Enumerative Combinatorics from an Algebraic-Geometric point of view

Christos Athanasiadis

University of Athens

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Outline

1. Three examples from combinatorics
2. An algebraic-geometric perspective
3. Gamma-positivity

Eulerian polynomials

We let

- $\mathfrak{S}_n$ be the group of permutations of $[n] := \{1, 2, \ldots, n\}$

and for $w \in \mathfrak{S}_n$

- $\mathrm{des}(w) \;:=\; \#\{i \in [n-1] : \, w(i) > w(i+1)\}$
- $\mathrm{exc}(w) \;:=\; \#\{i \in [n-1] : \, w(i) > i\}$

be the number of descents and excedances of w, respectively. The polynomial

$$A_n(x) \;:=\; \sum_{w \in \mathfrak{S}_n} x^{\mathrm{des}(w)} \;=\; \sum_{w \in \mathfrak{S}_n} x^{\mathrm{exc}(w)}$$

is the nth Eulerian polynomial.

Example

$$A_n(x) = \begin{cases} 1, & \text{if } n = 1 \\ 1 + x, & \text{if } n = 2 \\ 1 + 4x + x^2, & \text{if } n = 3 \\ 1 + 11x + 11x^2 + x^3, & \text{if } n = 4 \\ 1 + 26x + 66x^2 + 26x^3 + x^4, & \text{if } n = 5 \\ 1 + 57x + 302x^2 + 302x^3 + 57x^4 + x^5, & \text{if } n = 6. \end{cases}$$

Derangement polynomials

We let

- $\mathcal{D}_n$ be the set of derangements (permutations without fixed points) in the symmetric group $\mathfrak{S}_n$.

For instance,

- $\mathcal{D}_3 = \{ (2,3,1), (3,1,2) \}$.

The polynomial

$$d_n(x) := \sum_{w \in \mathcal{D}_n} x^{\mathrm{exc}(w)}$$

is the *n*th derangement polynomial.

Example

$$d_n(x) = \begin{cases} 0, & \text{if } n = 1 \\ x, & \text{if } n = 2 \\ x + x^2, & \text{if } n = 3 \\ x + 7x^2 + x^3, & \text{if } n = 4 \\ x + 21x^2 + 21x^3 + x^4, & \text{if } n = 5 \\ x + 51x^2 + 161x^3 + 51x^4 + x^5, & \text{if } n = 6 \\ x + 113x^2 + 813x^3 + 813x^4 + 113x^5 + x^6, & \text{if } n = 7. \end{cases}$$

Binomial Eulerian polynomials

The polynomial

$$\widetilde{A}_n(x) \;:=\; 1 \;+\; x\sum_{k=1}^{n}\binom{n}{k}A_k(x) \;=\; \sum_{k=0}^{n}\binom{n}{k}x^{n-k}A_k(x)$$

is the nth binomial Eulerian polynomial.

Example

$$\widetilde{A}_n(x) \;=\; \begin{cases} 1+x, & \text{if} \quad n=1 \\ 1+3x+x^2, & \text{if} \quad n=2 \\ 1+7x+7x^2+x^3, & \text{if} \quad n=3 \\ 1+15x+33x^2+15x^3+x^4, & \text{if} \quad n=4 \\ 1+31x+131x^2+131x^3+31x^4+x^5, & \text{if} \quad n=5 \\ 1+63x+473x^2+883x^3+473x^4+63x^5+x^6, & \text{if} \quad n=6. \end{cases}$$

Note: All these polynomials are symmetric and unimodal. There is an endless list of generalizations, refinements and variations with similar properties.

Symmetry and unimodality

Definition

A polynomial $f(x) \in \mathbb{R}[x]$ is

- symmetric (or palindromic) and
- unimodal

if for some $n \in \mathbb{N}$,

$$f(x) = p_0 + p_1 x + p_2 x^2 + \cdots + p_n x^n$$

with

- $p_k = p_{n-k}$ for $0 \leq k \leq n$ and
- $p_0 \leq p_1 \leq \cdots \leq p_{\lfloor n/2 \rfloor}$.

The number $n/2$ is called the center of symmetry.

Question: Can algebra or geometry shed light into such phenomena?

Face enumeration of simplicial complexes

We let

- Δ be a simplicial complex of dimension $n-1$
- $f_i(\Delta)$ be the number of i-dimensional faces.

Definition

The *h*-polynomial of Δ is defined as

$$h(\Delta, x) = \sum_{i=0}^{n} f_{i-1}(\Delta)\, x^i (1-x)^{n-i} = \sum_{i=0}^{n} h_i(\Delta)\, x^i.$$

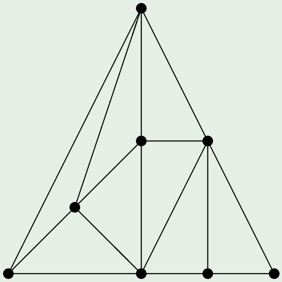
The sequence $h(\Delta) = (h_0(\Delta), h_1(\Delta), \ldots, h_n(\Delta))$ is the *h*-vector of Δ.

Note: $h(\Delta, 1) = f_{n-1}(\Delta)$.

Example

For the 2-dimensional complex

Δ =

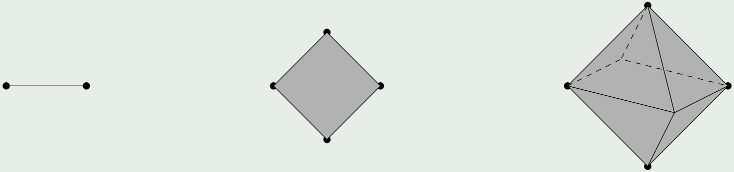


we have $f_0(\Delta) = 8$, $f_1(\Delta) = 15$ and $f_2(\Delta) = 8$ and hence

$$\begin{aligned} h(\Delta, x) &= (1-x)^3 + 8x(1-x)^2 + 15x^2(1-x) + 8x^3 \\ &= 1 + 5x + 2x^2. \end{aligned}$$

Example

The boundary complex Σ_n of the n-dimensional cross-polytope is a triangulation of the $(n-1)$-dimensional sphere:



We have

$$h(\Sigma_n, x) \;=\; (1+x)^n$$

for every $n \geq 1$.

The face ring

We let

- m be the number of vertices of Δ
- k be a field
- $S = k[x_1, x_2, \ldots, x_m]$
- $I_\Delta = \langle \prod_{i \in F} x_i : F \notin \Delta \rangle$ be the Stanley–Reisner ideal of Δ
- $$k[\Delta] \; = \; S/I_\Delta$$

  be the Stanley–Reisner ring (or face ring) of Δ.

Then $k[\Delta]$ is a graded k-algebra with Hilbert series

$$\sum_{i \geq 0} \dim_k(k[\Delta]_i)\, t^i \; = \; \frac{h(\Delta, t)}{(1-t)^n},$$

where $n - 1 = \dim(\Delta)$.

Theorem (Klee, Reisner, Stanley)

The polynomial $h(\Delta, x)$:

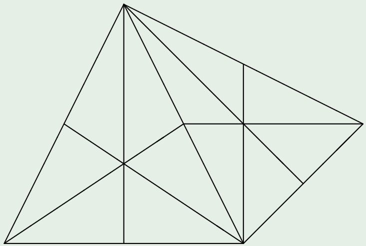
- *has nonnegative coefficients if Δ triangulates a ball or a sphere,*
- *is symmetric if Δ triangulates a sphere,*
- *is unimodal if Δ is the boundary complex of a simplicial polytope.*

Note: If Δ triangulates a ball or a sphere (more generally, if Δ is Cohen–Macaulay over k), then there exists a graded quotient $k(\Delta)$ of $k[\Delta]$ such that $h_i(\Delta) = \dim_k(k(\Delta)_i)$ for every i.

Example

We let

- V be an n-element set,
- 2^V be the simplex on the vertex set V,
- Δ be the first barycentric subdivision of the boundary complex of 2^V.

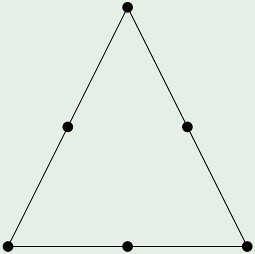


Then $h(\Delta, x) = A_n(x)$.

Example

For $n = 3$

$\Delta \quad =$



$$h(\Delta, x) \;=\; (1-x)^2 + 6x(1-x) + 6x^2 \;=\; 1 + 4x + x^2.$$

The local h-polynomial

We let

- V be an n-element set,
- Γ be a triangulation of the simplex 2^V on the vertex set V.

Definition (Stanley, 1992)

The local h-polynomial of Γ (with respect to V) is defined as

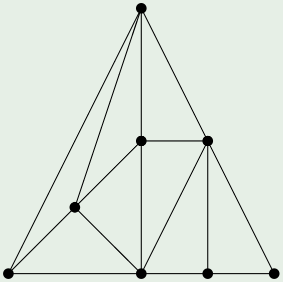
$$\ell_V(\Gamma, x) \; = \; \sum_{F \subseteq V} (-1)^{n-|F|} \, h(\Gamma_F, x),$$

where Γ_F is the restriction of Γ to the face F of the simplex 2^V.

Note: This polynomial plays a major role in Stanley's theory of subdivisions of simplicial (and more general) complexes.

Example

For the 2-dimensional triangulation

$\Gamma \quad =$ 

we have

$$\ell_V(\Gamma, x) = (1 + 5x + 2x^2) - (1 + 2x) - (1 + x) - 1 + 1 + 1 + 1 - 1 = 2x + 2x^2.$$

Theorem (Stanley, 1992)

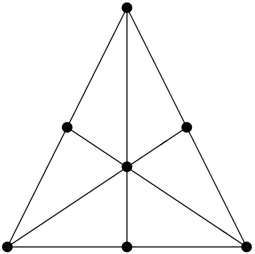
The polynomial $\ell_V(\Gamma, x)$

- *is symmetric,*
- *has nonnegative coefficients,*
- *is unimodal for every regular triangulation* Γ *of* 2^V.

Note: Stanley showed that there exists a graded S-module $L_V(\Gamma)$ whose Hilbert series equals $\ell_V(\Gamma, x)$.

Barycentric subdivision

For the barycentric subdivision Γ of the simplex 2^V on the vertex set V



Stanley showed that

$$\ell_V(\Gamma, x) \;=\; \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} A_k(x) \;=\; \sum_{w \in \mathcal{D}_n} x^{\mathrm{exc}(w)} \;=\; d_n(x).$$

The triangulation $\Sigma(\Gamma)$

We now turn attention to $\widetilde{A}_n(x)$. We let

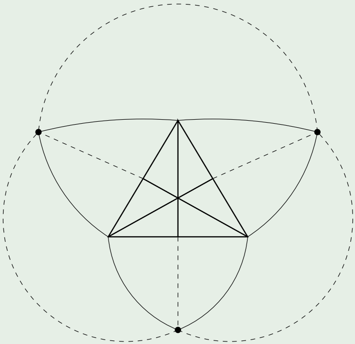
- V be an n-element set,
- Γ be a triangulation of the simplex 2^V on the vertex set V.

Then, there exists a triangulation $\Sigma(\Gamma)$ of Σ_n which restricts to Γ on one facet of Σ_n and satisfies

$$h(\Sigma(\Gamma), x) \;=\; \sum_{F \subseteq V} x^{n-|F|} \, h(\Gamma_F, x).$$

Example

For the barycentric subdivision Γ of 2^V we have



$$h(\Sigma(\Gamma), x) \ = \ \sum_{k=0}^{n} \binom{n}{k} x^{n-k} A_k(x) \ = \ \widetilde{A}_n(x).$$

h-vectors of triangulations of complexes

Recall that the link of a simplicial complex Δ at a face $F \in \Delta$ is defined as $\mathrm{link}_\Delta(F) := \{G \smallsetminus F : F \subseteq G \in \Delta\}$.

Proposition (Stanley, 1992)

For every triangulation Δ' of a pure simplicial complex Δ,

$$h(\Delta', x) \;=\; \sum_{F \in \Delta} \ell_F(\Delta'_F, x)\, h(\mathrm{link}_\Delta(F), x).$$

An application

We let again

- V be an n-element set,
- Γ be a triangulation of the simplex 2^V on the vertex set V.

Theorem (Kubitzke–Murai–Sieg, 2017)

$$\ell_V(\Gamma, x) \;=\; \sum_{F \subseteq V} \big(h(\Gamma_F, x) - h(\partial(\Gamma_F), x)\big) \cdot d_{n-|F|}(x).$$

Corollary (Kubitzke–Murai–Sieg, 2017)

We have

$$d_n(x) \;=\; \sum_{k=0}^{n-2} \binom{n}{k} d_k(x)(x + x^2 + \cdots + x^{n-1-k})$$

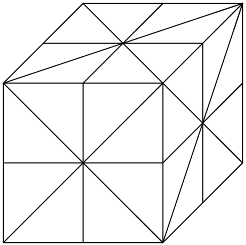
for $n \geq 2$. In particular, $d_n(x)$ is symmetric and unimodal for all n.

Question: Are there nice (symmetric and unimodal) analogues of Eulerian, derangement and binomial Eulerian polynomials for the hyperoctahedral group?

Eulerian polynomials of type B_n

We let

- $B_n = \{w = (w(1), w(2), \ldots, w(n)) : |w| \in \mathfrak{S}_n\}$ be the group of signed permutations of $[n]$
- Δ be the first barycentric subdivision of the boundary complex of the n-dimensional cube.



Then, $h(\Delta, 1) = 2^n n! = \# B_n$.

Moreover,

$$h(\Delta, x) \;=\; B_n(x) \;:=\; \sum_{w \in B_n} x^{\mathrm{des}_B(w)}$$

where

- $\mathrm{des}_B(w) \;:=\; \#\{i \in \{0, 1, \ldots, n-1\} : \; w(i) > w(i+1)\}$

for $w \in B_n$ as before, with $w(0) := 0$.

Example

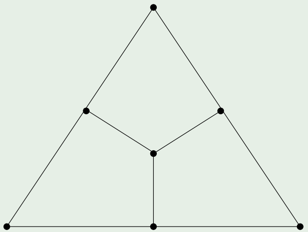
$$B_n(x) \;=\; \begin{cases} 1 + x, & \text{if } \; n = 1 \\ 1 + 6x + x^2, & \text{if } \; n = 2 \\ 1 + 23x + 23x^2 + x^3, & \text{if } \; n = 3 \\ 1 + 76x + 230x^2 + 76x^3 + x^4, & \text{if } \; n = 4 \\ 1 + 237x + 1682x^2 + 1682x^3 + 237x^4 + x^5, & \text{if } \; n = 5. \end{cases}$$

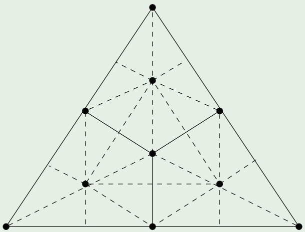
More barycentric subdivisions

We let

- V be an n-element set,
- K be the barycentric subdivision of the cubical barycentric subdivision of 2^V.

Example





$n = 3$

Consider the polynomial

$$d_n^+(x) \;=\; \ell_V(K,x).$$

Example

$$d_n^+(x) \;=\; \begin{cases} 0, & \text{if} \quad n=1 \\ 3x, & \text{if} \quad n=2 \\ 7x+7x^2, & \text{if} \quad n=3 \\ 15x+87x^2+15x^3, & \text{if} \quad n=4 \\ 31x+551x^2+551x^3+31x^4, & \text{if} \quad n=5 \\ 63x+2803x^2+8243x^3+2803x^4+63x^5, & \text{if} \quad n=6. \end{cases}$$

Note: The sum of the coefficients of $d_n^+(x)$ is equal to the number of even-signed derangements (signed permutations without fixed points of positive sign) in B_n.

For $w = (w_1, w_2, \ldots, w_n) \in B_n$ we let

$$\mathrm{fex}(w) \;=\; 2 \cdot \mathrm{exc}_A(w) + \mathrm{neg}(w),$$

where

- $\mathrm{exc}_A(w) \;:=\; \#\{i \in [n-1] : \; w(i) > i\}$
- $\mathrm{neg}(w) \;:=\; \#\{i \in [n] : \; w(i) < 0\}$.

Theorem (A, 2014)

We have

$$d_n^+(x) \;=\; \sum_{w \in \mathcal{D}_n^+} x^{\mathrm{fex}(w)/2}$$

where $\mathcal{D}_n^+$ is the set of even-signed derangements in B_n.

We now consider the polynomial

$$\widetilde{B}_n^+(x) \;:=\; h(\Sigma(K), x).$$

Savvidou showed that

$$h(K, x) \;=\; B_n^+(x) \;:=\; \sum_{w \in B_n^+} x^{\mathrm{des}_B(w)}$$

where B_n^+ is the set of $w \in B_n$ with positive last coordinate. Hence,

$$\widetilde{B}_n^+(x) \;=\; \sum_{k=0}^{n} \binom{n}{k} x^{n-k} B_k^+(x) \;=\; h(\Sigma(K), x)$$

is a symmetric and unimodal analogue of $\widetilde{A}_n(x)$ for the group B_n.

Example

$$\widetilde{B}^{+}_{\boldsymbol{n}}(x) \;=\; \begin{cases} 1 + x, & \text{if} \quad n = 1 \\ 1 + 5x + x^2, & \text{if} \quad n = 2 \\ 1 + 19x + 19x^2 + x^3, & \text{if} \quad n = 3. \end{cases}$$

Symmetric functions

We let

- $\mathbf{x} = (x_1, x_2, x_3, \ldots)$ be a sequence of commuting indeterminates,
- $h_n(\mathbf{x})$ be the complete homogeneous symmetric function in $\mathbf{x}$ of degree n, defined by

$$H(\mathbf{x}, z) \;:=\; \sum_{n \geq 0} h_n(\mathbf{x}) z^n \;=\; \prod_{i \geq 1} \frac{1}{1 - x_i z},$$

- $s_\lambda(\mathbf{x})$ be the Schur function in $\mathbf{x}$ corresponding to the partition λ.

We define polynomials $R_\lambda(t), P_\lambda(t), Q_\lambda(t)$ by

$$\frac{1-t}{H(\mathbf{x};tz)-tH(\mathbf{x};z)} = \sum_\lambda R_\lambda(t) s_\lambda(\mathbf{x})\, z^{|\lambda|},$$

$$\frac{(1-t)H(\mathbf{x},z)}{H(\mathbf{x};tz)-tH(\mathbf{x};z)} = \sum_\lambda P_\lambda(t) s_\lambda(\mathbf{x})\, z^{|\lambda|}$$

and

$$\frac{(1-t)H(\mathbf{x},z)H(\mathbf{x},tz)}{H(\mathbf{x};tz)-tH(\mathbf{x};z)} = \sum_\lambda Q_\lambda(t) s_\lambda(\mathbf{x})\, z^{|\lambda|}.$$

Note: The left-hand sides of these equations arise from algebraic-geometric and representation-theoretic considerations.

Note: We have

$$\sum_{\lambda \vdash n} f^\lambda R_\lambda(t) = d_n(t),$$

$$\sum_{\lambda \vdash n} f^\lambda P_\lambda(t) = A_n(t)$$

and

$$\sum_{\lambda \vdash n} f^\lambda Q_\lambda(t) = \widetilde{A}_n(t),$$

where f^λ is the number of standard Young tableaux of shape λ.

Theorem (Brenti, Gessel, Shareshian–Wachs, Stanley)

The polynomials $R_\lambda(t), P_\lambda(t)$ and $Q_\lambda(t)$ are symmetric and unimodal, with centers of symmetry $n/2$, $(n-1)/2$ and $n/2$, respectively, for every $\lambda \vdash n$. Moreover, there exist nonnegative integers $\xi_{\lambda,i}$, $\gamma_{\lambda,i}$ and $\widetilde{\gamma}_{\lambda,i}$ such that

$$R_\lambda(t) = \sum_{i=0}^{\lfloor n/2 \rfloor} \xi_{\lambda,i} t^i (1+t)^{n-2i},$$

$$P_\lambda(t) = \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \gamma_{\lambda,i} t^i (1+t)^{n-1-2i},$$

$$Q_\lambda(t) = \sum_{i=0}^{\lfloor n/2 \rfloor} \widetilde{\gamma}_{\lambda,i} t^i (1+t)^{n-2i}.$$

Gamma-positivity

Proposition (Brändén, 2004, Gal, 2005)

Suppose $f(x) \in \mathbb{R}[x]$ has nonnegative coefficients and only real roots and that it is symmetric, with center of symmetry $n/2$. Then,

$$f(x) = \sum_{i=0}^{\lfloor n/2 \rfloor} \gamma_i \, x^i (1+x)^{n-2i}$$

for some nonnegative real numbers $\gamma_0, \gamma_1, \ldots, \gamma_{\lfloor n/2 \rfloor}$.

Definition

The polynomial $f(x)$ is called γ-positive if there exist nonnegative real numbers $\gamma_0, \gamma_1, \ldots, \gamma_{\lfloor n/2 \rfloor}$ as above, for some $n \in \mathbb{N}$.

Thus, $A_n(x)$, $d_n(x)$ and $\widetilde{A}_n(x)$ are γ-positive for all n.

Example

$$A_n(x) = \begin{cases} 1, & \text{if } n = 1 \\ 1 + x, & \text{if } n = 2 \\ (1+x)^2 + 2x, & \text{if } n = 3 \\ (1+x)^3 + 8x(1+x), & \text{if } n = 4 \\ (1+x)^4 + 22x(1+x)^2 + 16x^2, & \text{if } n = 5 \\ (1+x)^5 + 52x(1+x)^3 + 186x^2(1+x), & \text{if } n = 6. \end{cases}$$

Note: Every γ-positive polynomial (even if it has nonreal roots) is symmetric and unimodal.

An index $i \in [n]$ is called a double descent of a permutation $w \in \mathfrak{S}_n$ if

$$w(i-1) > w(i) > w(i+1),$$

where $w(0) = w(n+1) = n+1$.

Theorem (Foata–Schützenberger, 1970)

We have

$$A_n(x) = \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \gamma_{n,i}\, x^i (1+x)^{n-1-2i},$$

where $\gamma_{n,i}$ *is the number of* $w \in \mathfrak{S}_n$ *which have no double descent and* $\mathrm{des}(w) = i$*. In particular,* $A_n(x)$ *is symmetric and unimodal.*

Recently, gamma-positivity attracted attention after the work of

- Bränden (2004, 2008) on P-Eulerian polynomials,
- Gal (2005) on flag triangulations of spheres.

Expositions can be found in:

- A, Gamma-positivity in combinatorics and geometry, `arXiv:1711.05983`.
- T.K. Petersen, Eulerian Numbers, Birkhaüser, 2015.

Note: Γ-positivity is known to hold for

- $B_n(x)$, $d_n^+(x)$ and $\widetilde{B}_n^+(x)$

for all n.

Example

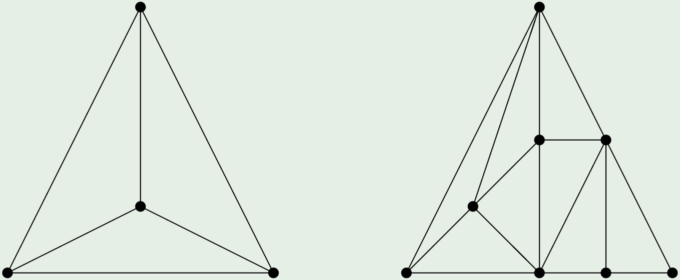
$$B_n(x) = \begin{cases} 1 + x, & \text{if } n = 1 \\ (1+x)^2 + 4x, & \text{if } n = 2 \\ (1+x)^3 + 20x(1+x), & \text{if } n = 3 \\ (1+x)^4 + 72x(1+x)^2, & \text{if } n = 4 \\ (1+x)^5 + 232x(1+x)^3 + 976x^2(1+x), & \text{if } n = 5 \\ (1+x)^6 + 716x(1+x)^4 + 7664x^2(1+x)^2, & \text{if } n = 6. \end{cases}$$

Flag complexes and Gal's conjecture

Definition

A simplicial complex Δ is called flag if it contains every simplex whose 1-skeleton is a subcomplex of Δ.

Example



not flag   flag

Example

For a 1-dimensional sphere Δ with m vertices we have

$$h(\Delta, x) \;=\; 1 + (m-2)x + x^2.$$

Note that $h(\Delta, x)$ is γ-positive $\Leftrightarrow$ $m \geq 4$ $\Leftrightarrow$ Δ is flag.

Conjecture (Gal, 2005)

The polynomial $h(\Delta, x)$ is γ-positive for every flag triangulation Δ of the sphere.

Note: This extends a conjecture of Charney–Davis (1995).

Example

The complex Σ_n is a flag and $h(\Sigma_n, x) = (1+x)^n$ for every $n \geq 1$.

Conjecture (A, 2012)

The polynomial $\ell_V(\Gamma, x)$ is γ-positive for every flag triangulation Γ of 2^V.

Note: This is stronger than Gal's conjecture. There is considerable evidence for both conjectures.

We recall our notation $\Sigma(\Gamma)$ and note that the formula

$$h(\Sigma(\Gamma), x) \;=\; \sum_{F \subseteq V} x^{n-|F|}\, h(\Gamma_F, x)$$

may be rewritten as

$$h(\Sigma(\Gamma), x) \;=\; \sum_{F \subseteq V} \ell_F(\Gamma_F, x)\, (1+x)^{n-|F|}.$$

Corollary

The γ-positivity of $h(\Sigma(\Gamma), x)$ is implied by that of the $\ell_F(\Gamma_F, x)$. In particular:

- *The γ-positivity of $\widetilde{A}_n(x)$ follows from that of $d_n(x)$.*
- *The γ-positivity of $\widetilde{B}_n^+(x)$ follows from that of $d_n^+(x)$.*

The analogue

$$\frac{(1-t)H(\mathbf{x},z)H(\mathbf{x},tz)}{H(\mathbf{x};tz)H(\mathbf{y};tz)-tH(\mathbf{x};z)H(\mathbf{y};tz)} \;=\; \sum_{\lambda,\mu} P_{\lambda,\mu}(t) s_\lambda(\mathbf{x}) s_\mu(\mathbf{y})\, z^{|\lambda|+|\mu|}$$

for the group B_n of

$$\frac{(1-t)H(\mathbf{x},z)}{H(\mathbf{x};tz)-tH(\mathbf{x};z)} \;=\; \sum_{\lambda} P_{\lambda}(t) s_\lambda(\mathbf{x})\, z^{|\lambda|}$$

was found by Dolgachev–Lunts and Stembridge (1994). We then have

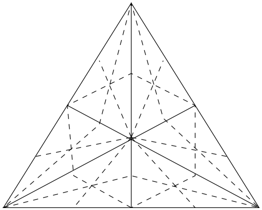
$$\sum_{(\lambda,\mu)\vdash n} \binom{n}{|\lambda|} f^\lambda f^\mu P_{\lambda,\mu}(t) \;=\; B_n(t)$$

for every $n \geq 1$.

Open Problem:

- Find the analogues of the identites involving R_λ and Q_λ for B_n.
- Find a combinatorial interpretation for $P_{\lambda,\mu}(t)$.
- Prove that the polynomials $P_{\lambda,\mu}(t)$ are γ-positive.

Consider the second barycentric subdivision Γ^2 of the simplex 2^V.



One can show that

$$\ell_V(\Gamma^2, x) \;=\; \sum \binom{n}{r_0, r_1, \ldots, r_k} d_k(x)\, d_{r_0}(x)\, A_{r_1}(x) \cdots A_{r_k}(x),$$

where the sum ranges over all $k \geq 0$ and over all sequences $(r_0, r_1, \ldots, r_k)$ of integers which satisfy $r_0 \geq 0$, $r_1, \ldots, r_k \geq 1$ and sum to n.

Note: This implies the γ-positivity of $\ell_V(\Gamma^2, x)$.

Exercise: The sum of the coefficients of $\ell_V(\Gamma^2, x)$ is equal to the number of pairs $(u, v) \in \mathfrak{S}_n \times \mathfrak{S}_n$ of permutations with no common fixed point.

Open Problem: Find a combinatorial interpretation for:

- $\ell_V(\Gamma^2, x)$,
- the corresponding γ-coefficients.

Thank you for your attention!