Enumerative Combinatorics from an Algebraic-Geometric point of view

Christos Athanasiadis

University of Athens

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Outline

1 Three examples from combinatorics

2 An algebraic-geometric perspective

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Eulerian polynomials

We let

- \( \mathcal{S}_n \) be the group of permutations of \([n] := \{1, 2, \ldots, n\}\)

and for \( w \in \mathcal{S}_n \)

- \( \text{des}(w) := \# \{ i \in [n-1] : w(i) > w(i + 1) \} \)
- \( \text{exc}(w) := \# \{ i \in [n-1] : w(i) > i \} \)

be the number of descents and excedances of \( w \), respectively. The polynomial

\[
A_n(x) := \sum_{w \in \mathcal{S}_n} x^{\text{des}(w)} = \sum_{w \in \mathcal{S}_n} x^{\text{exc}(w)}
\]

is the \( n \)th Eulerian polynomial.
Example

\[ A_n(x) = \begin{cases} 
1, & \text{if } n = 1 \\
1 + x, & \text{if } n = 2 \\
1 + 4x + x^2, & \text{if } n = 3 \\
1 + 11x + 11x^2 + x^3, & \text{if } n = 4 \\
1 + 26x + 66x^2 + 26x^3 + x^4, & \text{if } n = 5 \\
1 + 57x + 302x^2 + 302x^3 + 57x^4 + x^5, & \text{if } n = 6.
\end{cases} \]
Derangement polynomials

We let

- $\mathcal{D}_n$ be the set of derangements (permutations without fixed points) in the symmetric group $\mathfrak{S}_n$.

For instance,

- $\mathcal{D}_3 = \{(2, 3, 1), (3, 1, 2)\}$.

The polynomial

$$d_n(x) := \sum_{w \in \mathcal{D}_n} x^{\text{exc}(w)}$$

is the $n$th derangement polynomial.
Example

\[d_n(x) = \begin{cases} 
0, & \text{if } n = 1 \\
x, & \text{if } n = 2 \\
x + x^2, & \text{if } n = 3 \\
x + 7x^2 + x^3, & \text{if } n = 4 \\
x + 21x^2 + 21x^3 + x^4, & \text{if } n = 5 \\
x + 51x^2 + 161x^3 + 51x^4 + x^5, & \text{if } n = 6 \\
x + 113x^2 + 813x^3 + 813x^4 + 113x^5 + x^6, & \text{if } n = 7.
\]
Binomial Eulerian polynomials

The polynomial

$$\tilde{A}_n(x) := 1 + x \sum_{k=1}^{n} \binom{n}{k} A_k(x) = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} A_k(x)$$

is the \(n\)th binomial Eulerian polynomial.

**Example**

$$\tilde{A}_n(x) = \begin{cases} 
1 + x, & \text{if } n = 1 \\
1 + 3x + x^2, & \text{if } n = 2 \\
1 + 7x + 7x^2 + x^3, & \text{if } n = 3 \\
1 + 15x + 33x^2 + 15x^3 + x^4, & \text{if } n = 4 \\
1 + 31x + 131x^2 + 131x^3 + 31x^4 + x^5, & \text{if } n = 5 \\
1 + 63x + 473x^2 + 883x^3 + 473x^4 + 63x^5 + x^6, & \text{if } n = 6.
\end{cases}$$
Note: All these polynomials are symmetric and unimodal. There is an endless list of generalizations, refinements and variations with similar properties.
Symmetry and unimodality

Definition

A polynomial \( f(x) \in \mathbb{R}[x] \) is

- symmetric (or palindromic) and
- unimodal

if for some \( n \in \mathbb{N} \),

\[
f(x) = p_0 + p_1 x + p_2 x^2 + \cdots + p_n x^n
\]

with

- \( p_k = p_{n-k} \) for \( 0 \leq k \leq n \) and
- \( p_0 \leq p_1 \leq \cdots \leq p_{\lfloor n/2 \rfloor} \).

The number \( n/2 \) is called the center of symmetry.
Question: Can algebra or geometry shed light into such phenomena?
We let

- \( \Delta \) be a simplicial complex of dimension \( n - 1 \)
- \( f_i(\Delta) \) be the number of \( i \)-dimensional faces.

**Definition**

The \( h \)-polynomial of \( \Delta \) is defined as

\[
h(\Delta, x) = \sum_{i=0}^{n} f_{i-1}(\Delta) x^i (1 - x)^{n-i} = \sum_{i=0}^{n} h_i(\Delta) x^i.
\]

The sequence \( h(\Delta) = (h_0(\Delta), h_1(\Delta), \ldots, h_n(\Delta)) \) is the \( h \)-vector of \( \Delta \).

**Note:** \( h(\Delta, 1) = f_{n-1}(\Delta) \).
Example

For the 2-dimensional complex

\[ \Delta = \]

we have \( f_0(\Delta) = 8 \), \( f_1(\Delta) = 15 \) and \( f_2(\Delta) = 8 \) and hence

\[
h(\Delta, x) = (1 - x)^3 + 8x(1 - x)^2 + 15x^2(1 - x) + 8x^3
\]

\[
= 1 + 5x + 2x^2.
\]
Example

The boundary complex $\Sigma_n$ of the $n$-dimensional cross-polytope is a triangulation of the $(n - 1)$-dimensional sphere:

We have

$$h(\Sigma_n, x) = (1 + x)^n$$

for every $n \geq 1$. 
The face ring

We let

- $m$ be the number of vertices of $\Delta$
- $k$ be a field
- $S = k[x_1, x_2, \ldots, x_m]$
- $I_\Delta = \langle \prod_{i \in F} x_i : F \notin \Delta \rangle$ be the Stanley–Reisner ideal of $\Delta$

$$k[\Delta] = S/I_\Delta$$

be the Stanley–Reisner ring (or face ring) of $\Delta$.

Then $k[\Delta]$ is a graded $k$-algebra with Hilbert series

$$\sum_{i \geq 0} \dim_k(k[\Delta]_i) t^i = \frac{h(\Delta, t)}{(1 - t)^n},$$

where $n - 1 = \dim(\Delta)$. 
Theorem (Klee, Reisner, Stanley)

The polynomial $h(\Delta, x)$:

- has nonnegative coefficients if $\Delta$ triangulates a ball or a sphere,
- is symmetric if $\Delta$ triangulates a sphere,
- is unimodal if $\Delta$ is the boundary complex of a simplicial polytope.

Note: If $\Delta$ triangulates a ball or a sphere (more generally, if $\Delta$ is Cohen–Macaulay over $k$), then there exists a graded quotient $k(\Delta)$ of $k[\Delta]$ such that $h_i(\Delta) = \dim_k(k(\Delta)_i)$ for every $i$. 
We let

- \( V \) be an \( n \)-element set,
- \( 2^V \) be the simplex on the vertex set \( V \),
- \( \Delta \) be the first barycentric subdivision of the boundary complex of \( 2^V \).

Then \( h(\Delta, x) = A_n(x) \).
Example

For $n = 3$

\[
\Delta = h(\Delta, x) = (1 - x)^2 + 6x(1 - x) + 6x^2 = 1 + 4x + x^2.
\]
The local $h$-polynomial

We let

- $V$ be an $n$-element set,
- $\Gamma$ be a triangulation of the simplex $2^V$ on the vertex set $V$.

**Definition (Stanley, 1992)**

The local $h$-polynomial of $\Gamma$ (with respect to $V$) is defined as

$$\ell_V(\Gamma, x) = \sum_{F \subseteq V} (-1)^{n-|F|} h(\Gamma_F, x),$$

where $\Gamma_F$ is the restriction of $\Gamma$ to the face $F$ of the simplex $2^V$.

**Note:** This polynomial plays a major role in Stanley’s theory of subdivisions of simplicial (and more general) complexes.
Example

For the 2-dimensional triangulation

$$
\Gamma = \begin{align*}
&\end{align*}
$$

we have

$$
\ell_V(\Gamma, x) = (1 + 5x + 2x^2) - (1 + 2x) - (1 + x) - 1 + 1 + 1 + 1 - 1 = 2x + 2x^2.
$$
Theorem (Stanley, 1992)

The polynomial \( \ell_V(\Gamma, x) \)

- is symmetric,
- has nonnegative coefficients,
- is unimodal for every regular triangulation \( \Gamma \) of \( 2^V \).

Note: Stanley showed that there exists a graded \( S \)-module \( L_V(\Gamma) \) whose Hilbert series equals \( \ell_V(\Gamma, x) \).
Barycentric subdivision

For the barycentric subdivision $\Gamma$ of the simplex $2^V$ on the vertex set $V$

Stanley showed that

$$\ell_V(\Gamma, x) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} A_k(x) = \sum_{w \in D_n} x^{\text{exc}(w)} = d_n(x).$$
We now turn attention to $\tilde{A}_n(x)$. We let

- $V$ be an $n$-element set,
- $\Gamma$ be a triangulation of the simplex $2^V$ on the vertex set $V$.

Then, there exists a triangulation $\Sigma(\Gamma)$ of $\Sigma_n$ which restricts to $\Gamma$ on one facet of $\Sigma_n$ and satisfies

$$h(\Sigma(\Gamma), x) = \sum_{F \subseteq V} x^{n-|F|} h(\Gamma_F, x).$$
Example

For the barycentric subdivision $\Gamma$ of $2^V$ we have

$$h(\Sigma(\Gamma), x) = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} A_k(x) = \tilde{A}_n(x).$$
Recall that the link of a simplicial complex $\Delta$ at a face $F \in \Delta$ is defined as $\text{link}_\Delta(F) := \{ G \setminus F : F \subseteq G \in \Delta \}$.

**Proposition (Stanley, 1992)**

For every triangulation $\Delta'$ of a pure simplicial complex $\Delta$, 

$$h(\Delta', x) = \sum_{F \in \Delta} \ell_F(\Delta'_F, x) h(\text{link}_\Delta(F), x).$$
An application

We let again

- $V$ be an $n$-element set,
- $\Gamma$ be a triangulation of the simplex $2^V$ on the vertex set $V$.

**Theorem (Kubitzke–Murai–Sieg, 2017)**

$$\ell_V(\Gamma, x) = \sum_{F \subseteq V} (h(\Gamma_F, x) - h(\partial(\Gamma_F), x)) \cdot d_{n-|F|}(x).$$
Corollary (Kubitzke–Murai–Sieg, 2017)

We have

\[ d_n(x) = \sum_{k=0}^{n-2} \binom{n}{k} d_k(x)(x + x^2 + \cdots + x^{n-1-k}) \]

for \( n \geq 2 \). In particular, \( d_n(x) \) is symmetric and unimodal for all \( n \).
Question: Are there nice (symmetric and unimodal) analogues of Eulerian, derangement and binomial Eulerian polynomials for the hyperoctahedral group?
Eulerian polynomials of type $B_n$

We let

- $B_n = \{w = (w(1), w(2), \ldots, w(n)) : |w| \in S_n\}$ be the group of signed permutations of $[n]$
- $\Delta$ be the first barycentric subdivision of the boundary complex of the $n$-dimensional cube.

Then, $h(\Delta, 1) = 2^n n! = \#B_n$. 
Moreover,

\[ h(\Delta, x) = B_n(x) := \sum_{w \in B_n} x^{\text{des}_B(w)} \]

where

- \( \text{des}_B(w) := \# \{ i \in \{0, 1, \ldots, n - 1\} : w(i) > w(i + 1) \} \)

for \( w \in B_n \) as before, with \( w(0) := 0 \).

**Example**

\[
B_n(x) = \begin{cases} 
1 + x, & \text{if } n = 1 \\
1 + 6x + x^2, & \text{if } n = 2 \\
1 + 23x + 23x^2 + x^3, & \text{if } n = 3 \\
1 + 76x + 230x^2 + 76x^3 + x^4, & \text{if } n = 4 \\
1 + 237x + 1682x^2 + 1682x^3 + 237x^4 + x^5, & \text{if } n = 5.
\end{cases}
\]
More barycentric subdivisions

We let

- $\mathcal{V}$ be an $n$-element set,
- $\mathcal{K}$ be the barycentric subdivision of the cubical barycentric subdivision of $2^\mathcal{V}$.

Example

$n = 3$
Consider the polynomial

\[ d_n^+(x) = \ell_V(K, x). \]

**Example**

\[
\begin{align*}
    d_n^+(x) &= \begin{cases} 
        0, & \text{if } n = 1 \\
        3x, & \text{if } n = 2 \\
        7x + 7x^2, & \text{if } n = 3 \\
        15x + 87x^2 + 15x^3, & \text{if } n = 4 \\
        31x + 551x^2 + 551x^3 + 31x^4, & \text{if } n = 5 \\
        63x + 2803x^2 + 8243x^3 + 2803x^4 + 63x^5, & \text{if } n = 6.
    \end{cases}
\end{align*}
\]

**Note:** The sum of the coefficients of \( d_n^+(x) \) is equal to the number of even-signed derangements (signed permutations without fixed points of positive sign) in \( B_n \).
For \( w = (w_1, w_2, \ldots, w_n) \in B_n \) we let

\[
\text{fex}(w) = 2 \cdot \text{exc}_A(w) + \text{neg}(w),
\]

where

- \( \text{exc}_A(w) := \# \{i \in [n-1] : w(i) > i\} \)
- \( \text{neg}(w) := \# \{i \in [n] : w(i) < 0\} \).

**Theorem (A, 2014)**

We have

\[
d_n^+(x) = \sum_{w \in D_n^+} x^{\text{fex}(w)/2}
\]

where \( D_n^+ \) is the set of even-signed derangements in \( B_n \).
We now consider the polynomial

\[ \tilde{B}_n^+(x) := h(\Sigma(K), x). \]

Savvidou showed that

\[ h(K, x) = B_n^+(x) := \sum_{w \in B_n^+} x^{\text{des}_B(w)} \]

where \( B_n^+ \) is the set of \( w \in B_n \) with positive last coordinate. Hence,

\[ \tilde{B}_n^+(x) = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} B_k^+(x) = h(\Sigma(K), x) \]

is a symmetric and unimodal analogue of \( \tilde{A}_n(x) \) for the group \( B_n \).
Example

\[ \tilde{B}_n^+(x) = \begin{cases} 
1 + x, & \text{if } n = 1 \\
1 + 5x + x^2, & \text{if } n = 2 \\
1 + 19x + 19x^2 + x^3, & \text{if } n = 3.
\]
Symmetric functions

We let

- \( x = (x_1, x_2, x_3, \ldots) \) be a sequence of commuting indeterminates,
- \( h_n(x) \) be the complete homogeneous symmetric function in \( x \) of degree \( n \), defined by
  \[
  H(x, z) := \sum_{n \geq 0} h_n(x)z^n = \prod_{i \geq 1} \frac{1}{1 - x_i z},
  \]
- \( s_\lambda(x) \) be the Schur function in \( x \) corresponding to the partition \( \lambda \).
We define polynomials $R_\lambda(t), P_\lambda(t), Q_\lambda(t)$ by

$$
\frac{1 - t}{H(x; tz) - tH(x; z)} = \sum_\lambda R_\lambda(t) s_\lambda(x) z^{\lambda},
$$

$$
\frac{(1 - t)H(x, z)}{H(x; tz) - tH(x; z)} = \sum_\lambda P_\lambda(t) s_\lambda(x) z^{\lambda},
$$

and

$$
\frac{(1 - t)H(x, z)H(x, tz)}{H(x; tz) - tH(x; z)} = \sum_\lambda Q_\lambda(t) s_\lambda(x) z^{\lambda}.
$$

Note: The left-hand sides of these equations arise from algebraic-geometric and representation-theoretic considerations.
Note: We have

\[ \sum_{\lambda \vdash n} f^\lambda R_\lambda(t) = d_n(t), \]
\[ \sum_{\lambda \vdash n} f^\lambda P_\lambda(t) = A_n(t) \]

and

\[ \sum_{\lambda \vdash n} f^\lambda Q_\lambda(t) = \tilde{A}_n(t), \]

where \( f^\lambda \) is the number of standard Young tableaux of shape \( \lambda \).
Theorem (Brenti, Gessel, Shareshian–Wachs, Stanley)

The polynomials $R_\lambda(t)$, $P_\lambda(t)$ and $Q_\lambda(t)$ are symmetric and unimodal, with centers of symmetry $n/2$, $(n - 1)/2$ and $n/2$, respectively, for every $\lambda \vdash n$. Moreover, there exist nonnegative integers $\xi_{\lambda,i}$, $\gamma_{\lambda,i}$ and $\tilde{\gamma}_{\lambda,i}$ such that

\[
R_\lambda(t) = \sum_{i=0}^{\lfloor n/2 \rfloor} \xi_{\lambda,i} t^i (1 + t)^{n-2i},
\]

\[
P_\lambda(t) = \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \gamma_{\lambda,i} t^i (1 + t)^{n-1-2i},
\]

\[
Q_\lambda(t) = \sum_{i=0}^{\lfloor n/2 \rfloor} \tilde{\gamma}_{\lambda,i} t^i (1 + t)^{n-2i}.
\]
Gamma-positivity

Proposition (Bränden, 2004, Gal, 2005)

Suppose $f(x) \in \mathbb{R}[x]$ has nonnegative coefficients and only real roots and that it is symmetric, with center of symmetry $n/2$. Then,

$$f(x) = \sum_{i=0}^{\lfloor n/2 \rfloor} \gamma_i x^i (1 + x)^{n-2i}$$

for some nonnegative real numbers $\gamma_0, \gamma_1, \ldots, \gamma_{\lfloor n/2 \rfloor}$.

Definition

The polynomial $f(x)$ is called $\gamma$-positive if there exist nonnegative real numbers $\gamma_0, \gamma_1, \ldots, \gamma_{\lfloor n/2 \rfloor}$ as above, for some $n \in \mathbb{N}$.
Thus, $A_n(x)$, $d_n(x)$ and $\tilde{A}_n(x)$ are $\gamma$-positive for all $n$.

**Example**

$$A_n(x) = \begin{cases} 
1, & \text{if } n = 1 \\
1 + x, & \text{if } n = 2 \\
(1 + x)^2 + 2x, & \text{if } n = 3 \\
(1 + x)^3 + 8x(1 + x), & \text{if } n = 4 \\
(1 + x)^4 + 22x(1 + x)^2 + 16x^2, & \text{if } n = 5 \\
(1 + x)^5 + 52x(1 + x)^3 + 186x^2(1 + x), & \text{if } n = 6.
\end{cases}$$

**Note:** Every $\gamma$-positive polynomial (even if it has nonreal roots) is symmetric and unimodal.
An index \( i \in [n] \) is called a **double descent** of a permutation \( w \in \mathfrak{S}_n \) if

\[
w(i - 1) > w(i) > w(i + 1),
\]

where \( w(0) = w(n + 1) = n + 1 \).

**Theorem (Foata–Schützenberger, 1970)**

We have

\[
A_n(x) = \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \gamma_{n,i} x^i (1 + x)^{n-1-2i},
\]

where \( \gamma_{n,i} \) is the number of \( w \in \mathfrak{S}_n \) which have no double descent and \( \text{des}(w) = i \). In particular, \( A_n(x) \) is symmetric and unimodal.
Recently, gamma-positivity attracted attention after the work of

- Bränden (2004, 2008) on $P$-Eulerian polynomials,

Expositions can be found in:

Note: $\Gamma$-positivity is known to hold for

- $B_n(x)$, $d_n^+(x)$ and $\tilde{B}_n^+(x)$

for all $n$.

### Example

$$B_n(x) = \begin{cases} 
1 + x, & \text{if } n = 1 \\
(1 + x)^2 + 4x, & \text{if } n = 2 \\
(1 + x)^3 + 20x(1 + x), & \text{if } n = 3 \\
(1 + x)^4 + 72x(1 + x)^2, & \text{if } n = 4 \\
(1 + x)^5 + 232x(1 + x)^3 + 976x^2(1 + x), & \text{if } n = 5 \\
(1 + x)^6 + 716x(1 + x)^4 + 7664x^2(1 + x)^2, & \text{if } n = 6.
\end{cases}$$
Flag complexes and Gal’s conjecture

Definition
A simplicial complex $\Delta$ is called flag if it contains every simplex whose 1-skeleton is a subcomplex of $\Delta$.

Example

- Not flag
- Flag
Example

For a 1-dimensional sphere $\Delta$ with $m$ vertices we have

$$h(\Delta, x) = 1 + (m - 2)x + x^2.$$  

Note that $h(\Delta, x)$ is $\gamma$-positive $\iff$ $m \geq 4 \iff \Delta$ is flag.

Conjecture (Gal, 2005)

The polynomial $h(\Delta, x)$ is $\gamma$-positive for every flag triangulation $\Delta$ of the sphere.

Note: This extends a conjecture of Charney–Davis (1995).

Example

The complex $\Sigma_n$ is a flag and $h(\Sigma_n, x) = (1 + x)^n$ for every $n \geq 1$. 
Conjecture (A, 2012)

The polynomial $\ell_V(\Gamma, x)$ is $\gamma$-positive for every flag triangulation $\Gamma$ of $2^V$.

Note: This is stronger than Gal’s conjecture. There is considerable evidence for both conjectures.
We recall our notation $\Sigma(\Gamma)$ and note that the formula

$$h(\Sigma(\Gamma), x) = \sum_{F \subseteq V} x^{n-|F|} h(\Gamma_F, x)$$

may be rewritten as

$$h(\Sigma(\Gamma), x) = \sum_{F \subseteq V} \ell_F(\Gamma_F, x) (1 + x)^{n-|F|}.$$  

**Corollary**

The $\gamma$-positivity of $h(\Sigma(\Gamma), x)$ is implied by that of the $\ell_F(\Gamma_F, x)$. In particular:

- The $\gamma$-positivity of $\widetilde{A}_n(x)$ follows from that of $d_n(x)$.
- The $\gamma$-positivity of $\widetilde{B}_n^+(x)$ follows from that of $d_n^+(x)$. 

The analogue

\[
\frac{(1 - t)H(x, z)H(x, tz)}{H(x; tz)H(y; tz) - tH(x; z)H(y; tz)} = \sum_{\lambda, \mu} P_{\lambda, \mu}(t)s_\lambda(x)s_\mu(y) z^{|\lambda| + |\mu|}
\]

for the group \(B_n\) of

\[
\frac{(1 - t)H(x, z)}{H(x; tz) - tH(x; z)} = \sum_{\lambda} P_\lambda(t)s_\lambda(x) z^{|\lambda|}
\]

was found by Dolgachev–Lunts and Stembridge (1994). We then have

\[
\sum_{(\lambda, \mu) \vdash n} \binom{n}{|\lambda|} f^\lambda f^\mu P_{\lambda, \mu}(t) = B_n(t)
\]

for every \(n \geq 1\).
Open Problem:

- Find the analogues of the identities involving $R_\lambda$ and $Q_\lambda$ for $B_n$.
- Find a combinatorial interpretation for $P_{\lambda,\mu}(t)$.
- Prove that the polynomials $P_{\lambda,\mu}(t)$ are $\gamma$-positive.
Consider the second barycentric subdivision $\Gamma^2$ of the simplex $2^V$.

One can show that

$$\ell_V(\Gamma^2, x) = \sum \binom{n}{r_0, r_1, \ldots, r_k} d_k(x) d_{r_0}(x) A_{r_1}(x) \cdots A_{r_k}(x),$$

where the sum ranges over all $k \geq 0$ and over all sequences $(r_0, r_1, \ldots, r_k)$ of integers which satisfy $r_0 \geq 0$, $r_1, \ldots, r_k \geq 1$ and sum to $n$. 
Note: This implies the $\gamma$-positivity of $\ell_V(\Gamma^2, x)$.

Exercise: The sum of the coefficients of $\ell_V(\Gamma^2, x)$ is equal to the number of pairs $(u, v) \in S_n \times S_n$ of permutations with no common fixed point.

Open Problem: Find a combinatorial interpretation for:

- $\ell_V(\Gamma^2, x)$,
- the corresponding $\gamma$-coefficients.
Thank you for your attention!