Graduate students meeting on applied algebra and Combinatorics
University of Osnabrük

Strong persistence property of monomial ideals.

Jonathan Toledo
jtt@math.cinvestav.mx
Department of Mathematics of CINVESTAV-IPN

March 23, 2018.

Strong persistence property

Graphs with loops

Simple hypergraphs

Weighted ideals

Definition

An ideal I has the persistence property if $\mathrm{Ass}(I^k) \subseteq \mathrm{Ass}(I^{k+1})$ *for each k.*

Definition

An ideal I has the persistence property if $\operatorname{Ass}(I^k) \subseteq \operatorname{Ass}(I^{k+1})$ *for each k.*

Definition

An ideal I has the strong persistence property if $(I^{k+1} : I) = I^k$ *for each k.*

Definition

An ideal I has the persistence property if $\mathrm{Ass}(I^k) \subseteq \mathrm{Ass}(I^{k+1})$ *for each k.*

Definition

An ideal I has the strong persistence property if $(I^{k+1} : I) = I^k$ *for each k.*

Proposition (J. Herzog and A. A. Qureshi)

Strong persistence property implies persistence property.

Definition

Definition Let I be a monomial ideal of R with the unique minimal set of monomial generators $G(I) = \{u_1, ..., u_m\}$. Then, we say that Iis a unisplit monomial ideal, if there exists $i \in \mathbb{N}$ with $1 \leq i \leq m$, such that each monomial u_j has no common factor with u_i for all $j \in N$ with $1 \leq j \leq m$ and $j \neq i$.

Definition

Definition Let I be a monomial ideal of R with the unique minimal set of monomial generators $G(I) = \{u_1, ..., u_m\}$. Then, we say that $I$is a unisplit monomial ideal, if there exists $i \in \mathbb{N}$ with $1 \leq i \leq m$, such that each monomial u_j has no common factor with u_i for all $j \in N$ with $1 \leq j \leq m$ and $j \neq i$.

Proposition (M. Naserneja)

Every unisplit(separablse) monomial ideal of R satisfies the strong persistence property

Proposition

An ideal I has the strong persistence property if and only if $(I^r : I^s) = I^{r-s}$ for all $s \leq r$.

Proposition

An ideal I has the strong persistence property if and only if $(I^r : I^s) = I^{r-s}$ for all $s \leq r$.

Proposition

Let I ideal in an Noetherian domain, then there exists an integer s_0 such that $(I^r : I^s) = I^{r-s}$ for all $r - s \geq s_0$.

Proposition

An ideal I has the strong persistence property if and only if $(I^r : I^s) = I^{r-s}$ for all $s \leq r$.

Proposition

Let I ideal in an Noetherian domain, then there exists an integer s_0 such that $(I^r : I^s) = I^{r-s}$ for all $r - s \geq s_0$.

Proposition

If an ideal I has the strong persistence, then I^t also has it for each $t \geq 1$.

Example

An ideal I has the strong persistence property if
(1) I is a cover ideal of a perfect graph.
(2) I is a polymatroidal ideal[J. Herzog and A. A. Qureshi].

Example

An ideal I has the strong persistence property if
(1) I is a cover ideal of a perfect graph.
(2) I is a polymatroidal ideal[J. Herzog and A. A. Qureshi].

Proposition (S. Morey, J. Martínez-Bernal, and R. H. Villarreal)

Let I be the edge ideal of a simple graph G, then $(I^{k+1} : I) = I^k$ *for each* K.

Example

An ideal I has the strong persistence property if
(1) I is a cover ideal of a perfect graph.
(2) I is a polymatroidal ideal[J. Herzog and A. A. Qureshi].

Proposition (S. Morey, J. Martínez-Bernal, and R. H. Villarreal)

Let I be the edge ideal of a simple graph G, then $(I^{k+1} : I) = I^k$ *for each* K.

The **edge ideal** of a simple graph $G = (V, E)$ with $V = \{x_1, \ldots, x_n\}$ is the ideal $I(G)$ of $R = K[x_1, \ldots, x_n]$ generated by

$$\{x_i x_j \mid \{x_i, x_j\} \in E\}.$$

Definition

*A **graph with loops** is a triplet $\mathcal{G} = (V, E, L)$ where $G = (V, E)$ is a simple graph with $V = \{x_1, \ldots, x_n\}$ and $L \subseteq \{(x_i, x_i) \mid x_i \in V\}$, L is called the set of loops of $\mathcal{G}$.*

Definition

*A **graph with loops** is a triplet $\mathcal{G} = (V, E, L)$ where $G = (V, E)$ is a simple graph with $V = \{x_1, \ldots, x_n\}$ and $L \subseteq \{(x_i, x_i) \mid x_i \in V\}$, L is called the set of loops of $\mathcal{G}$.*

Definition

*The **edge ideal** of a graph with loops $\mathcal{G} = (V, E, L)$, denoted by $I(\mathcal{G})$, is the ideal $I(G) + (\{x_i^2 \mid (x_i, x_i) \in L\})$ where $I(G) = (\{x_i x_j \mid x_i x_j \in E\})$ is the edge ideal of $G = (V, E)$.*

Theorem

If $\mathcal{G}$ is a graph with loops, then $\left(I(\mathcal{G})^{k+1} \colon I(\mathcal{G})\right) = I(\mathcal{G})^k$ for each k.

Theorem

If $\mathcal{G}$ is a graph with loops, then $\left(I(\mathcal{G})^{k+1} : I(\mathcal{G})\right) = I(\mathcal{G})^k$ for each k.

Corollary

The edge ideal of a graph with loops has the persistence property.

A **simple hypergraph** or a **clutter** $\mathcal{C} = (V, E)$ consists of a finite set $V = \{x_1, \ldots, x_n\}$, called vertex set, and a edge set E consisting of subsets of V such that they do not have inclusion relations.

A **simple hypergraph** or a **clutter** $\mathcal{C} = (V, E)$ consists of a finite set $V = \{x_1, \dots, x_n\}$, called vertex set, and a edge set E consisting of subsets of V such that they do not have inclusion relations.
The **edge ideal** of a simple hypergraph $\mathcal{C} = (V, E)$ is the ideal $I(\mathcal{C})$ of $R = K[x_1, \dots, x_n]$ generated by

$$\left\{ \prod_{x_i \in e} x_i \mid e \in E \right\}.$$

A **simple hypergraph** or a **clutter** $\mathcal{C} = (V, E)$ consists of a finite set $V = \{x_1, \ldots, x_n\}$, called vertex set, and a edge set E consisting of subsets of V such that they do not have inclusion relations.
The **edge ideal** of a simple hypergraph $\mathcal{C} = (V, E)$ is the ideal $I(\mathcal{C})$ of $R = K[x_1, \ldots, x_n]$ generated by

$$\left\{ \prod_{x_i \in e} x_i \mid e \in E \right\}.$$

Edge ideal defines a bijection between the set of clutters on $V = \{x_1, \ldots, x_n\}$ and the set of squarefree monomial ideals of $R = K[x_1, \ldots, x_n]$.

Proposition

Let I be a squarefree monomial ideal, then $(I^2 : I) = I$.

Proposition

Let I be a squarefree monomial ideal, then $(I^2 : I) = I$.

Example

Let $\mathcal{C}_0$ be the clutter with vertex set $\{x_1, \ldots, x_6\}$ and whose edges are $x_1x_2x_3, x_1x_2x_4, x_1x_3x_5, x_1x_4x_6, x_1x_5x_6, x_2x_3x_6, x_2x_4x_5, x_2x_5x_6, x_3x_4x_5,$ $x_3x_4x_6$ $\mathcal{C}_0$ is an unmixed shellable. But $\left(I(\mathcal{C}_0)^3 \colon I(\mathcal{C}_0)\right) \neq I(\mathcal{C}_0)^2$, consequently $\mathcal{C}_0$ does not have the strong persistence property.

Theorem

A clutter has the strong persistence property if and only if some of its connected components has the strong persistence property.

Theorem

A clutter has the strong persistence property if and only if some of its connected components has the strong persistence property.

Lemma

Let $\mathcal{C}$ be a clutter such that there exists an edge $f \in E(\mathcal{C})$ such that the set $\{g \cap f \mid g \in E(C)\}$ is a chain, then $I(\mathcal{C})$ has the strong persistence property.

Theorem

A clutter has the strong persistence property if and only if some of its connected components has the strong persistence property.

Lemma

Let $\mathcal{C}$ be a clutter such that there exists an edge $f \in E(\mathcal{C})$ such that the set $\{g \cap f \mid g \in E(C)\}$ is a chain, then $I(\mathcal{C})$ has the strong persistence property.

Theorem

Let $\mathcal{C}$ be a König unmixed clutter. If $\mathcal{C}$ does not contain 4-cycle, then $\mathcal{C}$ has the strong persistence property.

Definition

Let $\mathcal{C}$ be a clutter and $x \notin V(\mathcal{C})$, the ***cone*** *over $\mathcal{C}$ is the clutter that has the vertex set $V(\mathcal{C}) \cup \{x\}$ and edge set $\left\{ f \cup \{x\} \mid f \in E(\mathcal{C}) \right\}$ and it is denoted by $\mathcal{C}x$.*

Definition

*Let $\mathcal{C}$ be a clutter and $x \notin V(\mathcal{C})$, the **cone** over $\mathcal{C}$ is the clutter that has the vertex set $V(\mathcal{C}) \cup \{x\}$ and edge set $\{f \cup \{x\} \mid f \in E(\mathcal{C})\}$ and it is denoted by $\mathcal{C}x$.*

Proposition

Let $\mathcal{C}$ be a clutter and $\mathcal{C}x$ its cone over $\mathcal{C}$, then we have that:

i) *$\mathcal{C}$ has the strong persistence property if and only if $\mathcal{C}x$ has the strong persistence property*

ii) *$\mathcal{C}$ has the persistence property if and only if $\mathcal{C}x$ has the persistence property.*

Definition

*Let $\mathcal{C} = (V, E)$ be a clutter and $x \in V$, the **deleting** of x is the clutter $\mathcal{C} \setminus x$ whose vertex and edge sets are $V \setminus \{x\}$ and $\{f \in E \mid x \notin f\}$, respectively.*

*Similarly, the **contraction** of x is the clutter $\mathcal{C} / x$ whose set of vertices and edges are $V \setminus \{x\}$ and $\min\{f \setminus \{x\} \mid f \in E\}$, respectively.*

Definition

*Let $\mathcal{C} = (V, E)$ be a clutter and $x \in V$, the **deleting** of x is the clutter $\mathcal{C} \setminus x$ whose vertex and edge sets are $V \setminus \{x\}$ and $\{f \in E \mid x \notin f\}$, respectively.*

*Similarly, the **contraction** of x is the clutter $\mathcal{C} / x$ whose set of vertices and edges are $V \setminus \{x\}$ and $\min\{f \setminus \{x\} \mid f \in E\}$, respectively.*

Proposition

Let $\mathcal{C}$ be a clutter and $x \in V(\mathcal{C})$, then

i) *$\mathcal{C} / x$ has the persistence property if $\mathcal{C}$ has the persistence property*
ii) *$\mathcal{C} / x$ has the strong persistence property if $\mathcal{C}$ has the strong persistence property*

Proposition

A clutter $\mathcal{C}$ with 3 edges has the strong persistence property.

Proposition

A clutter $\mathcal{C}$ with 3 edges has the strong persistence property.

Proposition

Let X be a set of vertices, $A \subseteq X$ and $x \notin X$, then the clutter C with edge set $\{X\} \cup \{xx_i \mid x_i \in A\}$ has the strong persistence property.

Proposition

If I is a squarefree monomial ideal in $K[x_1, x_2, x_3, x_4]$, then I has the strong persistence property.

Proposition

If I is a squarefree monomial ideal in $K[x_1, x_2, x_3, x_4]$*, then I has the strong persistence property.*

Conjecture

If I is a squarefree monomial ideal in $K[x_1, x_2, x_3, x_4, x_5]$*, then I has the strong persistence property.*

Definition

*A weight over a polynomial ring $R = K[x_1, \ldots, x_n]$ is a function $w : \{x_1, \ldots, x_n\} \to \mathbb{N}$ and $w_i = w(x_i)$ is called the **weight** of the variable x_i.*

Definition

*A weight over a polynomial ring $R = K[x_1, \ldots, x_n]$ is a function $w : \{x_1, \ldots, x_n\} \to \mathbb{N}$ and $w_i = w(x_i)$ is called the **weight** of the variable x_i.*

Definition

*Given a monomial ideal I and a weight w, the **weighted ideal** is the ideal $I_w = \big(h(m) \mid m \in G(I)\big)$ where h is the unique homomorphism $h : R \to R$ given by $x_i \mapsto x_i^{w_i}$.*

Theorem

Let I be a monomial ideal and w a weight, then

i) $\mathrm{Ass}(I_w^k) = \mathrm{Ass}(I^k)$ *for each k.*

ii) *I has the persistence property if and only if I_w has the persistence property.*

iii) *I has the strong persistence property if and only if I_w has the strong persistence property.*

References

i) E.Reyes and J.Toledo. On the strong persistence property for monomial ideals, Bulletin Mathematique de la Societe des Sciences Mathematiques de Roumanie, Volume 60 (108)/2017, Issue no. 3, pages 293-305.

Thanks