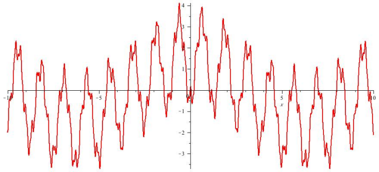
Sparse representations from moments

Bernard Mourrain

Inria Méditerranée, Sophia Antipolis
Bernard.Mourrain@inria.fr

Sparse representation of signals

Given a function or signal $f(t)$:



decompose it as

$$f(t) = \sum_{i=1}^{r'} (a_i cos(\mu_i t) + b_i sin(\mu_i t)) e^{\nu_i t} = \sum_{i=1}^{\mathbf{r}} \omega_{\mathbf{i}} e^{\zeta_{\mathbf{i}} t}$$

Prony's method (1795)

For the signal $f(t) = \sum_{i=1}^{\mathbf{r}} \omega_{\mathbf{i}} e^{\zeta_{\mathbf{i}} t}$, $(\omega_i, \zeta_i \in \mathbb{C})$,

- Evaluate f at $2\,r$ regularly spaced points: $\sigma_0 := f(0), \sigma_1 := f(1), \ldots$
- Compute a non-zero element $\mathbf{p} = [\mathbf{p_0}, \ldots, \mathbf{p_r}]$ in the kernel:

$$\begin{bmatrix} \sigma_0 & \sigma_1 & \ldots & \sigma_r \\ \sigma_1 & & & \sigma_{r+1} \\ \vdots & & & \vdots \\ \sigma_{r-1} & \ldots & \sigma_{2r-1} & \sigma_{2r-1} \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ \vdots \\ p_r \end{bmatrix} = 0$$

- Compute the roots $\xi_{\mathbf{1}} = \mathbf{e}^{\zeta_1}, \ldots, \xi_{\mathbf{r}} = \mathbf{e}^{\zeta_r}$ of $p(x) := \sum_{i=0}^{r} p_i x^i$.
- Solve the system

$$\begin{bmatrix} 1 & \ldots & \ldots & 1 \\ \xi_1 & & & \xi_r \\ \vdots & & & \vdots \\ \xi_1^{r-1} & \ldots & \ldots & \xi_r^{r-1} \end{bmatrix} \begin{bmatrix} \omega_{\mathbf{1}} \\ \omega_{\mathbf{2}} \\ \vdots \\ \omega_{\mathbf{r}} \end{bmatrix} = \begin{bmatrix} \sigma_0 \\ \sigma_1 \\ \vdots \\ \sigma_{r-1} \end{bmatrix}.$$

Symmetric tensor decomposition and Waring problem (1770)

Tensor decomposition problem: given a homogeneous polynomial Ψ of degree d in the variables $\overline{\mathbf{x}} = (x_0, x_1, \ldots, x_n)$:

$$\Psi(\overline{\mathbf{x}}) = \sum_{|\alpha|=d} \sigma_\alpha \binom{d}{\alpha} \overline{\mathbf{x}}^\alpha,$$

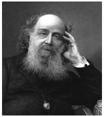
find a minimal decomposition of Ψ of the form

$$\Psi(\overline{\mathbf{x}}) = \sum_{i=1}^{r} \omega_i (\xi_{i,0} x_0 + \xi_{i,1} x_1 + \cdots + \xi_{i,n} x_n)^d$$

for $\xi_i = (\xi_{i,0}, \xi_{i,1}, \ldots, \xi_{i,n}) \in \mathbb{C}^{n+1}$ **,** $\omega_i \in \mathbb{C}$**.**

The minimal r in such a decomposition is called the **rank** of τ.

Sylvester approach (1851)



Theorem

The binary form $\Psi(x_0, x_1) = \sum_{i=0}^{d} \sigma_i \binom{d}{i} x_0^{d-i} x_1^i$ *can be decomposed as a sum of* r *distinct powers of linear forms*

$$\Psi = \sum_{k=1}^{r} \omega_k (\alpha_{\mathbf{k}} x_0 + \beta_{\mathbf{k}} x_1)^d$$

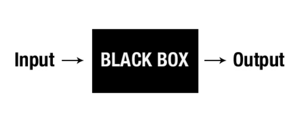
iff there exists a polynomial $p(x_0, x_1) := p_0 x_0^r + p_1 x_0^{r-1} x_1 + \cdots + p_r x_1^r$ *s.t.*

$$\begin{bmatrix} \sigma_0 & \sigma_1 & \ldots & \sigma_r \\ \sigma_1 & & & \sigma_{r+1} \\ \vdots & & & \vdots \\ \sigma_{d-r} & \ldots & \sigma_{d-1} & \sigma_d \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ \vdots \\ p_r \end{bmatrix} = 0$$

and of the form $p = c \prod_{k=1}^{r} (\beta_{\mathbf{k}} x_0 - \alpha_{\mathbf{k}} x_1)$.

Sparse interpolation

Given a black-box polynomial function $f(x)$



find what are the terms inside from output values.

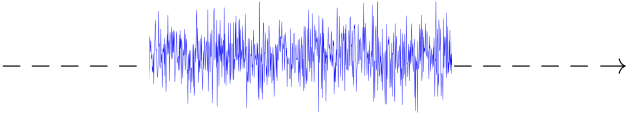
☞ Find $r \in \mathbb{N}, \omega_i \in \mathbb{C}, \alpha_i \in \mathbb{N}$ such that $f(x) = \sum_{i=1}^{r} \omega_i\, x^{\alpha_i}$.

- Choose $\varphi \in \mathbb{C}$
- Compute the sequence of terms $\sigma_0 = f(1), \ldots, \sigma_{2r-1} = f(\varphi^{2r-1})$;
- Construct the matrix $H = [\sigma_{i+j}]$ and its kernel $p = [p_0, \ldots, p_r]$ s.t.

$$\begin{bmatrix} \sigma_0 & \sigma_1 & \ldots & \sigma_r \\ \sigma_1 & & & \sigma_{r+1} \\ \vdots & & & \vdots \\ \sigma_{r-1} & \ldots & \sigma_{2r-1} & \sigma_{2r-1} \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ \vdots \\ p_r \end{bmatrix} = 0$$

- Compute the roots $\xi_{\mathbf{1}} = \varphi^{\alpha_1}, \ldots, \xi_{\mathbf{r}} = \varphi^{\alpha_r}$ of $p(x) := \sum_{i=0}^{r} p_i x^i$ and deduce the exponents $\alpha_{\mathbf{i}} = \log_\varphi(\xi_{\mathbf{i}})$.
- Deduce the weights $W = [\omega_i]$ by solving $V_\Xi W = [\sigma_0, \ldots, \sigma_{r-1}]$ where V_Ξ is the Vandermonde system of the roots $\xi_1, \ldots, \xi_r$.

Decoding



An algebraic code:

$$E = \{c(f) = [f(\xi_1), \ldots, f(\xi_m)] \mid f \in \mathbb{K}[x]; \deg(f) \leq d\}.$$

Encoding messages using the dual code:

$$C = E^{\perp} = \{\mathbf{c} \mid \mathbf{c} \cdot [f(\xi_1), \ldots, f(\xi_l)] = 0 \; \forall f \in V = \langle \mathbf{x}^{\mathbf{a}} \rangle \subset \mathbb{F}[\mathbf{x}]\}$$

Message received: $r = m + e$ for $m \in C$ where $e = [\omega_1, \ldots, \omega_m]$ is an error with $\omega_j \neq 0$ for $j = i_1, \ldots, i_r$ and $\omega_j = 0$ otherwise.

☞ **Find the error e.**

Berlekamp-Massey method (1969)

- Compute the syndrome $\sigma_k = c(x^k) \cdot r = c(x^k) \cdot e = \sum_{j=1}^{r} \omega_{i_j} \xi_{i_j}^k$.
- Compute the matrix

$$\begin{bmatrix} \sigma_0 & \sigma_1 & \dots & \sigma_r \\ \sigma_1 & & & \sigma_{r+1} \\ \vdots & & & \vdots \\ \sigma_{r-1} & \dots & \sigma_{2r-1} & \sigma_{2r-1} \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ \vdots \\ p_r \end{bmatrix} = 0$$

  and its kernel $p = [p_0, \dots, p_r]$.
- Compute the roots of the error locator polynomial $p(x) = \sum_{i=0}^{r} p_i \, x^i = p_r \prod_{j=1}^{r} (x - \xi_{i_j})$.
- Deduce the errors ω_{i_j}.

Sequences, series, duality (1D)

Sequences: $\sigma = (\sigma_k)_{k \in \mathbb{N}} \in \mathbb{K}^{\mathbb{N}}$ indexed by $k \in \mathbb{N}$.

Formal power series:

$$\sigma = \sum_{k=0}^{\infty} \sigma_k \frac{y^k}{k!} \in \mathbb{K}[[y]] \quad (\text{or } \sum_{k=0}^{\infty} \sigma_k z^k \in \mathbb{K}[[z]])$$

Linear functionals: $\mathbb{K}[x]^* = \{\Lambda : \mathbb{K}[x] \to \mathbb{K} \text{ linear}\}$.
Example:

- $p \mapsto$ coefficient of x^i in $p = \frac{1}{i!}\partial^i(p)(0)$
- $\mathbf{e}_\zeta : p \mapsto p(\zeta)$.

Series as linear functionals: For $\sigma = \sum_{k=0}^{\infty} \sigma_k \frac{y^k}{k!} \in \mathbb{K}[[y]]$,

$$\sigma : p = \sum_k p_k x^k \mapsto \langle \sigma | p \rangle = \sum_k \sigma_k p_k$$

$(\frac{y^k}{k!})$ is the dual basis of the monomial basis $(x^k)_{k \in \mathbb{N}}$.

Example: $\mathbf{e}_\zeta = \sum_{k=0}^{\infty} \zeta^k \frac{y^k}{k!} = e^{\zeta y} \in \mathbb{K}[[y]]$.

Structure of $\mathbb{K}[x]$-module: $p(x) \star \Lambda : q \mapsto \Lambda(p\, q)$.

$$\begin{aligned} x \star \sigma &= \sum_{k=1}^{\infty} \sigma_k \frac{y^{k-1}}{(k-1)!} = \partial(\sigma(y)) \\ p(x) \star \sigma &= p(\partial)(\sigma(y)). \end{aligned}$$

Sequences, series, duality (nD)

Multi-index sequences: $\sigma = (\sigma_\alpha)_{\alpha \in \mathbb{N}^n} \in \mathbb{K}^{\mathbb{N}^n}$ indexed by $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$.

Taylor series:

$$\sigma(\mathbf{y}) = \sum_{\alpha \in \mathbb{N}^n} \sigma_\alpha \frac{\mathbf{y}^\alpha}{\alpha!} \in \mathbb{K}[[y_1, \ldots, y_n]] \quad (\text{or} \sum_{\alpha \in \mathbb{N}^n} \sigma_\alpha \mathbf{z}^\alpha \in \mathbb{K}[[z_1, \ldots, z_n]])$$

where $\alpha! = \prod \alpha_i!$ for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}$.

Linear functionals: $\sigma \in R^* = \{\sigma : R \to \mathbb{K}, \textbf{linear}\}$

$$\sigma : p = \sum_\alpha p_\alpha \mathbf{x}^\alpha \mapsto \langle \sigma | p \rangle = \sum_\alpha \sigma_\alpha p_\alpha$$

The coefficients $\langle \sigma | \mathbf{x}^\alpha \rangle = \sigma_\alpha \in \mathbb{K}$, $\alpha \in \mathbb{N}^n$ are called the **moments** of σ.

Structure of R-module: $\forall p \in R, \sigma \in R^*$, $p \star \sigma : q \mapsto \langle \sigma | p\, q \rangle$.

☞ $$p \star \sigma = p(\partial_1, \ldots, \partial_n)(\sigma)(\mathbf{y})$$

Hankel operator: For $\sigma \in \mathbb{K}[[\mathbf{y}]]$,

$$\begin{array}{rcl} H_\sigma : \mathbb{K}[\mathbf{x}] & \rightarrow & \mathbb{K}[[\mathbf{y}]] \\ p & \mapsto & p \star \sigma = p(\partial_1, \ldots, \partial_n)(\sigma)(\mathbf{y}) \end{array}$$

σ is the **symbol** of H_σ.

Truncated Hankel operator: $V, V' \subset \mathbb{K}[\mathbf{x}]$,

$$H_\sigma^{V',V} : p \in V \rightarrow p \star \sigma_{|V'} \in V'^*$$

Example: $V = \langle \mathbf{x}^A \rangle, V' = \langle \mathbf{x}^B \rangle \subset \mathbb{K}[\mathbf{x}]$

$$H_\sigma^{A,B} = [\sigma_{\alpha+\beta}]_{\alpha \in A, \beta \in B}.$$

Ideal:

$$\begin{aligned} I_\sigma = \ker H_\sigma &= \{p \in \mathbb{K}[\mathbf{x}] \mid p \star \sigma = 0\}, \\ &= \{p = \sum_\alpha p_\alpha \mathbf{x}^\alpha \mid \forall \beta \in \mathbb{N}^n \sum_\alpha p_\alpha \sigma_{\alpha+\beta} = 0\} \end{aligned}$$

Linear recurrence relations on the sequence $\sigma = (\sigma_\alpha)_{\alpha \in \mathbb{N}^n}$.

Quotient algebra: $\mathcal{A}_\sigma = \mathbb{K}[x_1, \ldots, x_n]/I_\sigma$

☞ **Studied case:** $\dim \mathcal{A}_\sigma < \infty$

Structure of an Artinian algebra $\mathcal{A}$

Hypothesis: $\mathcal{A} = \mathbb{K}[\mathbf{x}]/I$ is Artinian , i.e. $\dim_{\mathbb{K}} \mathcal{A} < \infty$.
Hilbert nullstellensatz: $\mathcal{A} = \mathbb{K}[\mathbf{x}]/I$ Artinian $\Leftrightarrow \mathcal{V}_{\overline{\mathbb{K}}}(I) = \{\xi_1, \ldots, \xi_r\}$ is finite.

Assuming $\mathbb{K} = \overline{\mathbb{K}}$ is algebraically closed, we have

- $I = Q_1 \cap \cdots \cap Q_r$ where Q_i is m_{ξ_i}-primary where $\mathcal{V}_{\overline{\mathbb{K}}}(I) = \{\xi_1, \ldots, \xi_r\}$.
- $\mathcal{A} = \mathbb{K}[\mathbf{x}]/I = \mathcal{A}_1 \oplus \cdots \oplus \mathcal{A}_r,$ with
  - $\mathcal{A}_i = \mathbf{u}_i \, \mathcal{A} \sim \mathbb{K}[x_1, \ldots, x_n]/Q_i,$
  - $\mathbf{u}_i^2 = \mathbf{u}_i$, $\mathbf{u}_i \, \mathbf{u}_j = 0$ if $i \neq j$, $\mathbf{u}_1 + \cdots + \mathbf{u}_r = 1$.
- $\dim R/Q_i = \mu_i$ is the multiplicity of ξ_i.

Structure of the dual $\mathcal{A}^*$

Sparse series:

$$\mathcal{P}ol\mathcal{E}xp = \left\{ \sigma(\mathbf{y}) = \sum_{i=1}^{r} \omega_i(\mathbf{y})\, \mathbf{e}_{\xi_i}(\mathbf{y}) \mid \omega_i(\mathbf{y}) \in \mathbb{K}[\mathbf{y}], \right\}$$

where $\mathbf{e}_{\xi_i}(\mathbf{y}) = e^{\mathbf{y}\cdot\xi_i} = e^{y_1\xi_{1,i}+\cdots+y_n\xi_{n,i}}$ with $\xi_{i,j} \in \mathbb{K}$.

Inverse system generated by $\omega_1, \ldots, \omega_r \in \mathbb{K}[\mathbf{y}]$

$$\mathcal{D}(\omega_1, \ldots, \omega_r) = \langle \partial_{\mathbf{y}}^{\alpha}(\omega_i), \alpha \in \mathbb{N}^n \rangle$$

Theorem

For $\mathbb{K} = \overline{\mathbb{K}}$ algebraically closed,

$$\mathcal{A}^* = \oplus_{i=1}^{r} \mathcal{D}_i\, \mathbf{e}_{\xi_i}(\mathbf{y}) \quad \subset \quad \mathcal{P}ol\mathcal{E}xp$$

- $\mathcal{V}_{\overline{\mathbb{K}}}(I) = \{\xi_1, \ldots, \xi_r\}$
- $\mathcal{D}_i = \mathcal{D}(\omega_{i,1}, \ldots, \omega_{i,l_i}) = Q_i^{\perp}$ *with* $\omega_{i,j} \in \mathbb{K}[\mathbf{y}]$ *and* $I = Q_1 \cap \cdots \cap Q_r$
- $\mu(\omega_{i,1}, \ldots, \omega_{i,l_i}) := \dim_{\mathbb{K}}(\mathcal{D}_i) = \mu_i$ *multiplicity of* ξ_i.

The roots by eigencomputation

Hypothesis: $\mathcal{V}_{\mathbb{K}}(I) = \{\xi_1, \ldots, \xi_r\} \Leftrightarrow \mathcal{A} = \mathbb{K}[\mathbf{x}]/I$ Artinian.

$$\begin{array}{rcl} \mathcal{M}_a : \mathcal{A} & \to & \mathcal{A} \\ u & \mapsto & a\,u \end{array} \qquad \begin{array}{rcl} \mathcal{M}_a^{\mathsf{t}} : \mathcal{A}^* & \to & \mathcal{A}^* \\ \Lambda & \mapsto & a \star \Lambda = \Lambda \circ \mathcal{M}_a \end{array}$$

Theorem

- *The eigenvalues of $\mathcal{M}_a$ are $\{a(\xi_1), \ldots, a(\xi_r)\}$.*
- *The eigenvectors of all $(\mathcal{M}_a^{\mathsf{t}})_{a \in \mathcal{A}}$ are (up to a scalar) $\mathbf{e}_{\xi_i} : p \mapsto p(\xi_i)$.*

Proposition

If the roots are simple, the operators $\mathcal{M}_a$ are diagonalizable. Their common eigenvectors are, up to a scalar, interpolation polynomials $\mathbf{u}_i$ at the roots and idempotent in $\mathcal{A}$.

Example

Roots of polynomial systems

$$\left\{\begin{array}{lcl} f_1 & = & x_1^2 x_2 - x_1^2 \\ f_2 & = & x_1 x_2 - x_2 \end{array}\right. \qquad I = (f_1, f_2) \subset \mathbb{C}[\mathbf{x}]$$

$\mathcal{A} = \mathbb{C}[\mathbf{x}]/I \equiv \langle 1, x_1, x_2 \rangle \quad I = (x_1^2 - x_2, x_1 x_2 - x_2, x_2^2 - x_2)$

$$M_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} \qquad \begin{array}{c}\text{common} \\ \text{eigvecs of} \\ M_1^t, M_2^t \end{array} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$I = Q_1 \cap Q_2 \quad \text{where} \quad Q_1 = (x_1^2, x_2), \quad Q_2 = \mathbf{m}_{(1,1)} = (x_1 - 1, x_2 - 1)$

$I = Q_1^\perp \oplus Q_2^\perp \quad Q_1^\perp = \langle 1, y_1 \rangle = \langle 1, y_1 \rangle\, \mathbf{e}_{(0,0)}(\mathbf{y}) \quad Q_2^\perp = \langle 1 \rangle\, \mathbf{e}_{(1,1)}(\mathbf{y}) = \langle e^{y_1 + y_2} \rangle$

Solution of partial differential equations (with constant coeff.)

$$\left\{\begin{array}{lcl} \partial_{y_1}^2 \partial_{y_2} \sigma - \partial_{y_1}^2 \sigma & = & 0 \\ \partial_{y_1} \partial_{y_2} \sigma - \partial_{y_2} \sigma & = & 0 \end{array}\right. \qquad \begin{array}{l} f_1 \star \sigma = 0 \\ f_2 \star \sigma = 0 \end{array} \quad \Rightarrow \quad \sigma \in I^\perp = Q_1^\perp \oplus Q_2^\perp$$

$$\sigma = a + b\, y_1 + c\, e^{y_1 + y_2} \quad a, b, c \in \mathbb{C}$$

Hankel operators and quotient algebra

Hankel operator: For $\sigma \in \mathbb{K}[[\mathbf{y}]]$,

$$\begin{array}{rcl} H_\sigma : \mathbb{K}[\mathbf{x}] & \rightarrow & \mathbb{K}[[\mathbf{y}]] \\ p & \mapsto & p \star \sigma = p(\partial_1, \ldots, \partial_n)(\sigma)(\mathbf{y}) \end{array}$$

Quotient algebra: $\mathcal{A}_\sigma = \mathbb{K}[x_1, \ldots, x_n]/I_\sigma$

- $\mathcal{A} = \mathbb{K}[x_1, \ldots, x_n]/I$ **Artinian** iff $\dim_{\mathbb{K}} \mathcal{A} < \infty$.
- $\mathcal{A}$ **Gorenstein** iff $\exists \sigma \in \mathcal{A}^*$ such that $\mathcal{A}^* = \mathcal{A} \star \sigma$ is a free $\mathcal{A}$.

☞ **Isomorphism with the dual space $\mathcal{A}_\sigma^*$:**

$$\begin{array}{ccccc} 0 \rightarrow I_\sigma \rightarrow \mathbb{K}[\mathbf{x}] & \xrightarrow{H_\sigma} & \mathcal{A}_\sigma^* \rightarrow 0 \\ p & \mapsto & p \star \sigma \end{array}$$

Correspondence between **sequences** $\sigma \in \mathbb{K}^{\mathbb{N}^n}$ **with** $\mathrm{rank} H_\sigma < \infty$ and **Artinian Gorenstein algebras** $\mathcal{A}_\sigma := \mathbb{K}[\mathbf{x}]/I_\sigma$.

Univariate series:

Kronecker (1881)

The Hankel operator $H_\sigma : (p_m) \in \mathbb{C}^{\mathbb{N},finite} \mapsto (\sum_m \sigma_{m+n} p_m)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$ is of **finite rank** *r iff $\exists \omega_1, \ldots, \omega_{r'} \in \mathbb{C}[y]$ and $\xi_1, \ldots, \xi_{r'} \in \mathbb{C}$ distincts s.t.*

$$\sigma(y) = \sum_{n\in\mathbb{N}} \sigma_n \frac{y^n}{n!} = \sum_{i=1}^{r'} \omega_i(y) \mathbf{e}_{\xi_i}(y) \text{ with } \sum_{i=1}^{r'} (\deg(\omega_i)+1) = r.$$

Multivariate series:

Theorem (Generalized Kronecker Theorem)

H_σ is of rank r iff $\sigma = \sum_{i=1}^{r'} \omega_i(\mathbf{y})\, \mathbf{e}_{\xi_i}(\mathbf{y}) \in \mathcal{P}ol\mathcal{E}xp$ with $r = \sum_{i=1}^{r'} \mu(\omega_i)$. In this case, we have

- $\mathcal{V}_{\mathbb{C}}(I_\sigma) = \{\xi_1, \ldots, \xi_{r'}\}$.
- $I_\sigma = Q_1 \cap \cdots \cap Q_{r'}$ *with* $Q_i^\perp = \mathcal{D}(\omega_i)\, \mathbf{e}_{\xi_i}(\mathbf{y})$.
- $\mathcal{A}_\sigma^* = \mathcal{A}_\sigma \star \sigma$ *(free $\mathcal{A}_\sigma$-module of rank 1).*
- $(a, b) \mapsto \langle \sigma | ab \rangle$ *is non-degenerate in $\mathcal{A}_\sigma$.*

The structure of $\mathcal{A}_\sigma$

For $\sigma = \sum_{i=1}^{\mathbf{r}} \omega_{\mathbf{i}}\, \mathbf{e}_{\xi_{\mathbf{i}}}$, with $\omega_{\mathbf{i}} \in \mathbb{C} \setminus \{0\}$ and $\xi_i \in \mathbb{C}^n$ distinct.

- rank $H_\sigma = \mathbf{r}$ and the multiplicity of the points $\xi_{\mathbf{1}}, \ldots, \xi_{\mathbf{r}}$ in $\mathcal{V}(I_\sigma)$ is 1.
- For B, B' be of size r, $H_\sigma^{B',B}$ invertible iff B and B' are bases of $\mathcal{A}_\sigma = \mathbb{K}[\mathbf{x}]/I_\sigma$.
- The matrix M_i of multiplication by x_i in the basis B of $\mathcal{A}_\sigma$ is such that

$$\mathbf{H}_\sigma^{\mathbf{B}',\mathbf{x_i B}} = \mathbf{H}_{\mathbf{x_i}\star\sigma}^{\mathbf{B}',\mathbf{B}} = \mathbf{H}_\sigma^{\mathbf{B}',\mathbf{B}}\, \mathbf{M_i}$$

- The common **eigenvectors** of M_i are (up to a scalar) the Lagrange **interpolation polynomials** $\mathbf{u}_{\xi_{\mathbf{i}}}$ at the points $\xi_{\mathbf{i}}$, $i = 1, \ldots, r$.

$$\mathbf{u}_{\xi_{\mathbf{i}}}(\xi_{\mathbf{j}}) = \begin{cases} 1 & \text{if}\, i = j, \\ 0 & \text{otherwise} \end{cases} \quad {\mathbf{u}_{\xi_{\mathbf{i}}}}^2 \equiv \mathbf{u}_{\xi_{\mathbf{i}}},\ \sum_{i=1}^{\mathbf{r}} \mathbf{u}_{\xi_{\mathbf{i}}} \equiv 1.$$

- The common **eigenvectors** of M_i^t are (up to a scalar) the vectors $[B(\xi_{\mathbf{i}})]$, $i = 1, \ldots, r$.

Decomposition algorithm

Input: The first coefficients $(\sigma_\alpha)_{\alpha\in A}$ of the series

$$\sigma = \sum_{\alpha\in\mathbb{N}^n} \sigma_\alpha \frac{\mathbf{y}^\alpha}{\alpha!} = \sum_{i=1}^{r} \omega_i \mathbf{e}_{\xi_i}(\mathbf{y})$$

1. Compute bases $B, B' \subset \langle \mathbf{x}^A \rangle$ s.t. that $H^{B',B}$ invertible and $|B| = |B'| = r = \dim \mathcal{A}_\sigma$;
2. Deduce the tables of multiplications $M_i := (H_\sigma^{B',B})^{-1} H_\sigma^{B',x_i B}$
3. Compute the eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_r$ of $\sum_i l_i M_i$ for a generic $\mathbf{l} = l_1 x_1 + \cdots + l_n x_n$;
4. Deduce the points $\xi_i = (\xi_{i,1}, \ldots, \xi_{i,n})$ s.t. $M_j \mathbf{v}_i - \xi_{i,j} \mathbf{v}_i = 0$ and the weights $\omega_i = \frac{1}{\mathbf{v}_i(\xi_i)} \langle \sigma | \mathbf{v}_i \rangle$.

Output: The decomposition $\sigma = \sum_{i=1}^{r} \frac{1}{\mathbf{v}_i(\xi_i)} \langle \sigma | \mathbf{v}_i \rangle \mathbf{e}_{\xi_i}(\mathbf{y})$.

Multivariate Prony method (1)

Let $h(t_1, t_2) = 2 + 3\,2^{t_1}\,2^{t_2} - 3^{t_1}$, $\sigma = \sum_{\alpha \in \mathbb{N}^2} h(\alpha) \frac{\mathbf{y}^\alpha}{\alpha!} = 2\mathbf{e}_{(1,1)}(\mathbf{y}) + 3\mathbf{e}_{(2,2)}(\mathbf{y}) - \mathbf{e}_{(3,1)}(\mathbf{y})$.

- Take $B = \{1, x_1, x_2\}$ and compute

$$H_0 := H_\sigma^{B,B} = \begin{bmatrix} h(0,0) & h(1,0) & h(0,1) \\ h(1,0) & h(2,0) & h(1,1) \\ h(0,1) & h(1,1) & h(0,2) \end{bmatrix} = \begin{bmatrix} 4 & 5 & 7 \\ 5 & 5 & 11 \\ 7 & 11 & 13 \end{bmatrix},$$

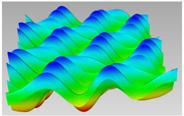
$$H_1 := H_\sigma^{B,x_1 B} = \begin{bmatrix} 5 & 5 & 7 \\ 5 & -1 & 17 \\ 811 & 178 & 23 \end{bmatrix}, \; H_2 := H_\sigma^{B,x_2 B} = \begin{bmatrix} 7 & 11 & 13 \\ 11 & 17 & 23 \\ 13 & 23 & 25 \end{bmatrix}.$$

- Compute the generalized eigenvectors of $(aH_1 + bH_2, H_0)$:

$$U = \begin{bmatrix} 2 & -1 & 0 \\ -1/2 & 0 & 1/2 \\ -1/2 & 1 & -1/2 \end{bmatrix} \text{ and } H_0\,U = \begin{bmatrix} 2 & 3 & -1 \\ 2 \times 1 & 3 \times 2 & -1 \times 3 \\ 2 \times 1 & 3 \times 2 & -1 \times 1 \end{bmatrix}.$$

- This yields the weights $2, 3, -1$ and the roots $(1,1), (2,2), (3,1)$.

Multivariate Prony method (2)



$h(t_1, t_2) := \sum_{i=1}^{r} \omega_i e^{f_1 t_1 + f_2 t_2}$ with

$$\mathbf{f} := \begin{bmatrix} -0.5 & 1.0 + 3.141592654\, i \\ 0.1 + 21.36283005\, i & 1.5 + 32.67256360\, i \\ 0.1 + 21.36283005\, i & -0.5 + 79.16813488\, i \\ -2.5 + 145.7698991\, i & -10.0 + 517.1061508\, i \end{bmatrix} \omega := \begin{bmatrix} 1.375328890 + 0.9992349291\, i \\ 1.046162168 + 0.3399186938\, i \\ 0.9 \\ -9.2 \end{bmatrix}$$

For the sampling $[\frac{1}{50}, \frac{1}{170}]$, $B = \{1, x_1, x_2, x_1 x_2\}$, the SVD of $H_\sigma^{B,B}$ is

$$[33.1196344300301391, 14.3767453860219057, 0.244096952193142480, 0.0230734326225932214]$$

and the computed decomposition is

$$\mathbf{f}^* = \begin{bmatrix} -2.49999999703636711 + 145.769899153890435\, i & -9.9999999913514852 + 517.106150711515852\, i \\ 0.0999940670173818935 + 21.3628392917863437\, i & -0.500045063743692286 + 79.1681566575291527\, i \\ 0.100028305341504586 + 21.3628527756206275\, i & 1.50002358381760881 + 32.6725933609709571\, i \\ -0.499926454593063452 + 0.0000142466247443506387\, i & 1.00008814016387371 + 3.14161379568963772\, i \end{bmatrix}$$

$$\omega^* = \begin{bmatrix} -9.19999999613861696 - 0.00000000772422142913953280\, i \\ 0.899999936743709261 - 0.00000156202814849404348\, i \\ 1.04615643213670850 + 0.339923495269889020\, i \\ 1.37533468654902213 + 0.999231697828891208\, i \end{bmatrix}$$

Sparse interpolation

$$f(\mathbf{x}) = \sum_{i=1}^{r} \omega_i \, \mathbf{x}^{\alpha_i} \quad \Rightarrow \quad \sigma = \sum_{\gamma} f(\varphi^{\gamma}) \frac{\mathbf{y}^{\gamma}}{\gamma!} = \sum_{i=1}^{r} \omega_i \, \mathbf{e}_{\varphi^{\alpha_i}}(\mathbf{y})$$

Example: $f(x_1, x_2) = x_1^{33} x_2^{12} - 5\, x_1 x_2^{45} + 101.$

- Compute $\sigma_\alpha = f(\varphi_1^{\alpha_1}, \varphi_2^{\alpha_2})$ for $\alpha_1 + \alpha_2 \leq 3$ and $\varphi_1 = \varphi_2 = e^{\frac{2\mathbf{i}\pi}{50}}$.
- Compute the Hankel matrix $H_\sigma^{1,2}$:

$$\begin{bmatrix} 97.00000 & 97.01771 + 3.93695\,\mathbf{i} & 95.50360 - 1.47099\,\mathbf{i} & 98.46280 + 4.88062\,\mathbf{i} & 97.42748 + 1.82098\,\mathbf{i} & 9 \\ 97.01771 + 3.93695\,\mathbf{i} & 98.46280 + 4.88062\,\mathbf{i} & 97.42748 + 1.82098\,\mathbf{i} & 102.35770 + 3.77300\,\mathbf{i} & 99.50853 + 5.29465\,\mathbf{i} & 9 \\ 95.50360 - 1.47099\,\mathbf{i} & 97.42748 + 1.82098\,\mathbf{i} & 95.73130 - .33862\,\mathbf{i} & 99.50853 + 5.29465\,\mathbf{i} & 95.42134 + 1.47250\,\mathbf{i} & 9 \end{bmatrix}$$

- Deduce the decomposition of $\sigma = \sum_{i=1}^{3} \omega_i \mathbf{e}_{\xi_i}$:

$$\Xi = \begin{bmatrix} 0.99211 + 0.12533\mathbf{i} & 0.80902 - 0.58779\mathbf{i} \\ 1.00000 + 4.86234e^{-11}\mathbf{i} & 1.00000 - 6.91726e^{-10}\mathbf{i} \\ -0.53583 - 0.84433\mathbf{i} & 0.06279 + 0.99803\mathbf{i} \end{bmatrix} \omega = \begin{bmatrix} -5.00000 - 4.43772e^{-7}\mathbf{i} \\ 101.00000 + 4.65640e^{-7}\mathbf{i} \\ 1.00000 - 1.92279e^{-8}\mathbf{i} \end{bmatrix}$$

- and the exponents $\frac{50\,\Xi}{2\pi i}$ mod 50 of the terms of f:

$$\begin{bmatrix} 1.00000 - 0.414119e^{-7}\,\mathbf{i} & -5.00000 + 0.270858e^{-6}\,\mathbf{i}, \\ 0.386933e^{-9} + 0.137963e^{-8}\,\mathbf{i} & -0.550458e^{-8} - 0.38761e^{-8}\,\mathbf{i} \\ -17.00000 - 0.100085e^{-6}\,\mathbf{i} & 12.00000 + 0.700984e^{-6}\,\mathbf{i} \end{bmatrix}$$

Symmetric tensor and apolarity

Apolar product: For $f = \sum_{|\alpha|=d} f_\alpha \binom{d}{\alpha} \overline{\mathbf{x}}^\alpha$, $g = \sum_{|\alpha|=d} g_\alpha \binom{d}{\alpha} \overline{\mathbf{x}}^\alpha \in \mathbb{K}[\overline{\mathbf{x}}]_d$,

$$\langle f, g \rangle_d = \sum_{|\alpha| \le d} f_\alpha \, g_\alpha \binom{d}{\alpha}.$$

Property: $\langle f, (\xi_0 x_0 + \cdots + \xi_n x_n)^d \rangle = f(\xi_0, \ldots, \xi_n)$

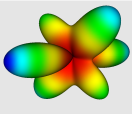
Duality:

$$\tau = \sum_{\alpha \in \mathbb{N}^{n+1}, |\alpha|=d} \tau_\alpha \binom{d}{\alpha} \mathbf{x}^\alpha \quad \Rightarrow \quad \tau^* = \sum_{\alpha' \in \mathbb{N}^n, |\alpha'| \le d} \tau_{\overline{\alpha'}} \frac{\mathbf{y}^{\alpha'}}{\alpha'!}$$

where $\overline{(\alpha_1, \ldots, \alpha_n)} = (d - \sum_i \alpha_i, \alpha_1, \ldots, \alpha_n) \in \mathbb{N}^{n+1}$

$\tau(\mathbf{x}) = \sum_{i=1}^{\mathbf{r}} \omega_{\mathbf{i}} \, (1 + \xi_{\mathbf{i},\mathbf{1}} \, x_1 + \cdots + \xi_{\mathbf{i},\mathbf{n}} x_n)^d \Leftrightarrow \tau^*(\mathbf{y}) = \sum_{i=1}^{\mathbf{r}} \omega_{\mathbf{i}} \, \mathbf{e}_{\xi_{\mathbf{i}}}(\mathbf{y}) + (\mathbf{y})^{d+1}$

Symmetric tensor decomposition



$$
\begin{aligned}
\tau &= (\mathbf{x_0} - \mathbf{x_1} + 3\,\mathbf{x_2})^4 + (\mathbf{x_0} + \mathbf{x_1} + \mathbf{x_2})^4 - 3\,(\mathbf{x_0} + 2\,\mathbf{x_1} + 2\,\mathbf{x_2})^4 \\
&= -{x_0}^4 - 24\,{x_0}^3 x_1 - 8\,{x_0}^3 x_2 - 60\,{x_0}^2 {x_1}^2 - 168\,{x_0}^2 x_1 x_2 - 12\,{x_0}^2 {x_2}^2 \\
&\quad -96\,x_0 {x_1}^3 - 240\,x_0 {x_1}^2 x_2 - 384\,x_0 x_1 {x_2}^2 + 16\,x_0 {x_2}^3 - 46\,{x_1}^4 - 200\,{x_1}^3 x_2 \\
&\quad -228\,{x_1}^2 {x_2}^2 - 296\,x_1 {x_2}^3 + 34\,{x_2}^4 \\
\tau^* &= \mathbf{e}_{(-1,3)}(\mathbf{y}) + \mathbf{e}_{(1,1)}(\mathbf{y}) - 3\mathbf{e}_{(2,2)}(\mathbf{y}) \qquad (\textit{by apolarity})
\end{aligned}
$$

$$
H_{\tau^*}^{2,2} := \begin{bmatrix}
-1 & -2 & -6 & -2 & -14 & -10 \\
-2 & -2 & -14 & 4 & -32 & -20 \\
-6 & -14 & -10 & -32 & -20 & -24 \\
-2 & 4 & -32 & 34 & -74 & -38 \\
-14 & -32 & -20 & -74 & -38 & -50 \\
-10 & -20 & -24 & -38 & -50 & -46
\end{bmatrix}
$$

For $B = \{1, x_2, x_1\}$,

$$
H_{\tau^*}^{B,B} = \begin{bmatrix}
-1 & -2 & -6 \\
-2 & -2 & -14 \\
-6 & -14 & -10
\end{bmatrix},
H_{\tau^*}^{B,x_1 B} = \begin{bmatrix}
-6 & -14 & -10 \\
-14 & -32 & -20 \\
-10 & -20 & -24
\end{bmatrix},
H_{\tau^*}^{B,x_2 B} = \begin{bmatrix}
-2 & -2 & -14 \\
-2 & 4 & -32 \\
-14 & -32 & -20
\end{bmatrix}
$$

- The matrix of multiplication by x_1 in $B = \{1, x_2, x_1\}$ is

$$M_1 = (H_{\tau^*}^{B,B})^{-1} H_{\tau^*}^{B,x_1 B} = \begin{bmatrix} 0 & -2 & -2 \\ 0 & \frac{1}{2} & \frac{3}{2} \\ 1 & \frac{5}{2} & \frac{3}{2} \end{bmatrix}.$$

- Its eigenvalues are $[-1, 1, 2]$ and the eigenvectors:

$$U := \begin{bmatrix} 0 & -2 & -1 \\ \frac{1}{2} & \frac{3}{4} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}.$$

that is the polynomials

$$U(x) = \left[\ \frac{1}{2} x_2 - \frac{1}{2} x_1 \quad -2 + \frac{3}{4} x_2 + \frac{1}{4} x_1 \quad -1 + \frac{1}{2} x_2 + \frac{1}{2} x_1 \ \right].$$

- We deduce the weights and the frequencies:

$$H_{\tau^*}^{[1,x_1,x_2],U} = \begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{-3} \\ \mathbf{1} \times \mathbf{-1} & \mathbf{1} \times \mathbf{1} & \mathbf{-3} \times \mathbf{2} \\ \mathbf{1} \times \mathbf{3} & \mathbf{1} \times \mathbf{1} & \mathbf{-3} \times \mathbf{2} \end{bmatrix}.$$

Weights: $1, 1, -3$; frequencies: $(-1, 3), (1, 1), (2, 2)$.

Decomposition: $\tau^*(\mathbf{y}) = \mathbf{e}_{(-1,3)}(\mathbf{y}) + \mathbf{e}_{(1,1)}(\mathbf{y}) - 3\,\mathbf{e}_{(2,2)}(\mathbf{y}) + (\mathbf{y})^4$

A general framework

- $\mathfrak{F}$ the functional space, in which the "signal" lives.
- $S_1, \ldots, S_n : \mathfrak{F} \to \mathfrak{F}$ commuting linear operators: $S_i \circ S_j = S_j \circ S_i$.
- $\Delta : h \in \mathfrak{F} \mapsto \Delta[h] \in \mathbb{C}$ a linear functional on $\mathfrak{F}$.

Generating series associated to $h \in \mathfrak{F}$:

$$\sigma_h(\mathbf{y}) = \sum_{\alpha \in \mathbb{N}^n} \Delta[S^\alpha(h)] \frac{\mathbf{y}^\alpha}{\alpha!} = \sum_{\alpha \in \mathbb{N}^n} \sigma_\alpha \frac{\mathbf{y}^\alpha}{\alpha!}.$$

- Eigenfunctions:

$$S_j(E) = \xi_j E, j = 1, \ldots, n \Rightarrow \sigma_E = \mathbf{e}_\xi(\mathbf{y}).$$

- Generalized eigenfunctions:

$$S_j(E_k) = \xi_j E_k + \sum_{k' < k} m_{j,k'} E_{k'}) \Rightarrow \sigma_{E_k} = \omega_i(\mathbf{y}) \mathbf{e}_\xi(\mathbf{y}).$$

☞ **If $h \mapsto \sigma_h$ is injective $\Rightarrow$ unique decomposition of f as a linear combination of generalized eigenfunctions.**

Other applications

- Decomposition of measures as sums of spikes from moments (images, spectroscopy, radar, astronomy, ...)
- Decomposition of convolution operators of finite rank
- Sparse interpolation of PolyLog functions
- Polynomial optimisation and convex relaxations
- Vanishing ideal of points: $\sigma = \sum_{i=1}^{r} \mathbf{e}_{\xi_i}(\mathbf{y})$
- Change of ordering for Grobner basis or basis representation for zero-dimensional (Gorenstein) ideals: $\sigma_\alpha = \langle u, N(\mathbf{x}^\alpha)\rangle$,
- ...

Challenges, open questions

- Numerical stability, correction of errors,
- Efficient construction of basis, complexity,
- Super-resolution, collision of points,
- Super-extrapolation,
- Best low rank approximation,
- ...

Thanks for your attention