K[G]$ is Gorenstein if and only if $x_1 \cdots x_m y_1 \cdots y_n \in K[G]$ and one has $|N(T)| = |T| + 1$ for every subset $T \subset X$ such that $G_{T \cup N(T)}$ is connected and that $G_{(X \cup Y) \setminus (T \cup N(T))}$ is a connected graph with at least one edge.
Definition

Let $G$ be a graph on $V$. Then we say that $G$ is 2-connected if $G$ together with $G \setminus \{v\}$ for all $v \in V$ are connected.

Proposition

If $\mathcal{P}$ is a convex polyomino on $[m] \times [n]$, then the bipartite graph $G_{\mathcal{P}}$ is 2-connected.
Let $G$ be a graph and $T \subset V(G)$. The set

$$N(T) = \{y \in V(G) \mid \{x, y\} \in E(G) \text{ for some } x \in T\}$$

represents the set of the neighbors of the subset $T \subset V(G)$. Let $\mathcal{P}$ be a convex polyomino on $[m] \times [n]$. We set $X = \{x_1, \ldots, x_m\}$ and $Y = \{y_1, \ldots, y_n\}$ and, if needed, we identify the point $(x_i, y_j)$ in the plane with the vertex $(i, j) \in V(\mathcal{P})$. 
Definition

Let $T \subset X$. The set $N_Y(T) = \{y \in Y \mid (x, y) \in V(\mathcal{P}) \text{ for some } x \in T\}$ is called a neighbor vertical interval if $N_Y(T) = \{y_a, y_{a+1}, \ldots, y_b\}$ with $a < b$ and for every $i \in \{a, a+1, \ldots, b-1\}$ there exists $x \in T$ such that $[(x, y_i), (x, y_{i+1})]$ is an edge in $\mathcal{P}$. 

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Properties of the coordinate ring of a convex polyomino
Definition

Let $U \subset Y$. The set $N_X(U) = \{x \in X \mid (x,y) \in V(P) \text{ for some } y \in U\}$ is called a neighbor horizontal interval if $N_X(U) = \{x_a, x_{a+1}, \ldots, x_b\}$ with $a < b$ and for every $i \in \{a, a+1, \ldots, b-1\}$ there exists $y \in U$ such that $[(x_i,y), (x_{i+1},y)]$ is an edge in $P$.
Proposition

Let $\mathcal{P}$ be a convex polyomino on $[m] \times [n]$ and $G_\mathcal{P}$ its associated bipartite graph. Then we have

$$x_1 \cdots x_m y_1 \cdots y_n \in K[G_\mathcal{P}]$$

if and only if

$$|N_Y(T)| \geq |T| \text{ for every } T \subseteq X \text{ and }$$

$$|N_X(U)| \geq |U| \text{ for every } U \subseteq Y.$$
Gorenstein convex polyominoes

Figure: Perfect matching for $G_{\mathcal{P}}$
Proposition

Let $\mathcal{P}$ be a convex polyomino on $[m] \times [n]$ and $G := G_\mathcal{P}$ its associated bipartite graph. For each $\emptyset \neq T \subset X,$

$$N_Y(T) \text{ is a neighbor vertical interval if and only if } G_{T \cup N(T)} \text{ is a connected graph}$$

and

$$N_X(Y \setminus N_Y(T)) = X \setminus T \text{ is a neighbor horizontal interval if and only if } G_{(X \cup Y) \setminus (T \cup N_Y(T))} \text{ is a connected graph with at least one edge.}$$
Theorem

Let $\mathcal{P}$ be a convex polyomino on $[m] \times [n]$. Then $K[\mathcal{P}]$ is Gorenstein if and only if the following conditions are fulfilled:

1. $|U| \leq |N_X(U)|$, $\forall$ $U \subset Y$ and $|T| \leq |N_Y(T)|$, $\forall$ $T \subset X$;
2. For every $\emptyset \neq T \subsetneq X$ with properties
   1. $N_Y(T)$ is a neighbor vertical interval and
   2. $N_X(Y \setminus N_Y(T)) = X \setminus T$ is a neighbor horizontal interval,
   one has $|N_Y(T)| = |T| + 1$. 
Gorenstein convex polyominoes

\[ K[P] \text{ is not Gorenstein} \quad \text{and} \quad K[P] \text{ is Gorenstein} \]
Gorenstein convex polyominoes

Let $\mathcal{P}$ be a convex polyomino on $[m] \times [n]$. Then $\mathcal{P}$ is called a stack polyomino if all cells of the first line of $[(1,1), (m,n)]$ are in $\mathcal{P}$. 

Gorenstein stack polyomino
Let \( \mathcal{P} \) be a stack polyomino on \([m] \times [n]\). We consider \( H_{\mathbb{K}[\mathcal{P}]}(t) \) to be the Hilbert series of \( \mathbb{K}[\mathcal{P}] \). Then

\[
H_{\mathbb{K}[\mathcal{P}]}(t) = \frac{Q(t)}{(1-t)^d}
\]

where \( Q(t) \in \mathbb{Z}[t] \) and \( d \) is the Krull dimension of \( \mathbb{K}[\mathcal{P}] \). It is known that

\[
\text{reg}(\mathbb{K}[\mathcal{P}]) = \text{deg}(Q(t)) = \dim(\mathbb{K}[\mathcal{P}]) + a(\mathbb{K}[\mathcal{P}]),
\]

since \( \mathbb{K}[\mathcal{P}] \) is a Cohen-Macaulay ring. The \( a \)-invariant \( a(\mathbb{K}[\mathcal{P}]) \) of \( \mathbb{K}[\mathcal{P}] \) is defined to be the degree of the Hilbert series of \( \mathbb{K}[\mathcal{P}] \), which by definition is equal to \( \text{deg}(Q(t)) - d \).
Let $G_P$ be the bipartite graph attached to $P$. In this section, we consider $G_P$ as a digraph with all its arrows leaving the vertex set $Y$.

**Figure:** A stack polyomino and its associated digraph
The regularity of $K[\mathcal{P}]$

**Definition**

If $T \subset X \cup Y$, then

$\delta^+(T) = \{ e = (z, w) \in E(G_{\mathcal{P}}) \mid z \in T \text{ and } w \not\in T \}$ is the set of edges leaving the vertex set $T$ and

$\delta^-(T) = \{ e = (z, w) \in E(G_{\mathcal{P}}) \mid z \not\in T \text{ and } w \in T \}$ is the set of edges entering the vertex set $T$.

The set $\delta^+(T)$ is called a directed cut of the digraph $G_{\mathcal{P}}$ if $\emptyset \neq T \subsetneq X \cup Y$ and $\delta^-(T) = \emptyset$. 
In the digraph of the following figure, let

$$T_1 = \{x_3, y_2, y_3\} \text{ and } T_2 = \{x_3, y_1, y_2\}.$$  

Then $\delta^+(T_2)$ is a directed cut, while $\delta^+(T_1)$ is not.
The regularity of $\mathbb{K}[\mathcal{P}]$

Since $\mathbb{K}[\mathcal{P}] \cong \mathbb{K}[G_{\mathcal{P}}]$,

$$\delta^+(T) = \{(x,y) \in V(\mathcal{P}) \mid x \notin T \text{ and } y \in T\}$$

and

$$\delta^-(T) = \{(x,y) \in V(\mathcal{P}) \mid x \in T \text{ and } y \notin T\}$$

for every $T \subset X \cup Y$.

**Lemma**

Let $\mathcal{P}$ be a stack polyomino on $[m] \times [n]$, $G_{\mathcal{P}}$ its associated digraph and $T \subset X \cup Y$. Then $\delta^+(T)$ is a directed cut of the digraph $G_{\mathcal{P}}$ if and only if $T = T^x \cup T^y$ with $T^x \subset X$, $T^y \subset Y$ and $N_Y(T^x) \subset T^y$. 

**Proposition (C.E. Valencia, R.H. Villarreal, Proposition 4.2)**

Let $G$ be a connected bipartite graph with $V(G) = X \cup Y$. If $G$ is a digraph with all its arrows leaving the vertex set $Y$, then

$-a(K[G]) = \text{the maximum number of disjoint directed cuts}$.
The regularity of $K[\mathcal{P}]$

**Proposition**

*If $\mathcal{P}$ is a stack polyomino on $[m] \times [n]$, then*

$$-a(K[\mathcal{P}]) = \max\{m, n\}.$$ 

**Corollary**

*If $\mathcal{P}$ is a stack polyomino on $[m] \times [n]$, then*

$$\text{reg}(K[\mathcal{P}]) = m + n - 1 - \max\{m, n\} = \min\{m, n\} - 1.$$
The regularity of $K[\mathcal{P}]$

Figure: $\text{reg}(K[\mathcal{P}]) = 3$
Let $\mathcal{P}$ be a stack polyomino on $[m] \times [n]$.
The Hilbert series $H_{\mathbb{K}[\mathcal{P}]}(t)$ of $\mathbb{K}[\mathcal{P}]$ is given by

$$H_{\mathbb{K}[\mathcal{P}]}(t) = \frac{Q(t)}{(1-t)^d},$$

where $Q(t) \in \mathbb{Z}[t]$ and $d = \dim(\mathbb{K}[\mathcal{P}]) = m + n - 1$.
The multiplicity of $\mathbb{K}[\mathcal{P}]$, denoted $e(\mathbb{K}[\mathcal{P}])$, is given by $Q(1)$. 
The multiplicity of $\mathbb{K}[\mathcal{P}]$

For every $i \in [m]$, 

$$\text{height}(i) = \max\{j \in [n] \mid (i,j) \in V(\mathcal{P})\}.$$ 

We give a total order on the variables $x_{ij}$, with $(i,j) \in V(\mathcal{P})$: 

$$x_{ij} > x_{kl} \text{ if and only if}$$ 

$$(\text{height}(i) > \text{height}(k)) \text{ or (height}(i) = \text{height}(k) \text{ and } i > k)$$ 

$$\text{or (}i = k \text{ and } j > l).$$ 

Let $<$ be the reverse lexicographical order induced by this order of variables.
It is known that the polyomino ideal $I_P$ has a reduced Gröbner basis with respect to $<$ consisting of all inner 2-minors of $P$. We may view $\text{in}_<(I_P)$ as the Stanley-Reisner ideal of a simplicial complex, denoted $\Delta_P$, on the vertex set $V(P)$.

It is known that $\Delta_P$ is a pure shellable simplicial complex.
The multiplicity of $\mathbb{K}[\mathcal{P}]$

Let $\mathcal{P}$ be the polyomino of the figure. Then
\[\text{in}_<(I_{\mathcal{P}}) = (x_{11}x_{32}, x_{21}x_{32}, x_{21}x_{12}, x_{21}x_{13}, x_{22}x_{13})\]
and
\[\Delta_{\mathcal{P}} = \langle F_1 = \{(1,1), (2,1), (2,2), (2,3), (3,1)\} ;
F_2 = \{(1,1), (1,2), (2,2), (2,3), (3,1)\} ;
F_3 = \{(1,1), (1,2), (1,3), (2,3), (3,1)\} ;
F_4 = \{(1,2), (2,2), (2,3), (3,1), (3,2)\} ;
F_5 = \{(1,2), (1,3), (2,3), (3,1), (3,2)\}\rangle.

The order of the variables:

\[x_{23} > x_{22} > x_{21} > x_{13} > x_{12} > x_{11} > x_{32} > x_{31}\]
The multiplicity of $K[\mathcal{P}]$

Since

$$H_{K[\mathcal{P}]}(t) = H_{S/\text{in}_<(I_{\mathcal{P}})}(t) = H_{K[\Delta \mathcal{P}]}$$,

we obtain

$$e(K[\mathcal{P}]) = |\mathcal{F}(\Delta \mathcal{P})|,$$

where $\mathcal{F}(\Delta \mathcal{P})$ denotes the set of the facets of $\Delta \mathcal{P}$. 

The multiplicity of $\mathbb{K}[\mathcal{P}]$

**Definition**

Let $\Delta$ be a simplicial complex on the vertex set $V$ and $v \in V$. The **link of $v$ in $\Delta$** is the simplicial complex

$$\text{lk}(v) = \{ F \in \Delta \mid v \notin F \text{ and } F \cup \{v\} \in \Delta \}$$

and the **deletion of $v$** is the simplicial complex

$$\text{del}(v) = \{ F \in \Delta \mid v \notin F \}.$$
Let $x_{ij}$ be the smallest variable in $S$ and fix $\nu = (i, \text{height}(i)) \in V(P)$. If $i = 1$, then we denote by $P_1$ the polyomino obtained from $P$ by deleting the cell which contains the vertex $\nu$. Otherwise, $P_1$ is given by deleting the cell which contains the vertex $(m, \text{height}(m))$. 
Lemma

We have $|\mathcal{F}(\Delta \mathcal{P}_1)| = |\mathcal{F}(\text{del}(v))|$. 

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Let $\mathcal{P}_2$ be the polyomino obtained from $\mathcal{P}$ by deleting all the cells of $\mathcal{P}$ which lie below the horizontal edge interval containing the vertex $v$.

\begin{align*}
\mathcal{P} & \quad \mathcal{P}_2 \\
\begin{array}{c}
\text{V} \\
\text{v}
\end{array} & \quad \begin{array}{c}
\text{v}
\end{array}
\end{align*}

Lemma

We have $|\mathcal{F}(\Delta \mathcal{P}_2)| = |\mathcal{F}(\text{lk}(v))|$. 
The multiplicity of \( K[P] \)

**Theorem**

Let \( P \) be a stack polyomino on \([m] \times [n]\) and \( v = (i,j) \in V(P) \) with the properties:

1. \( x_{i_1} \) is the smallest variable in \( S \) and
2. \( j = \text{height}(i) \).

Let \( P_1 \) and \( P_2 \) be the polyominoes presented above. Then

\[
e(K[P]) = e(K[P_1]) + e(K[P_2]).
\]
The multiplicity of $\mathbb{K}[\mathcal{P}]$

Figure: $e(\mathbb{K}[\mathcal{P}]) = 14$
Let $\mathcal{P}$ be the following convex polyomino and $v = (5, 3)$. The link of $v$ is the cone of the vertex $(5, 2)$ with the simplicial complex which we may associate to the collection of cells displayed in the right figure.

Figure: The order of the variables is $x_{24} > x_{23} > x_{22} > x_{14} > x_{13} > x_{12} > x_{43} > x_{42} > x_{41} > x_{33} > x_{32} > x_{31} > x_{53} > x_{52}$
Thank you for your attention!