Properties of the coordinate ring of a convex polyomino

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Graduate Student Meeting on Applied Algebra and Combinatorics

March 2018

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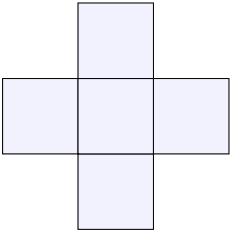
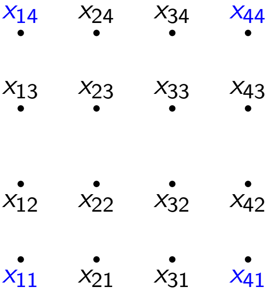
Outline

Preliminaries

A. Qureshi, *Ideals generated by 2-minors, collections of cells and stack polyominoes*, J. Algebra **357** (2012), 279–303.

The coordinate ring of a convex polyomino was introduced by Qureshi.





Preliminaries

In order to define polyominoes and polyomino ideals, we give some terminology.
On $\mathbb{N}^2$, we consider the natural partial order defined as follows:

$$(i,j) \leq (k,l) \text{ if and only if } i \leq k \text{ and } j \leq l.$$

Let $a = (i,j),\ b = (k,l) \in \mathbb{N}^2$ and $a \leq b$.
The set

$$[a,b] = \{c \in \mathbb{N}^2 \mid a \leq c \leq b\}$$

represents an interval in $\mathbb{N}^2$.

Preliminaries

The interval

$$C = [a, a+(1,1)]$$

is called a cell in $\mathbb{N}^2$ with lower left corner a.

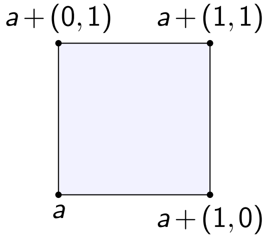


Figure: A cell in $\mathbb{N}^2$

Preliminaries

Let $\mathscr{P}$ be a finite collection of cells in $\mathbb{N}^2$.
Two cells A and B of $\mathscr{P}$ are connected by a path in $\mathscr{P}$, if there is a sequence of cells of $\mathscr{P}$ given by $A = A_1, A_2, \cdots, A_{n-1}, A_n = B$ such that $A_i \cap A_{i+1}$ is an edge of A_i and A_{i+1} for $i \in \{1, \cdots, n-1\}$.

Definition

A collection of cells $\mathscr{P}$ is called a polyomino if any two cells of $\mathscr{P}$ are connected by a path in $\mathscr{P}$.

Preliminaries

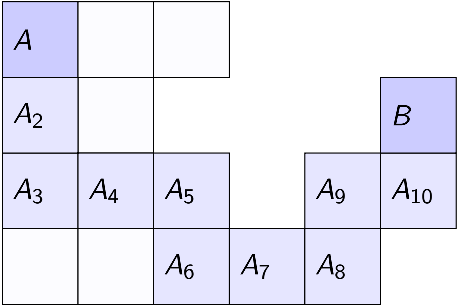
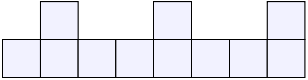
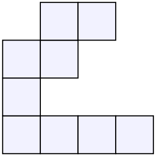


Figure: A polyomino

Preliminaries



A column convex polyomino



A row convex polyomino

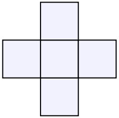
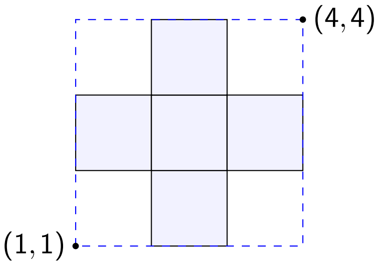


Figure: A convex polyomino

Preliminaries

Let $\mathscr{P}$ be a convex polyomino. After a possible translation, we consider $[(1,1),(m,n)]$ to be the smallest interval which contains the vertices of $\mathscr{P}$.



We say that $\mathscr{P}$ is a convex polyomino on $[m] \times [n]$, where $[m] = \{1, \ldots, m\}$ and $[n] = \{1, \ldots, n\}$.

Preliminaries

Let $\mathscr{P}$ be a convex polyomino on $[m] \times [n]$. Fix a field $\mathbb{K}$ and a polynomial ring

$$S = \mathbb{K}[x_{ij} \mid (i,j) \in V(\mathscr{P})],$$

where $V(\mathscr{P})$ is the set of the vertices of $\mathscr{P}$.
The polyomino ideal $I_{\mathscr{P}} \subset S$ is generated by all binomials

$$x_{il}x_{kj} - x_{ij}x_{kl}$$

for which $[(i,j),(k,l)]$ is an interval in $\mathscr{P}$.
The $\mathbb{K}$-algebra $S/I_{\mathscr{P}}$ is denoted $\mathbb{K}[\mathscr{P}]$ and is called the coordinate ring of $\mathscr{P}$.

Preliminaries

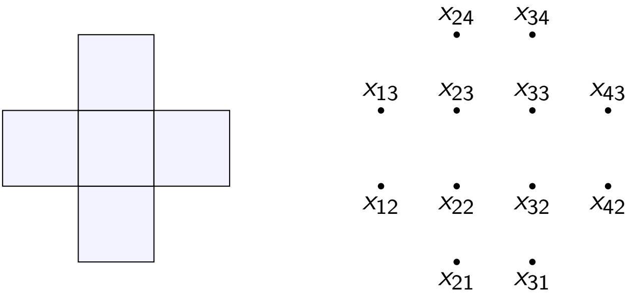


Figure: For the "cross", $I_{\mathscr{P}}$ has 11 generators.

Preliminaries

The ring

$$R = \mathbb{K}[x_i y_j \mid (i,j) \in V(\mathscr{P})] \subset \mathbb{K}[x_1, \cdots, x_m, y_1, \cdots, y_n]$$

can be viewed as an edge ring of a bipartite graph $G_{\mathscr{P}}$ with vertex set

$$V(G_{\mathscr{P}}) = X \cup Y, \text{ where } X = \{x_1, \cdots, x_m\} \text{ and } Y = \{y_1, \cdots, y_n\}$$

and edge set

$$E(G_{\mathscr{P}}) = \{\{x_i, y_j\} \mid (i,j) \in V(\mathscr{P})\}.$$

Preliminaries

A. Qureshi, *Ideals generated by 2-minors, collections of cells and stack polyominoes*, J. Algebra **357** (2012), 279–303.

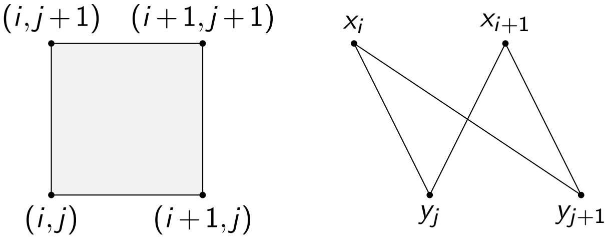


Figure: The bipartite graph attached to a cell in $\mathbb{N}^2$

$\mathbb{K}[\mathscr{P}]$ can be identified with $\mathbb{K}[G_{\mathscr{P}}]$.

Gorenstein convex polyominoes

A. Qureshi, *Ideals generated by 2-minors, collections of cells and stack polyominoes*, J. Algebra **357** (2012), 279–303.

Let $\mathscr{P}$ be a convex polyomino on $[m] \times [n]$.

Theorem (A. Qureshi, Theorem 2.2)

$\mathbb{K}[\mathscr{P}]$ *is a Cohen-Macaulay domain with* $\dim \mathbb{K}[\mathscr{P}] = m + n - 1$.

Gorenstein convex polyominoes

H. Ohsugi, T. Hibi, *Special simplices and Gorenstein toric rings*, J. Combinatorial Theory **113** (2006), 718–725.

Let $X = \{x_1, \dots, x_m\}$ and $Y = \{y_1, \dots, y_n\}$.

Theorem (H. Ohsugi, T. Hibi, Theorem 2.1)

We consider G to be a bipartite graph on $X \cup Y$ and suppose that G is 2-connected. Then $\mathbb{K}[G]$ is Gorenstein if and only if $x_1 \cdots x_m y_1 \cdots y_n \in \mathbb{K}[G]$ and one has $|N(T)| = |T| + 1$ for every subset $T \subset X$ such that $G_{T \cup N(T)}$ is connected and that $G_{(X \cup Y) \setminus (T \cup N(T))}$ is a connected graph with at least one edge.

Gorenstein convex polyominoes

Definition

Let G be a graph on V. Then we say that G is 2-connected if G together with $G_{V\setminus\{v\}}$ for all $v \in V$ are connected.

Proposition

If $\mathscr{P}$ is a convex polyomino on $[m] \times [n]$, then the bipartite graph $G_{\mathscr{P}}$ is 2-connected.

Gorenstein convex polyominoes

Let G be a graph and $T \subset V(G)$. The set

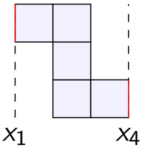
$$N(T) = \{y \in V(G) \mid \{x,y\} \in E(G) \text{ for some } x \in T\}$$

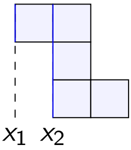
represents the set of the neighbors of the subset $T \subset V(G)$.
Let $\mathscr{P}$ be a convex polyomino on $[m] \times [n]$.
We set $X = \{x_1, \cdots, x_m\}$ and $Y = \{y_1, \cdots, y_n\}$ and, if needed, we identify the point (x_i, y_j) in the plane with the vertex $(i,j) \in V(\mathscr{P})$.

Gorenstein convex polyominoes

Definition

Let $T \subset X$. The set $N_Y(T) = \{y \in Y \mid (x,y) \in V(\mathscr{P})$ for some $x \in T\}$ is called a neighbor vertical interval if $N_Y(T) = \{y_a, y_{a+1}, \cdots, y_b\}$ with $a < b$ and for every $i \in \{a, a+1, \cdots, b-1\}$ there exists $x \in T$ such that $[(x,y_i),(x,y_{i+1})]$ is an edge in $\mathscr{P}$.

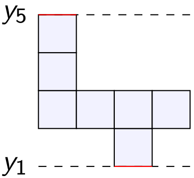


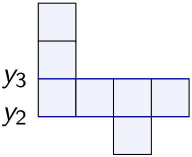


Gorenstein convex polyominoes

Definition

Let $U \subset Y$. The set $N_X(U) = \{x \in X \mid (x,y) \in V(\mathscr{P})$ for some $y \in U\}$ is called a neighbor horizontal interval if $N_X(U) = \{x_a, x_{a+1}, \cdots, x_b\}$ with $a < b$ and for every $i \in \{a, a+1, \cdots, b-1\}$ there exists $y \in U$ such that $[(x_i, y), (x_{i+1}, y)]$ is an edge in $\mathscr{P}$.





Gorenstein convex polyominoes

Proposition

Let $\mathscr{P}$ be a convex polyomino on $[m] \times [n]$ and $G_{\mathscr{P}}$ its associated bipartite graph. Then we have

$$x_1 \cdots x_m y_1 \cdots y_n \in \mathbb{K}[G_{\mathscr{P}}]$$

if and only if

$$|N_Y(T)| \geq |T| \text{ for every } T \subset X \text{ and}$$

$$|N_X(U)| \geq |U| \text{ for every } U \subset Y.$$

Gorenstein convex polyominoes

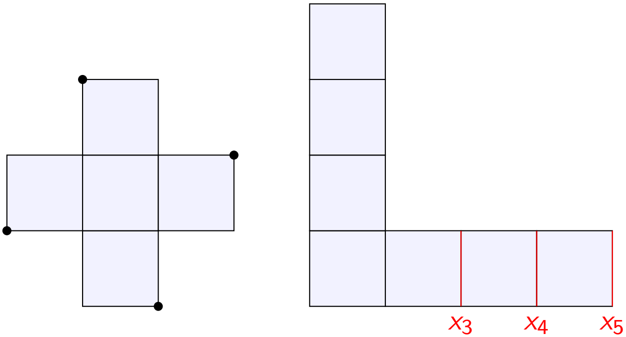


Figure: Perfect matching for $G_{\mathscr{P}}$

Gorenstein convex polyominoes

Proposition

Let $\mathscr{P}$ be a convex polyomino on $[m]\times[n]$ and $G := G_{\mathscr{P}}$ its associated bipartite graph. For each $\emptyset \neq T \subsetneq X$,

$N_Y(T)$ *is a neighbor vertical interval if and only if* $G_{T\cup N(T)}$ *is a connected graph*

and

$N_X(Y\setminus N_Y(T)) = X\setminus T$ *is a neighbor horizontal interval if and only if* $G_{(X\cup Y)\setminus(T\cup N_Y(T))}$ *is a connected graph with at least one edge.*

Gorenstein convex polyominoes

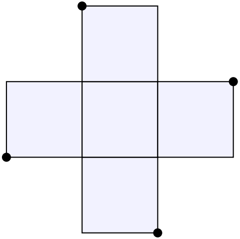
Theorem

Let $\mathscr{P}$ be a convex polyomino on $[m] \times [n]$. Then $\mathbb{K}[\mathscr{P}]$ is Gorenstein if and only if the following conditions are fulfilled:

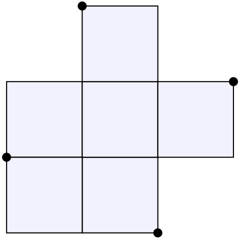
1. $|U| \leq |N_X(U)|, \forall\ U \subset Y$ *and* $|T| \leq |N_Y(T)|\ \forall\ T \subset X$;
2. *For every* $\emptyset \neq T \subsetneq X$ *with properties*
   1. $N_Y(T)$ *is a neighbor vertical interval and*
   2. $N_X(Y \setminus N_Y(T)) = X \setminus T$ *is a neighbor horizontal interval,*

   one has $|N_Y(T)| = |T| + 1$.

Gorenstein convex polyominoes



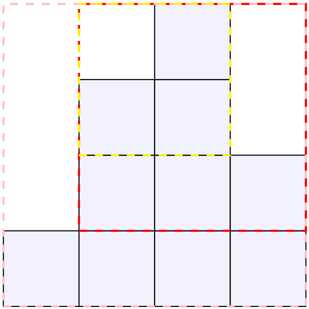
$\mathbb{K}[\mathscr{P}]$ is not Gorenstein



$\mathbb{K}[\mathscr{P}]$ is Gorenstein

Gorenstein convex polyominoes

Let $\mathscr{P}$ be a convex polyomino on $[m] \times [n]$.
Then $\mathscr{P}$ is called a stack polyomino if all cells of the first line of $[(1,1),(m,n)]$ are in $\mathscr{P}$.



Gorenstein stack polyomino

The regularity of $\mathbb{K}[\mathscr{P}]$

Let $\mathscr{P}$ be a stack polyomino on $[m] \times [n]$. We consider $H_{\mathbb{K}[\mathscr{P}]}(t)$ to be the Hilbert series of $\mathbb{K}[\mathscr{P}]$. Then

$$H_{\mathbb{K}[\mathscr{P}]}(t) = \frac{Q(t)}{(1-t)^d}$$

where $Q(t) \in \mathbb{Z}[t]$ and d is the Krull dimension of $\mathbb{K}[\mathscr{P}]$.
It is known that

$$\operatorname{reg}(\mathbb{K}[\mathscr{P}]) = \deg(Q(t)) = \dim(\mathbb{K}[\mathscr{P}]) + a(\mathbb{K}[\mathscr{P}]),$$

since $\mathbb{K}[\mathscr{P}]$ is a Cohen-Macaulay ring. The a-invariant $a(\mathbb{K}[\mathscr{P}])$ of $\mathbb{K}[\mathscr{P}]$ is defined to be the degree of the Hilbert series of $\mathbb{K}[\mathscr{P}]$, which by definition is equal to $\deg(Q(t)) - d$.

The regularity of $\mathbb{K}[\mathscr{P}]$

Let $G_{\mathscr{P}}$ be the bipartite graph attached to $\mathscr{P}$. In this section, we consider $G_{\mathscr{P}}$ as a digraph with all its arrows leaving the vertex set Y.

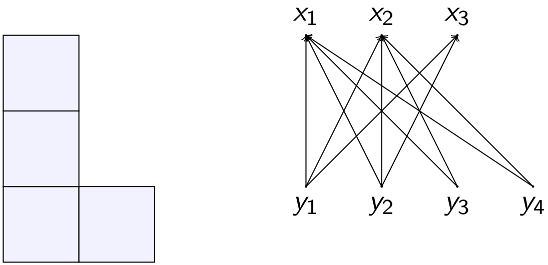


Figure: A stack polyomino and its associated digraph

The regularity of $\mathbb{K}[\mathscr{P}]$

Definition

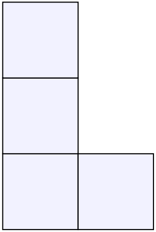
If $T \subset X \cup Y$, then
$\delta^+(T) = \{e = (z,w) \in E(G_{\mathscr{P}}) \mid z \in T \text{ and } w \notin T\}$ is the set of edges leaving the vertex set T and
$\delta^-(T) = \{e = (z,w) \in E(G_{\mathscr{P}}) \mid z \notin T \text{ and } w \in T\}$ is the set of edges entering the vertex set T.
The set $\delta^+(T)$ is called a directed cut of the digraph $G_{\mathscr{P}}$ if $\emptyset \neq T \subsetneq X \cup Y$ and $\delta^-(T) = \emptyset$.

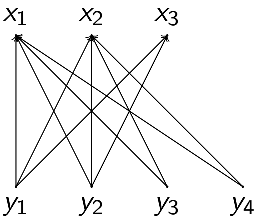
The regularity of $\mathbb{K}[\mathscr{P}]$

In the digraph of the following figure, let

$$T_1 = \{x_3, y_2, y_3\} \text{ and } T_2 = \{x_3, y_1, y_2\}.$$

Then $\delta^+(T_2)$ is a directed cut, while $\delta^+(T_1)$ is not.





The regularity of $\mathbb{K}[\mathscr{P}]$

Since $\mathbb{K}[\mathscr{P}] \cong \mathbb{K}[G_{\mathscr{P}}]$,

$$\delta^+(T) = \{(x,y) \in V(\mathscr{P}) \mid x \notin T \text{ and } y \in T\}$$

and

$$\delta^-(T) = \{(x,y) \in V(\mathscr{P}) \mid x \in T \text{ and } y \notin T\}$$

for every $T \subset X \cup Y$.

Lemma

Let $\mathscr{P}$ be a stack polyomino on $[m] \times [n]$, $G_{\mathscr{P}}$ its associated digraph and $T \subset X \cup Y$. Then $\delta^+(T)$ is a directed cut of the digraph $G_{\mathscr{P}}$ if and only if $T = T^x \cup T^y$ with $T^x \subset X$, $T^y \subset Y$ and $N_Y(T^x) \subset T^y$.

The regularity of $\mathbb{K}[\mathscr{P}]$

C.E. Valencia, R.H. Villarreal, *Canonical modules of certain edge rings*, European J. Combin. **24** (2003), 471–487.

Proposition (C.E. Valencia, R.H. Villarreal, Proposition 4.2)

Let G be a connected bipartite graph with $V(G) = X \cup Y$. If G is a digraph with all its arrows leaving the vertex set Y, then

$$-a(\mathbb{K}[G]) = \textit{the maximum number of disjoint directed cuts}.$$

The regularity of $\mathbb{K}[\mathscr{P}]$

Proposition

If $\mathscr{P}$ is a stack polyomino on $[m] \times [n]$, then

$$-a(\mathbb{K}[\mathscr{P}]) = \max\{m, n\}.$$

Corollary

If $\mathscr{P}$ is a stack polyomino on $[m] \times [n]$, then

$$\operatorname{reg}(\mathbb{K}[\mathscr{P}]) = m + n - 1 - \max\{m, n\} = \min\{m, n\} - 1.$$

The regularity of $\mathbb{K}[\mathscr{P}]$

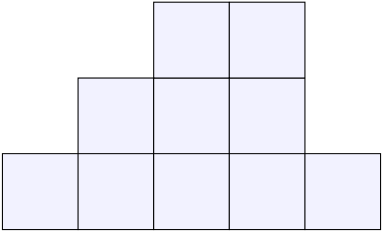


Figure: $\mathrm{reg}(\mathbb{K}[\mathscr{P}]) = 3$

The multiplicity of $\mathbb{K}[\mathscr{P}]$

Let $\mathscr{P}$ be a stack polyomino on $[m] \times [n]$.
The Hilbert series $H_{\mathbb{K}[\mathscr{P}]}(t)$ of $\mathbb{K}[\mathscr{P}]$ is given by

$$H_{\mathbb{K}[\mathscr{P}]}(t) = \frac{Q(t)}{(1-t)^d},$$

where $Q(t) \in \mathbb{Z}[t]$ and $d = \dim(\mathbb{K}[\mathscr{P}]) = m+n-1$.
The multiplicity of $\mathbb{K}[\mathscr{P}]$, denoted $e(\mathbb{K}[\mathscr{P}])$, is given by $Q(1)$.

The multiplicity of $\mathbb{K}[\mathscr{P}]$

For every $i \in [m]$,

$$\text{height}(i) = \max\{j \in [n] \mid (i,j) \in V(\mathscr{P})\}.$$

We give a total order on the variables x_{ij}, with $(i,j) \in V(\mathscr{P})$:

$$x_{ij} > x_{kl} \text{ if and only if}$$

$$(\text{height}(i) > \text{height}(k)) \text{ or } (\text{height}(i) = \text{height}(k) \text{ and } i > k)$$
$$\text{or } (i = k \text{ and } j > l).$$

Let $<$ be the reverse lexicographical order induced by this order of variables.

The multiplicity of $\mathbb{K}[\mathscr{P}]$

It is known that the polyomino ideal $I_{\mathscr{P}}$ has a reduced Gröbner basis with respect to $<$ consisting of all inner 2-minors of $\mathscr{P}$. We may view $\text{in}_<(I_{\mathscr{P}})$ as the Stanley-Reisner ideal of a simplicial complex, denoted $\Delta_{\mathscr{P}}$, on the vertex set $V(\mathscr{P})$.

It is known that $\Delta_{\mathscr{P}}$ is a pure shellable simplicial complex.

The multiplicity of $\mathbb{K}[\mathscr{P}]$

Let $\mathscr{P}$ be the polyomino of the figure. Then

$\mathrm{in}_<(I_{\mathscr{P}}) = (x_{11}x_{32}, x_{21}x_{32}, x_{21}x_{12}, x_{21}x_{13}, x_{22}x_{13})$ and

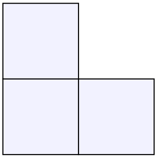
$\Delta_{\mathscr{P}} = \langle F_1 = \{(1,1),(2,1),(2,2),(2,3),(3,1)\};$

$F_2 = \{(1,1),(1,2),(2,2),(2,3),(3,1)\};$

$F_3 = \{(1,1),(1,2),(1,3),(2,3),(3,1)\};$

$F_4 = \{(1,2),(2,2),(2,3),(3,1),(3,2)\};$

$F_5 = \{(1,2),(1,3),(2,3),(3,1),(3,2)\}\rangle.$



The order of the variables:

$x_{23} > x_{22} > x_{21} > x_{13} > x_{12} > x_{11} > x_{32} > x_{31}$

The multiplicity of $\mathbb{K}[\mathscr{P}]$

Since

$$H_{\mathbb{K}[\mathscr{P}]}(t) = H_{S/\mathrm{in}_<(I_{\mathscr{P}})}(t) = H_{\mathbb{K}[\Delta_{\mathscr{P}}]},$$

we obtain

$$e(\mathbb{K}[\mathscr{P}]) = |\mathfrak{F}(\Delta_{\mathscr{P}})|,$$

where $\mathfrak{F}(\Delta_{\mathscr{P}})$ denotes the set of the facets of $\Delta_{\mathscr{P}}$.

The multiplicity of $\mathbb{K}[\mathscr{P}]$

Definition

Let Δ be a simplicial complex on the vertex set V and $v \in V$. The link of v in Δ is the simplicial complex

$$\mathrm{lk}(v) = \{F \in \Delta \mid v \notin F \text{ and } F \cup \{v\} \in \Delta\}$$

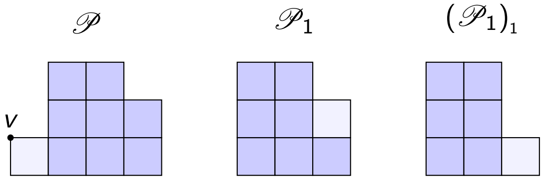
and the deletion of v is the simplicial complex

$$\mathrm{del}(v) = \{F \in \Delta \mid v \notin F\}.$$

The multiplicity of $\mathbb{K}[\mathscr{P}]$

Let x_{ij} be the smallest variable in S and fix $v = (i, \text{height}(i)) \in V(\mathscr{P})$.
If $i = 1$, then we denote by $\mathscr{P}_1$ the polyomino obtained from $\mathscr{P}$ by deleting the cell which contains the vertex v.
Otherwise, $\mathscr{P}_1$ is given by deleting the cell which contains the vertex $(m, \text{height}(m))$.

The multiplicity of $\mathbb{K}[\mathscr{P}]$

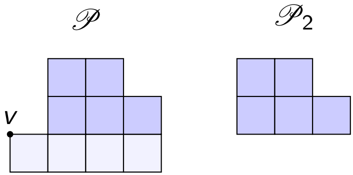


Lemma

We have $|\mathfrak{F}(\Delta_{\mathscr{P}_1})| = |\mathfrak{F}(\mathrm{del}(v))|$.

The multiplicity of $\mathbb{K}[\mathscr{P}]$

Let $\mathscr{P}_2$ be the polyomino obtained from $\mathscr{P}$ by deleting all the cells of $\mathscr{P}$ which lie below the horizontal edge interval containing the vertex v.



Lemma

We have $|\mathfrak{F}(\Delta_{\mathscr{P}_2})| = |\mathfrak{F}(\mathrm{lk}(v))|$.

The multiplicity of $\mathbb{K}[\mathscr{P}]$

Theorem

Let $\mathscr{P}$ be a stack polyomino on $[m] \times [n]$ and $v = (i,j) \in V(\mathscr{P})$ with the properties:

1. x_{i1} *is the smallest variable in* S *and*
2. $j = \text{height}(i)$.

Let $\mathscr{P}_1$ and $\mathscr{P}_2$ be the polyominoes presented above. Then

$$e(\mathbb{K}[\mathscr{P}]) = e(\mathbb{K}[\mathscr{P}_1]) + e(\mathbb{K}[\mathscr{P}_2]).$$

The multiplicity of $\mathbb{K}[\mathscr{P}]$

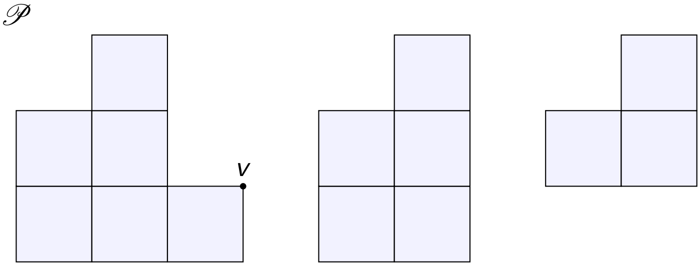


Figure: $e(\mathbb{K}[\mathscr{P}]) = 14$

The multiplicity of $\mathbb{K}[\mathscr{P}]$

Let $\mathscr{P}$ be the following convex polyomino and $v = (5,3)$. The link of v is the cone of the vertex $(5,2)$ with the simplicial complex which we may associate to the collection of cells displayed in the right figure.

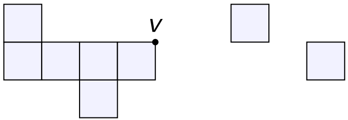


Figure: The order of the variables is $x_{24} > x_{23} > x_{22} > x_{14} > x_{13} > x_{12} > x_{43} > x_{42} > x_{41} > x_{33} > x_{32} > x_{31} > x_{53} > x_{52}$


Thank you for your attention!