Christopher Borger (joint work with Benjamin Nill)

Defectivity of families of full-dimensional point configurations

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DFG-Graduiertenkolleg
MATHEMATISCHE
KOMPLEXITÄTSREDUKTION

Systems of Laurent polynomials

Let $f_1, \ldots, f_k \in \mathbb{C}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$ and consider the system of equations

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Example

Suppose $f(x) = ax^2 + bx + c$. Then we have:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Laurent polynomials with fixed support

Definition

A Laurent polynomial $f \in \mathbb{C}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$ has *support* $A \subset \mathbb{Z}^n$ if it is of the form

$$f = \sum_{a \in A} c_a x^a,$$

for coefficients $c_a \in \mathbb{C}$.

Mixed Discriminants

Let $A_1, \ldots, A_k \subset \mathbb{Z}^n$ and consider systems of the form

$$\left(f_1 := \sum_{a_1 \in A_1} c_{a_1} x^{a_1}\right) = \cdots = \left(f_k := \sum_{a_k \in A_k} c_{a_k} x^{a_k}\right) = 0.$$

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$$\Sigma_{A_1,\ldots,A_k} := \overline{\{\text{coeff. leading to non-degenerate multiple roots in } (\mathbb{C}^*)^n\}}.$$

If $\Sigma_{A_1,\ldots,A_k}$ is a hypersurface the *mixed discriminant* $\Delta_{A_1,\ldots,A_k}$ is the (up to scalar multiples) unique irreducible polynomial such that $\{\Delta_{A_1,\ldots,A_k} = 0\} = \Sigma_{A_1,\ldots,A_k}$.

Multiple roots

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A *multiple root* of a system of Laurent polynomials $f_1, \ldots, f_k \in \mathbb{C}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$ is an element $c \in (\mathbb{C}^*)^n$ such that

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2. $\nabla f_1(c), \ldots, \nabla f_k(c)$ are linearly dependent.

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A multiple root c is called *non-degenerate* if any proper subset of $\nabla f_1(c), \dots, \nabla f_k(c)$ is linearly independent.

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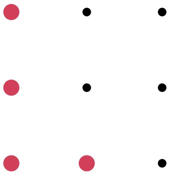
$$A := \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 \end{pmatrix} \qquad f(x, y) = a_{00} + a_{01}y + a_{02}y^2 + a_{10}x$$



$$\nabla f(x, y) = (a_{10}, a_{01} - 2a_{02}y)$$

$$\Sigma_A = \{a_{10} = 0\} \cap \left\{a_{01}^2 - 4a_{00}a_{02} = 0\right\}$$

A necessary condition

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Theorem

Let $k \leqslant n+1$ *and* $A_1, \ldots, A_k \subset \mathbb{Z}^n$ *be full-dimensional configurations that form a spanning family (that is containing* 0 *and such that* $\langle A_1 \rangle_{\mathbb{Z}} + \cdots + \langle A_k \rangle_{\mathbb{Z}} = \mathbb{Z}^n$*). If* $A_1, \ldots, A_k$ *is a defective family, then*

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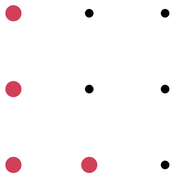

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Corollary

Let $A_1, \ldots, A_n \subset \mathbb{Z}^n$ be a spanning family of full-dimensional configurations. Then $A_1, \ldots, A_n$ is defective if and only if all A_i are translates of the vertices of the same unimodular simplex.

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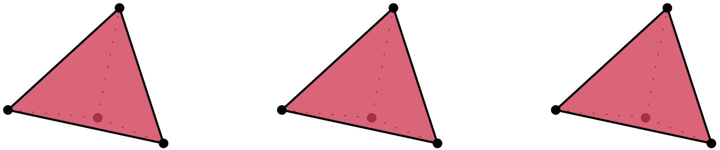

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Theorem (Di Rocco, Dickenstein, Morrison 2018+)

A family of full-dimensional configurations $A_1, \ldots, A_k \subset \mathbb{Z}^n$ is defective if and only if the Cayley sum $A_1 * \cdots * A_k \subset \mathbb{Z}^{n+k-1}$ is defective. In the non-defective case both discriminants are the same.

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$$\begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix} \circledast \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \cong \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

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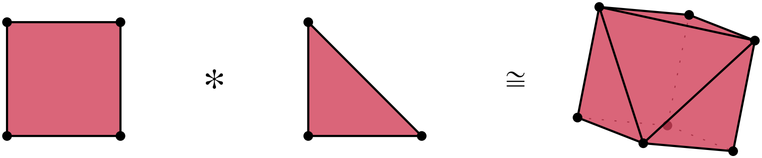

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In general:

$$A_1 * \cdots * A_k := A_1 \times \{0\} \cup A_2 \times \{e_1\} \cup \cdots \cup A_k \times \{e_{k-1}\}.$$

Main proof ingredients

Theorem (Furukawa, Ito, 2016)

A spanning configuration $A \subset \mathbb{Z}^{n+k-1}$ *is defective if and only if* $A \cong B_1 * \cdots * B_r$ *such that there is a lattice projection*

$$\pi \colon \mathbb{Z}^{n+k-r} \to \mathbb{Z}^{n+k-r-c},$$

such that

$$\pi(B_1) * \cdots * \pi(B_r) \text{ is of join type,}$$

where $c < r - 1$.

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Definition (Cayley sum of join type)

For $B_1 * \cdots * B_k \subset \mathbb{Z}^n$ we say that the Cayley sum $B_1 * \cdots * B_k$ is *of join type*, if

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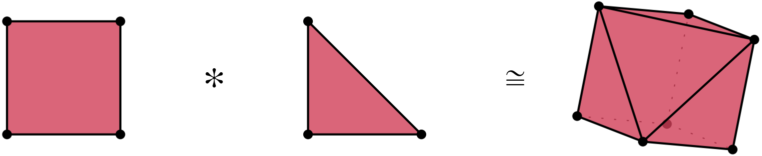

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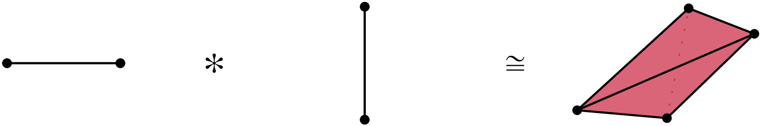

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Different Cayley decompositions

Lemma

Let $A_1, \dots, A_k \subset \mathbb{Z}^n$ be full-dimensional configurations and $B_1, \dots, B_r \subset \mathbb{Z}^{n+k-1-r}$ non-empty configurations, such that

$$A_1 * \cdots * A_k \cong B_1 * \cdots * B_r \subset \mathbb{Z}^{n+k-1}.$$

1. *One has $\dim(B_i) \geqslant \min(k-1, n)$ for all $i \in [r]$.*
2. *If furthermore $\dim(B_i) < n$ for all $i \in [r]$, also the following inequality holds:*

$$\dim(B_1) + \cdots + \dim(B_r) \geqslant n + 1 + r(k-2).$$

Sketch of proof

Assume $A_1, \ldots, A_k \subset \mathbb{Z}^n$ is a spanning family of full-dimensional defective configurations.

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- $A_1 * \cdots * A_k \cong B_1 * \cdots * B_r$ such that there is a projection π of codimension $c < r - 1$ such that $\pi(B_1) * \cdots * \pi(B_r)$ is of join type (i.p. $\dim(B_1) + \cdots + \dim(B_r) \leqslant n + k - 1 + (r-1)(r-3)$).

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- By lemma we have $n + 1 + r(k-2) \leqslant \dim(B_1) + \cdots + \dim(B_r)$.

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- By lemma we have $n + 1 + r(k-2) \leqslant \dim(B_1) + \cdots + \dim(B_r)$.
- Both bounds in particular imply $k < r$.
- This implies $\mathrm{int}_{\mathbb{Z}}(k(\mathrm{conv}(A_1 * \cdots * A_k))) = \varnothing$ or equivalently $\mathrm{int}_{\mathbb{Z}}(\mathrm{conv}(A_1) + \cdots + \mathrm{conv}(A_k)) = \varnothing$.

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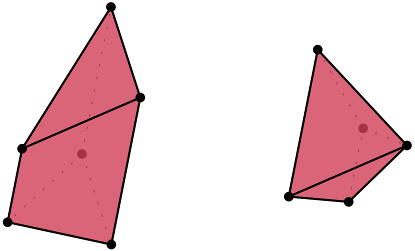
  Up to finitely many exceptions the full-dimensional families $A_1, A_2 \subset \mathbb{Z}^3$ fulfilling the condition are those having a common projection onto the same unimodular simplex:



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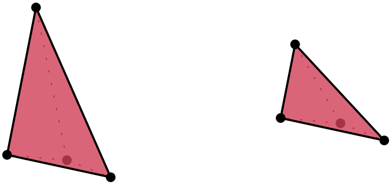
- How non-sufficient is the necessary condition? E.g. for $n = 3, k = 2$ we have:
  Only those $A_1, A_2 \subset \mathbb{Z}^3$ that are equivalent to tuples $L_1 * \{0\} * \{0\}, L_2 * \{0\} * \{0\}$ for $L_1, L_2 \subset \mathbb{Z}^1$ are actually defective:



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- What is a good sufficient condition?
- What about relaxing the *full-dimensional*, e.g. to *irreducible*?

Thank you for your attention!

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$$\begin{aligned}\Delta_{A_1,A_2} = & a_{00}^2 b_{11}^2 - 2a_{00}a_{01}b_{10}b_{11} - 2a_{00}a_{10}b_{01}b_{11} - 2a_{00}a_{11}b_{00}b_{11} \\ & + 4a_{00}a_{11}b_{01}b_{10} + a_{01}^2 b_{10}^2 + 4a_{01}a_{10}b_{00}b_{11} \\ & - 2a_{01}a_{10}b_{01}b_{10} - 2a_{01}a_{11}b_{00}b_{10} \\ & + a_{10}^2 b_{01}^2 - 2a_{10}a_{11}b_{00}b_{01} + a_{11}^2 b_{00}^2.\end{aligned}$$