The classification of empty lattice 4-simplices

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Empty lattice d-simplices

- A d-polytope is the convex hull of a finite set of points in some $\mathbb{R}^d$. Its *dimension* is the dimension of its affine span. (E.g., 2-polytopes = Convex polygons, etc.)
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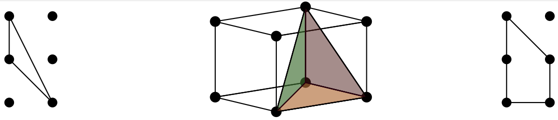
Definition

A *lattice polytope* $P \subset \mathbb{R}^d$ is a polytope with integer vertices. It is:

- *hollow* if it has no integer points in its interior.
- *empty* if it has no integer points other than its vertices.

In particular, an *empty d-simplex* is the convex hull of $d+1$ affinely independent integer points and not containing other integer points.



Empty 2 and 3-simplices and hollow 2-polytope.

Volume, width

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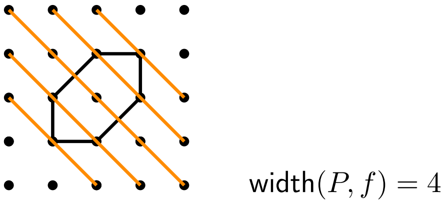
- The *width* of $P \subset \mathbb{R}^d$ with respect to a linear functional $f : \mathbb{R}^d \to \mathbb{R}$ equals the difference $\max_{x \in P} f(x) - \min_{x \in P} f(x)$.



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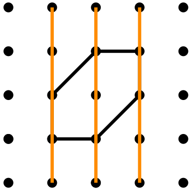
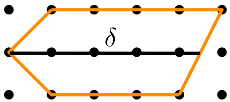
  We call *(lattice) width* of P the minimum width of P with respect to integer functionals.



width$(P) = 2$

Diameter

We call *rational (lattice) diameter* of P to the maximum length of a rational segment contained in P (with "length" measured with respect to the lattice).



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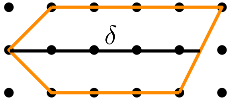

$\text{diam}(P) = 4.5$

- It equals the inverse of the *first successive minimum* of $P - P$. In particular, Minkowski's First Theorem implies:

$$\text{Vol}(P) \le d!\,\text{diam}(P)^d.$$

- Not to be mistaken with the (integer) lattice diameter = max. lattice length of an *integer* segment in P.

What do we know about empty lattice d-simplices?

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- The only empty 1-simplex is the unit segment.
- The only empty 2-simplex is the unimodular triangle ($\simeq$ Pick's Theorem).
- Empty lattice 3-simplices are completely classified:

Theorem (White 1964)

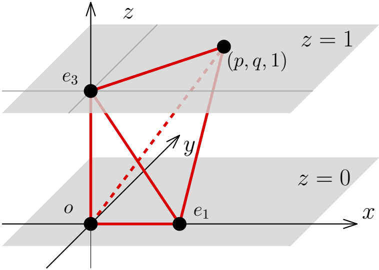
Every empty tetrahedron of determinant q is equivalent to

$$T(p,q) := \operatorname{conv}\{(0,0,0),(1,0,0),(0,0,1),(p,q,1)\}$$

for some $p \in \mathbb{Z}$ with $\gcd(p,q) = 1$. Moreover, $T(p,q) \cong_{\mathbb{Z}} T(p',q)$ if and only if $p' = \pm p^{\pm 1} \pmod{q}$.

What do we know about empty lattice 3-simplices

In particular, they all have width 1, i.e., they are between two parallel lattice hyperplanes.



In this picture, they have width 1 with respect to the functional $f(x, y, z) = z$.

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3. The amount of empty 4-simplices of width greater than 2 is **finite**:

Proposition (Blanco-Haase-Hofmann-Santos, 2016)

1. For each d, there is a $w^\infty(d)$ such that for every $n \in \mathbb{N}$ all but finitely many d-polytopes with n lattice points have width $\leq w^\infty(d)$.
2. $w^\infty(4) = 2$.

What do we know about empty lattice 4-simplices?

Theorem (Haase-Ziegler, 2000)

Among the 4-dimensional empty simplices with width greater than two and determinant $D \leq 1000$,

1. *All simplices of width 3 have determinant $D \leq 179$, with a (unique) smallest example, of determinant $D = 41$, and a (unique) example of determinant $D = 179$.*
2. *There is a unique class of width 4, with determinant $D = 101$,*
3. *There are no simplices of width $w \geq 5$,*

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Conjecture (Haase-Ziegler, 2000)

The above list is complete. That is, there are no empty 4-simplices of width > 2 and determinant > 179.

Theorem (I.V.-Santos, 2018)

This conjecture is true.

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There is no hollow 4-simplex of width > 2 with determinant greater than 5058.

Part I and Part II of the talk

- **Part I: Empty 4-simplices of width greater than two**
  The proof of the conjecture follows from the combination of a theoretical Theorem 1 and the Theorem 2 based on an enumeration:

Theorem 1 (I.V.-Santos, 2018)

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Theorem 2 (I.V.-Santos, 2018)

Up to determinant ≤ 7600, all empty 4-simplices of width larger than two have determinant in $[41, 179]$ and are as described explicitly by Haase and Ziegler.

- **Part II: The complete classification of empty 4-simplices** We show new results regarding the classification of the infinite families of width two

Let P be a empty lattice 4-simplex of width greater than two. We separate in two cases:

Case 1 *There is an integer projection $\pi : P \to Q$ to a hollow 3-polytope Q.* Then, Q will also have width greater than two, and there are only the following five hollow 3-polytopes of width greater than two (Averkov, Krümpelmann and Weltge, 2015).

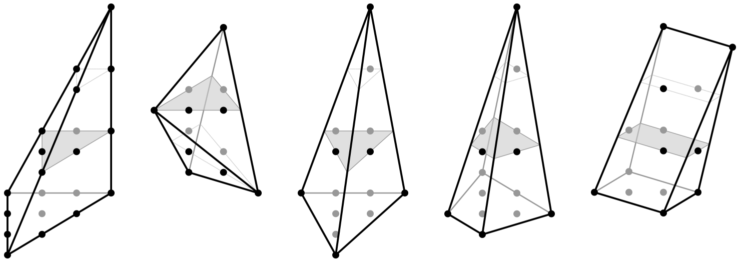

Figure : The five hollow lattice 3-polytopes of width greater than two. Their normalized volumes are 27, 25, 27, 27 and 27. respectively.

Theorem 1: case "P can be projected to a hollow polytope"

We can show that in this case:

Proposition

If a hollow 4-simplex P of width at least three can be projected to a hollow lattice 3-polytope Q, then

$$\mathrm{Vol}(P) \leq \mathrm{Vol}(Q) \leq 27.$$

Sketch of proof: The volume of P equals the volume of Q times the length of the maximum fiber in P. This fiber is projecting to a lattice point and P is hollow, which implies the fiber to have length at most one. □

Thm 1: case "P cannot be projected to a hollow polytope"

Case 2 *There is no integer projection of P to a hollow 3-polytope*
We use the following lemma:

Lemma

Let $\pi : P \to Q$ be an integer projection of a **hollow** *d-simplex P onto a* **non-hollow** *lattice $(d-1)$-polytope Q. Let:*

- *δ be the maximum length of a fiber (π^{-1} of a point) in P.*
- *$0 \leq r < 1$ be the maximum dilation factor such that Q contains a homothetic hollow copy Q_r of itself.*

Then:

1. $\mathrm{Vol}(P) \leq \delta\, \mathrm{Vol}(Q)$.
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Then:

1. $\mathrm{Vol}(P) \leq \delta \, \mathrm{Vol}(Q)$.
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- r measures whether Q is "close to hollow" ($r \simeq 1$) or "far from hollow" ($r \simeq 0$)
- In what follows we project P along the direction with δ=diam(P). Part (2) says "if Q is far from hollow then diam(P) is small"

The Lemma

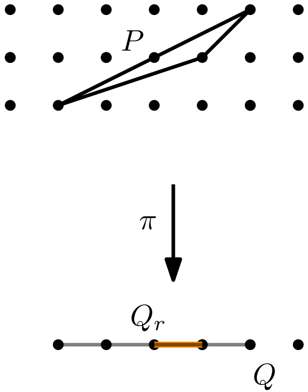

Figure : Projection of an empty (d)-simplex into an $(d-1)$-polytope

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So, let $\pi : P \to Q$ be the projection along the direction giving the diameter of P, so that the δ in the theorem equals the lattice diameter of P. We have a dichotomy:

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- If Q is "far from hollow" then we use Minkowski's First Thm. $\mathrm{Vol}(P-P) \le d!2^d\delta^d$. Together with $\mathrm{Vol}(P-P) = \binom{2d}{d}\ \mathrm{Vol}(P)$ (Rogers-Shephard for a simplex):

$$\mathrm{Vol}(P) = \frac{\mathrm{Vol}(P-P)}{\binom{8}{4}} \le \frac{24 \cdot 16}{\binom{8}{4}}\delta^4 = 5.48\delta^4.$$

  E.g., with $r \le 0.81$, $\delta \le 1/0.19$.

$$\mathrm{Vol}(P) \le 5.48 \cdot \delta^4 < 4210.$$

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So, let $\pi : P \to Q$ be the projection along the direction giving the diameter of P, so that the δ in the theorem equals the lattice diameter of P. We have a dichotomy:

- If Q is "close to hollow" then we use the Lemma:

$$\mathrm{Vol}(P) = \delta\,\mathrm{Vol}(Q) = \frac{\delta}{r^3}\,\mathrm{Vol}(Q_r), \quad \text{where :}$$

1. r is bounded away from 0: by the previous case we can assume $r \geq 0.81$.
2. Q_r is hollow of width at least $3r \geq 2.5$, which implies $\mathrm{Vol}(Q_r) \leq 32\frac{5^3}{3^3} = 148.148$ (see next slide).
3. $\delta \leq 42$ (we skip this).

...so we get an upper bound on $\mathrm{Vol}(P)$.

A bound on the volume of Q_r

Proposition (I.V.-Santos, 2018, generalizing Averkov. Krümpelmann and Weltge. 2015)

Let $w \geq 2.5$. Then, the following holds for any lattice-free convex body K in dimension three of width at least w:

(a)
$$\mathrm{Vol}(K) \leq \frac{48w^3}{(w-1)^3} \leq 222.22\ldots.$$

(b) If K is a lattice 3-polytope with at most five points:
$$\mathrm{Vol}(K) \leq \frac{32w^3}{(w-1)^3} \leq 148.148\ldots.$$

An upper bound for the volume of empty 4-simplices

Putting the bounds for $r \geq 0.81$, $\mathrm{Vol}(Q_r) \leq 148.148\ldots$ and $\delta \leq 42$ together we get:

$$\mathrm{Vol}(P) \leq \frac{\delta}{r^3}\,\mathrm{Vol}(Q_r) \leq 42\frac{1}{0.81^3}32\frac{5^3}{6^3} \leq 10751.$$

But these three bounds are not independent since:

- $1 - r \leq \delta^{-1}$ (e.g., if $r \simeq 5/6$ then $\delta \lesssim 6$).
- $\mathrm{Vol}(Q_r) \leq 32\left(\frac{3r}{3r-1}\right)^3$ (e.g., if $r \simeq 1$ then $\mathrm{Vol}(Q_r) \simeq 108$).

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Optimizing the three parameters together we get ($r \leq 0.81$, $\delta \leq 1/0.19$).

$$\mathrm{Vol}(P) \leq 5058.$$

An upper bound for the volume of empty 4-simplices

Summing up:

- If P projects to a hollow 3-polytope then

$$\mathrm{Vol}(P) \leq 27$$

- If P does not project to a hollow 3-polytope we have the following cases:

  1. Q is "far from hollow" $(r \leq 0.81)$ then

  $$\mathrm{Vol}(P) \leq 4210$$

  2. If Q is "close to hollow" $(r \geq 0.81)$ then

  $$\mathrm{Vol}(P) \leq 5058$$

Theorem 2: Two different enumeration algorithms

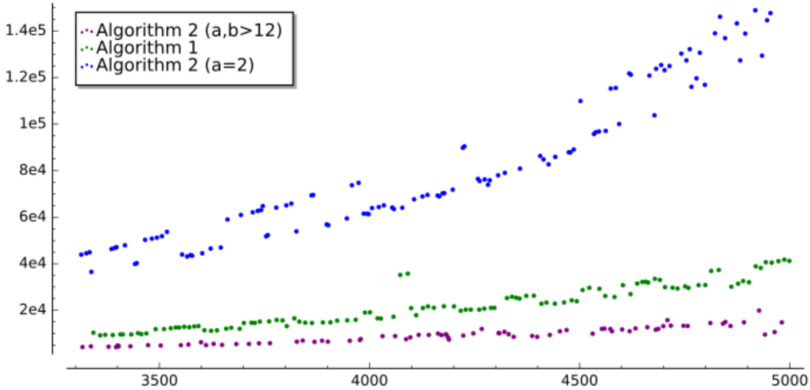
To enumerate all empty 4-simplices of a given determinant D we use one of two algorithms:

Algorithm 1: **If D has less than 5 prime factors.** It is a complete enumeration of all posibilitys after fixing one of the facets of the simplex.

Algorithm 2: **If D has at least 2 prime factors**. Create the simplices by decomposing the volume $D = ab$ with a and b relatively prime and combining the simplices with volumes a and b.

For some values of D both algorithms can be used, or different factorizations of D can be chosen in Algorithm 2. Experimentally, we observe that Algorithm 2 is much slower than Algorithm 1 if $a \ll b$, and slightly faster than Algorithm 1 if $a \simeq b$:

Computational data



Computation time (sec.) for the list of all empty lattice 4-simplices of a given determinant

Part II: Empty 4-simplices of width one and two

We have identified all empty lattice 4-simplices of with greater than two. How to classify the rest of empty lattice 4-simplices:

- Those of width 1 can be classified as they form a 3-parameter family, similiar to the White Theorem in dimension 3.

$$\operatorname{conv}\{(0,0,0,0),(1,0,0,0),(0,1,0,0),(0,0,0,1),(a,b,V,1)\},$$

  with $\gcd(a,b,V)=1$.

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Theorem (Not true (Barile et al. 2011))

All except for finitely many empty 4-simplices belong to the classes (of cyclic quotient singularities) classified by Mori-Morrison-Morrison (1988), and hence have width at most two.

We still have some information of those of width 2.

Classification of empty 4-simplices

At the end, we have found some new families that can complete the classification of empty 4-simplices of width 2, and so, the classification of empty 4-simplices.

Theorem (I.V.-Santos, '18+)

All except for finitely many empty 4-simplices belong to one of the following cases:

- *The three-parameter family of empty 4-simplices of width one.*
- *Two 2-parameter families of empty 4-simplices projecting to the second dilation of a unimodular triangle (one listed by Mori et al., the other not).*
- *The 29 Mori 1-parameter families (they project to 29 hollow "primitive" 3-polytopes).*
- *23 additional 1-parameter families that project to 23 "non-primitive" hollow 3-polytopes.*

Finitely many empty 4-simplices

At the end, we have found some new families that can complete the classification of empty 4-simplices of width 2, and so, the classification of empty 4-simplices.

Theorem (I.V.-Santos, '18+)

There are exactly 2461 (classes of) empty 4-simplices that do not belong to any of the infinite families shown in the theorem before. These empty 4-simplices correspond to those that do not project to a hollow $d-1$-polytope. Their determinants range from 24 to 419.

Remark

The empty 4-simplices of width greater than 3 explicity discribed in Part I of this talk are 180 cases of these 2461.

Thanks for your attention

The article for part I you can check it in arXiv:1704.07299 and also accepted for publication in TAMS

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